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Advances in the Mathematical Sciences

Geometry of Differential Equations

A. Khovanskiĭ
A. Varchenko
V. Vassiliev
Editors

American Mathematical Society

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Geometry of
Differential Equations

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(*Formerly Advances in Soviet Mathematics*)

Geometry of Differential Equations

A. Khovanskiĭ
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American Mathematical Society
Providence, Rhode Island

ADVANCES IN THE MATHEMATICAL SCIENCES
EDITORIAL COMMITTEE
V. I. ARNOLD
S. G. GINDIKIN
V. P. MASLOV

1991 *Mathematics Subject Classification.* Primary 58Fxx; Secondary 14-xx, 32-xx, 70-xx.

ABSTRACT. This volume contains articles written by V. I. Arnold's colleagues on the occasion of his 60th birthday. The articles are mostly devoted to various aspects of geometry of differential equations and relations to global analysis and Hamiltonian mechanics. The book can be used by researchers and graduate students working in differential equations and geometry.

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Preface

This book is devoted to the geometry of differential equations; it contains articles written by V. I. Arnold's colleagues on the occasion of his 60th birthday.

Arnold's main interest all through his life has been the geometry of differential equations. Examples of this are his work on invariant tori in celestial mechanics, Maslov's index, and Lagrangian and Legendrian geometry (including his work on singularity theory). His basic goal was to understand and simplify a phenomenon, and to replace computations by geometric constructions.

Arnold likes to quote Hermann Weyl that the devil of algebra and the angel of geometry fight for the soul of every theory.

Examples of such a simplification are Arnold's books on ordinary differential equations and methods of classical mechanics. Everyone admires his books, although there are no limits for perfection. When one of us was an undergraduate student, he accompanied Arnold and I. M. Gelfand at Moscow State University. At some point the conversation turned to Arnold's books, and Gelfand said: "Dima, you absolutely don't know how to write books. Please come to me with one of your books, we'll take a page and I'll show how to rewrite this page properly." For us Arnold's books (as well as Gelfand's) were examples of brilliance, and for us it was just thrilling to listen to these remarks.

A characteristic of Arnold's style is his attraction to examples. Start with an example and you will learn the theory. He usually started his lectures in the U.S.A. with the words: "We in Russia always start with examples." Again according to Gelfand, theories come and go, but examples stay forever.

Arnold is a natural leader. He has an overwhelming passion to communicate his ideas, his point of view, his problems. We recall the famous Arnold lists of problems on singularity theory, differential equations, and symplectic geometry. Dozens of mathematicians consider themselves Arnold's students.

P. Eberlein told us about the first time he met Arnold, at a conference in Pennsylvania in 1989. In the conference room the blackboard was unusually easy to slide, and nobody warned Arnold in advance. When he filled up the blackboard and pushed it up, the crashing upper thrust spread chalk dust on Arnold and all around the wall, and vibrated vigorously. Arnold looked at the blackboard and said: "I think this demonstrates the power of Russian mathematics."

Arnold's brilliant personality and always sharp, sometimes controversial statements leave no one indifferent. We use this occasion to once again congratulate V. I. Arnold on his 60th birthday and wish him many more happy and fruitful years.

Now we say several words about the content of the book.

The development of symplectic topology was greatly influenced by the Arnold conjectures that provide Morse type estimates on the number of intersection points of Lagrangian submanifolds. In their paper in this collection, Ya. Eliashberg and M. Gromov show that stable Morse theory, which includes much more than just Morse inequalities, is also applicable to the theory of Lagrangian intersections (and its analogue in contact geometry). Eliashberg and Gromov establish a link between symplectic topology and pseudo-isotopy theory via the method of generating functions. They show, in particular, that algebro-K-theoretic invariants that appear in pseudo-isotopy theory provide important information for studying the topology of spaces of Lagrangian and Legendrian embeddings, and for other problems in symplectic topology. The authors systematically use a finite-dimensional approach, known as the method of generating functions, instead of the theory of pseudo-holomorphic curves, Floer homology theory, and other infinite-dimensional methods.

The principal motivation of Yu. I. Manin's paper is the desire to understand the quantum cohomology of $\mathbf{P}^2$ and to find an algebro-geometric object which could be called the mirror of $\mathbf{P}^2$, thus extending the scope of the mirror duality discovered for Calabi–Yau manifolds. He gives an algebraic-geometric picture of all Painlevé VI equations, which in particular suggests that the mirror of $\mathbf{P}^2$ can be thought of as a pencil of elliptic curves with labeled sections of order two and an additional, possibly transcendental, multisection. More precisely, the Picard–Fuchs equation for the periods of the mirror dual Calabi–Yau family is replaced in his paper by a "non-homogeneous Picard–Fuchs equation" satisfied by the Abelian integral from zero to this additional multisection.

The Euler equations for a rigid body (in body momentum representation) as well as the Euler equations for an ideal incompressible fluid (in spatial representation), can be generalized to the Lie–Poisson equations on the dual of a Lie algebra. This set-up and its counterpart, the Euler–Poincaré equations on the Lie algebra, are the basic ingredients used in the fundamental work of Arnold on hydrodynamical stability. Hernán Cendra, Darryl D. Holm, Jerrold E. Marsden and Tudor S. Ratiu develop the variational structure of the general Euler–Poincaré equations and show how this is related to the general theory of Lagrangian reduction. They also explain the abstract notion of circulation that gives rise to a general Kelvin–Noether theorem.

The Arnold–Hilbert problem on the zeroes of Abelian integrals was solved by A. N. Varchenko using the theory of fewnomials. In the paper of A. Gabrielov and A. Khovanskii the following local version of fewnomials theory is studied. A differential ring of analytic functions in several complex variables is called a ring of Noetherian functions if it is finitely generated as a ring and contains the ring of all polynomials. An effective bound is given on the multiplicity of an isolated solution of a system of n equations $f_i = 0$ where f_i belong to a ring of Noetherian functions in n complex variables. In the one-dimensional case, such an estimate was already known and has applications in number theory and control theory. The multi-dimensional case presented in the paper provides a solution of an old problem concerning finiteness properties of transcendental functions defined by algebraic partial differential equations.

Ya. G. Sinai in his paper studies statistical properties of local convex hulls of random processes whose probability distributions are invariant under groups of scaling transformations. It turns out that randomness introduces non-smoothness and some fractal structure in their shape.

Oleg Viro became interested in real algebraic geometry under the influence of Arnold's pioneering paper on the subject written in the early 1970s. In his paper, Viro obtains new restrictions on the topology of real algebraic hypersurfaces of a given degree. He remarks that even the rather lean homology of the projective space provides additional ways of linking and knotting surfaces. Various graphs describing the topology of the configuration of connected components of a hypersurface in projective space are introduced in his paper. Viro obtains new restrictions in terms of these graphs by using the Bezout theorem. Upper bounds on the number of non-contractible components of a real algebraic surface of a given degree in $\mathbb{R}P^3$ also are discussed in the paper.

The paper by A. Weinstein and P. Xu is devoted to the deformation theory of symplectic manifolds. The authors show that the Hochschild cohomology of the algebra obtained by formal deformation quantization on a symplectic manifold is isomorphic to the algebra of formal power series with coefficients in the de Rham cohomology of the manifold.

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