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Pseudoperiodic Topology

Vladimir Arnold
Maxim Kontsevich
Anton Zorich
Editors

American Mathematical Society

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Pseudoperiodic Topology

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Advances in the Mathematical Sciences — 46

(*Formerly Advances in Soviet Mathematics*)

Pseudoperiodic Topology

Vladimir Arnold
Maxim Kontsevich
Anton Zorich
Editors

American Mathematical Society
Providence, Rhode Island

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ABSTRACT. This book is an account of the present state of the art in pseudoperiodic topology, which is a young branch of mathematics, born at the boundary between the ergodic theory of dynamical systems, topology, and number theory, and related also to the theory of algorithms, convex integer polyhedra, Morse inequalities, real algebraic geometry, statistical physics, and algebraic number theory.

The book will be useful for researchers and graduate students working in dynamical systems, topology, and number theory.

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Preface

A *pseudoperiodic function* is the sum of a linear function and a periodic function. A *pseudoperiodic variety* is the set of common zeros of several pseudoperiodic functions.

An example of a pseudoperiodic curve is provided by the intersection of a periodic surface in three-space with a plane (which is virtually incommensurable with the periods lattice). Such curves occur in solid state physics (where the role of the periodic surface is played by the Fermi surface and the direction of the plane is defined by the magnetic field).

The present book is an account of the present state of the art in pseudoperiodic topology, which is a young branch of mathematics, born at the boundary between the ergodic theory of dynamical systems, topology, and number theory, and related also to algorithm theory, convex integer polyhedra, Morse inequalities, real algebraic geometry, statistical physics, and algebraic number theory. The authors represent the Moscow mathematical school.

The beginning of pseudoperiodic topology goes back to the classical works of Henri Poincaré on vector fields on tori (continued by A. Denjoy, A. Weil, C. L. Siegel, T. Cherry, and others). Following the ideas of Poincaré, A. N. Kolmogorov [**1**] proved in 1953 that the level lines of a closed one-form on the 2-torus, having no critical points, can be straightened by a suitable diffeomorphism of the torus (such that the form on the covering plane becomes the differential of a linear function).

This was the starting point of his celebrated works on invariant tori in Hamiltonian systems, now included in KAM-theory.

The study of the topology of level varieties of closed (but not exact) 1-forms is evidently related to pseudoperiodic topology.

From another point of view, such a form defines a *pseudo-Hamiltonian dynamical system* on a surface (the Hamilton function being replaced by a closed 1-form).

Studying the ergodic properties of such pseudo-Hamiltonian flows on surfaces of genus greater than one, I was led in 1963 to the model of *interval exchange* with its slow mixing. (This simple model was introduced in [**2**, Chapter 6, Section 2 ("Unsolved problems")]; see Russian Math. Surveys **18** (1963), p. 178, lines 15–18. Strangely, it seems that it had not been considered in earlier works on dynamical systems).

Studying the asymptotics of ergodic mean stabilization in this model in computer experiments (as well as in the original problem on pseudo-Hamiltonian flows), A. Zorich has discovered some striking phenomena.

Experiments have shown that some asymptotic characteristics of the memory loss in this system are universal: they are the same for almost all lengths of intervals, depending only on their number and on the order in which the intervals are permuted. This is one of few cases where an important new mathematical phenomenon has been discovered by computer experimentation.

The corresponding conjectures and theorems (due to A. Zorich and M. Kontsevich, and based on the study of the geodesic flows on Teichmüller spaces) are described below in Zorich's article, "How do the leaves of a closed 1-form wind around a surface?"

The problem on the intersection curve of a plane with a periodic surface in 3-space led S. P. Novikov (around 1981) to the remarkable conjecture that every infinite connected component of such a curve lives in a strip of finite width around a straight line of the plane.

The astonishing feature of this conjecture is the appearance of this line, which has no simple description in terms of the data of the problem and which is rather sensitive to a change in these data. Long attempts by Novikov and his students (especially S. Tsarev and I. Dynnikov) to prove this conjecture have led to the following striking answer: the conjecture is true "almost always", but in some "exceptional" cases there are counterexamples.

This situation is typical for pseudoperiodic topology: the Diophantine properties of the plane with respect to the period lattice are crucial, and the answer to this problem (as well as in the ergodic theory of continued fractions and the metric theory of Diophantine approximations) is universal not for all pseudoperiodic objects, but for almost all of them (in the Lebesgue measure sense).

This is also true in the problem of the ergodic properties of pseudo-Hamiltonian flows with singular points (which has been studied by Ya. G. Sinai and K. M. Khanin)—see the survey "Pseudoperiodic mappings" by D. A. Panov in the present book and my 1992 survey [**3**]. It seems that the situation is similar in many other problems of pseudoperiodic topology.

Perhaps this is the case for the other important conjecture by S. P. Novikov (discussed in A. Pajitnov's article "C^0-generic properties of boundary operators in the Novikov complex"), the conjecture on the exponential growth of the number of instantons (of the trajectories of the gradient of a pseudoperiodic function, on the manifold where the cyclic group acts, connecting one critical point to far-translated versions of the other).

This conjecture (which is still unproved, even for "generic" functions) has triggered a series of papers by several authors, including my theory on the asymptotics of intersections in dynamical systems (described in three papers, dedicated to the 60th birthdays of J. Moser, of S. Smale and of J. Milnor; see [**4, 5**] and the paper quoted by Pajitnov as [**2**]).

However, in this theory also, the crucial questions are open. This is, in particular, true for the conjecture on the typically exponential grows rate of the number of periodic orbits with the period. The problem on the possibility of superexponential growth in analytic systems is still open (despite the spectacular result by O. Kozlovski on the intersections growth rate).

The Gauss–Wiman–Kuzmin statistics of the elements of the continuous fraction expansion of a random irrational number (controlling the pseudoperiodic topology of the systems on a 2-torus) led me in 1988–1993 to a series of conjectures on "higher-dimensional continued fractions". Such a fraction is neither an algorithm nor a set of Diophantine approximations, but rather a certain infinite polyhedral hypersurface with integer vertices defined by a simplicial cones. It is the boundary of the convex hull of the semigroup of interior integer points of the cone.

The problems that I had then formulated included, among others, the generalization of the Lagrange theorem on periodic continued tractions (proved later by E. Korkina) as well as the computation of the statistics of different asymptotic characteristics of these polyhedra for generic simplicial cones. For instance, I suggested studying the mean number of vertices on a face, the distribution of the number of integer points on the edges and on the faces, and so on.

In formulating all these problems (published later in [**6**]), I had not realized that the very *existence* of the required universal statistics needs to be proved, not just the form of the statistics.

The existence proof is given in the remarkable article by M. Kontsevich and Yu. Sukhov in this book. It is notable that their article leaves open the most interesting problems on the character of the universal statistics whose existence they prove. We still do not know, say, whether the integer length of an edge of a random simplicial cone is shifted toward the longer or toward the shorter intervals as compared to the integer length of a random interval, whether there are more triangular than quadrangular faces, nor whether there are more faces of small integer area than of those with larger area (which, I hope, is true).

There is no proof of the coincidence of these universal statistics with the asymptotic averages obtained from simplicial cones defined by integral planes with bounded coefficients or defined by the eigenplanes of bounded integer matrices. There is no information on the asymptotic statistics of the triangulations of tori, corresponding to the "periodic" higher-dimensional continued fractions (even in the case of the 2-torus associated to cubical algebraic numbers fields).

One might regard the article of S. Gusein-Zade in this book as the first step in the pseudoperiodic version of Newton polyhedra theory and in real algebraic geometry. This domain (providing the "Morse inequality per volume unit" and "pseudoperiodic Harnak inequalities") will undoubtedly have important further development (relating it also to the statistics of quasicrystalls, as explained in [**7**]).

The theory of pseudoperiodic functions and varieties appears unexpectedly in many domains of mathematics. For instance, Hilbert [**8**] motivated his celebrated problem on smoothness in Lie group theory by the existence of analytic pseudoperiodic equations whose solutions are not smooth. Today we would call his equation the homological equation in the pseudoperiodic theory of circle rotation:

$$f(x+\alpha) - f(x) = g(x),$$

where g is the given and f the unknown 2π-periodic function (the mean value of g being equal to zero).

This equation has an analytic solution for almost every α. However, for some rotation angles α (incommensurable with 2π) the solutions of this equation (with an analytic right hand side g) are only finitely smooth (and sometimes even discontinuous).

This difficulty is typical for all the problems of pseudoperiodic topology. For almost all values of the parameters (in the sense of Lebesgue measure) the problems have natural and nice solutions. However, for some exceptional values of the parameter (forming everywhere dense sets) many kinds of pathology occur.

The authors of this book have done much to show how modern mathematics begets, from this sea of pathological counterexamples, remarkable general and universal laws, whose discovery would be unthinkable and whose formulation would be impossible in the naive set-theoretical setting.

V. I. Arnold

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