

Preface

This book is intended for you if you have already had a course in calculus—maybe a high school Advanced Placement class—and yet want to engage the concepts and ideas of single-variable calculus at a deeper level. If you are such a student, then you would not want a repeat of the same material but you also may be hesitant to declare your understanding of calculus topics complete.

I have been teaching an honors—or an alternative—second semester calculus class at Pomona College since the fall of 1991. My students have been and are great. They are eager to engage new material and work hard and they already have had a solid course in calculus. They know how to do routine derivatives and integrals and how to solve standard calculus problems. However, they are not so sure when I ask them: Why is finding the area under a curve related to finding slopes of tangents? What is so special about differentiable functions? What does $2^{\sqrt{3}}$ mean and how do you multiply 2 by itself $\sqrt{3}$ times? Can you estimate $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{10000}}$ or $\ln 2$ to a prescribed degree of accuracy? Could you imagine any way that calculus could be used to estimate the number of prime numbers between 10^9 and 10^{11} ?

On the first day of class, I also ask my students to tell me about areas of current research in mathematics. My students easily come up with many intellectually exciting open research questions in many of the sciences. They can also think of worthwhile research projects in the humanities and the social sciences but usually very little about mathematics emerges. What do mathematicians work on? What makes them excited about their subject? Do they actually do new research? What does it mean to do research in mathematics?

My goal has been to put together a collection that will give you an opportunity to think deeply and conceptually about calculus and, at the same time, give you some experience with the whole enterprise of mathematics.

The theme of the book is approximations. Calculus is such a powerful intellectual tool since it enables us to approximate complicated functions with simpler ones. In fact, replacing a function locally with a linear—or higher order—approximation is at the heart of calculus. We will be approximating functions and integrals, and we will be using linearization—replacing a function locally with a line—for many theoretical purposes, but the real test case for us will be approximating the number of primes up to a number x . Challenging calculus with the task of approximating the number of primes may seem unfair. After all, calculus is about things that change continuously and primes are a distinctly discrete phenomenon. Some experimentation and a few theorems will convince you that there is not that much rhyme or reason in the way new primes appear and yet we promise an interesting role for calculus.

More specifically, in seeking a smooth function that approximates the number of primes up to x we are led to the so-called Prime Number Theorem. In fact,

we introduce the natural log function because it is exactly what we need to state the Prime Number Theorem. We use calculus techniques and the Prime Number Theorem—which we do not prove—to gain more information about primes. For example, is there always a prime between n and $2n$? Does the series of the reciprocals of primes diverge? In so doing we see in action an interplay between the discrete and the continuous, and we get to use and develop many of the standard calculus techniques. Most major topics of calculus such as derivatives, integrals, fundamental theorem of calculus, transcendental functions, sequences and series, Taylor polynomials and series, differential equations, theorems like the Mean Value Theorem, and convergence are treated, but we allow ourselves to take many detours and do other unusual topics as well: Curvature, Padé approximants, the logarithmic integral, public key cryptography, and the qualitative analysis of the logistic equation are a few.

I want to emphasize that this is not a technical guide to approximation theory but rather an excuse to invite beginners to analytical thinking. In a course on numerical methods or approximation theory the real question is whether the method under consideration is worth using and the best possible. My aim here is to keep coming back to a few main ideas—linearization, Taylor polynomials, approximating an integral using areas of rectangles, induction, the mean value theorem, and the fundamental theorem of calculus—in many different contexts to demonstrate the versatility and power of the main ideas of calculus.

The style of the book is also different from a traditional mathematics text. You, the student, are asked to take an active role in developing the material. You do this by solving problems, and so this is a “problem solving” approach to learning about the ideas of calculus. Almost all topics are broken down to a series of problems that lead you toward an understanding of the material. Many of the problems are open ended and do not have a “right” answer. They invite you to investigate a certain pattern and suggest a conjecture. In so doing, I want you not only to participate in finding solutions but also in formulating questions. The mathematical enterprise has many aspects. Discovery of patterns, formulation of questions and conjectures, and logical reasoning and proof. In traditional calculus classes, the student’s role is mainly to apply the learned algorithms and ideas to a series of (often repetitive) problems. Here, the hope is that you will also experience the joy of discovery which is one of the main attractions of the subject to professional mathematicians. Beginning with discrete topics has the advantage that we can introduce the idea of a proof in a more concrete setting, and we can also quickly lead the students to open problems in mathematics. It is one of my goals to show that mathematics is alive and growing and contrary to public opinion it was not completed between the time of the Greeks and that of Newton.

When I teach from this material, at any given time, the students are working on a couple of different subjects or strands. In one area we are just starting to look at examples and making conjectures, in another we are developing the theory, and in a third area we may be looking at applications. This at times is a bit confusing and it also requires a judicious choice of problems. However, it allows us to cover a significant amount of material. The students usually enjoy seeing these different “threads” come together as the course progresses. In fact, in deciding in which order to cover the material—and you want to avoid getting bogged down in the first few chapters—you may want to think of this as three interrelated but somewhat

independent books: The first one is on induction, very elementary number theory, primes and their distribution (roughly Chapters 1–3, 6, and 12); the second one is on derivatives, integrals, and approximations (roughly Chapters 4, 5, 7–1, 13, 17, and 18); and the third one is on sequences and series (Chapters 14–16).

I don't believe much is to be gained by doing examples of problems in the text and then having you copy the process. In real life very few mathematical problems come neatly packaged at the end of a chapter with supporting examples. On the other hand, I do want you to read the text which attempts to give you the right attitude toward the concepts. I have tried to resist the temptation to present a complete or mathematically slick treatment and have only presented material that I want you to read. I also have to warn you that doing the problems is a challenging task and can be at times quite frustrating. This is quite normal. In fact, if you do not find at least some of the problems quite challenging, then you are reading the wrong book. However, with much perseverance and some guidance (there is also a hint section at the back of the book), almost any student can make their way through this material.

Prerequisites. These problems assume that you have had a first semester calculus class. We do review the basic ideas and concepts of calculus in Chapters 4 and 5, and throughout the book we keep coming back to the main ideas and concepts of calculus. Hence there will be ample opportunity to enhance and deepen your understanding of one-variable calculus. However, we do not spend much time practicing various algorithms for solving standard calculus problems, and, for example, you are expected to know how to differentiate reasonable looking functions, already have a basic idea of the properties of a derivative—e.g., a positive derivative means that the function is increasing—know how to do simple integration by substitution, and believe that integration can be used to find areas. It is fine if you don't know exactly why any of these things work or why they are interesting. As long as you are curious and willing to challenge yourself we can get started.

Mathematical Rigor and Proof. Many mathematicians believe that proofs are at the heart of mathematics and without rigorous definitions and proofs we are not doing mathematics. Proofs not only provide justification but they allow us to “look under the hood” so to speak. A good proof does much more than tell us that a statement follows from the assumptions. It gives us intuition, it reveals the hidden and subtle aspects of our definitions and our assumptions, it gives a road map for constructing good examples, it confronts us with new questions, and it points to possible future directions. Getting trained in the art of deciphering and constructing proofs certainly is an important part of the rite of passage to mathematical culture and the mathematical community.

My approach to introducing you to proofs is not a linear but rather a spiral one. We are not developing an axiomatic treatment of calculus and hence we do not prove every fact that we use. However, I try to be honest about when proofs are necessary as well as when rigor is lacking. As the book progresses we continue to come back to previously unexamined assumptions and concepts and delve into them deeper.

In the first three chapters, when doing discrete mathematics, I ask you to do rigorous proofs, and both induction and proof by contradiction are introduced and used. This will be natural since we find patterns that clearly need a proof

and without a proof we would not be convinced of the longevity of the pattern. Even here I don't insist that everything be proved. Sometimes we see a pattern, conjecture that it continues, realize that a proof is needed, but go on and use the result without an actual proof (e.g., you will state as a conjecture and then use the theorem that makes RSA cryptosystem work, but a proof will not be given).

The spiral metaphor will be in full force in the treatment of calculus topics. We will *not* begin with a rigorous definition of limits and derivatives but rather rely on your prior understanding of these concepts. It is my belief that—in the first pass through calculus—the emphasis should be on the intuition that underlies much of calculus and rigor should be introduced as needed. This attitude matches the historical development of calculus well. After all, mathematicians developed tremendously powerful mathematics between when Fermat—in the 1630s—found tangent lines as a limit of secant lines and when Cauchy—in the 1820s—gave what we consider a satisfactory definition of limits and derivatives. As my colleague and historian of mathematics Judith Grabiner [41, p. 195] has said: “The derivative was first *used*; it was then *discovered*; it was then *explored and developed*; and it was finally *defined*.” I have attempted something similar in this book. While limits of functions are not defined properly at first, limits of sequences and series—in Chapters 14–16—are treated rigorously. In Chapter 5 we first give an intuitive proof of the Fundamental Theorem of Calculus and then go back and fill in many of the theoretical holes.

It is my hope that, as you work your way through the book, you will gain an appreciation for the role and importance of mathematical rigor as well as the value of effectively and clearly communicating mathematical arguments.

How to Use This Book as a Text. There is enough material in this book for a number of different courses. When I teach a second semester single variable calculus course, I do enough from the earlier chapters to give the students some experience with prime numbers, writing proofs in paragraph form, and with the basic ideas of linearization, Taylor polynomials, the fundamental theorem of calculus, and approximating integrals with inscribed and circumscribed rectangles—I cover only some of the tangential topics presented in Chapters 1–5. Then I concentrate on transcendental functions, the mean value theorem, error in Taylor polynomials, sequences and series, and differential equations. I usually begin with Chapters 1 and 4 simultaneously, and, often, the students continue to work on projects and problems from one chapter while we start on a new one.

In an introduction to proof class, the more theoretical material, and especially Chapters 8, 10 and 14–17, can be emphasized.

The book is meant to encourage students to participate in the development of the material. There are many (optional) projects scattered throughout that can become the subject of expository student papers. I have put most of the proofs in the problems—as opposed to in the text—since I think that it is important that the students fully engage at least some of them. Reading someone else's proof can be tedious, but doing your own is often exhilarating.

Detailed syllabi for possible courses are available on the book's web page hosted by the AMS. The URL for the web page is on the back cover of the book.

Computer Software. The book contains many references to Maple. To do some of the problems you actually need a symbolic mathematical software like

Maple. However, most of the themes of the book can easily be developed without computers, and in fact I did not use any computers the first few times that I taught this material.

Acknowledgments. Some of the problems and a few of the approaches in this book are mine, but I have learned most of them from others. It is impossible to recall where I have encountered each problem or topic (some useful and inspiring sources have been Aleksandrov [4], Fitzpatrick [31], Hughes-Hallett et al. [46], Shahriari [74], Taylor and Mann [86], and Wilf [92]), but a few sources do need special mention. I learned much from my discussions with Robert Young in 1985–86 while he was busy writing his magnificent book [93]. Even though the final product ended up being very different, I started developing this material after trying to imagine an actual calculus class based on Bob’s book. 1985 is also when the calculus book by George Simmons [78] appeared. I have enjoyed reading this book and I have learned many gems from it. I was developing this material as the “calculus reform” movement was gathering speed. Even though my attitude toward proofs and rigor may be a bit different from the dominant “reformed” perspective, the effects of the discussions and ideas that have been in the air because of this effort can be seen throughout this book. In particular, I learned from this movement the importance of developing intuition and the fact that “approximations” can be the central theme of calculus.

When I look at the finished product and the book’s conceptual emphasis as well as its do-everything-through-challenge-problems style, I am struck by how much I continue to be influenced by how I, myself, learned calculus. My first pass through calculus was in the summer after eleventh grade when I came home asking my dad how to answer a challenge posed by my German language teacher. He had asked our class—of mostly recent high school graduates who had just finished a year of calculus—to find the derivative of x^x . In a span of three or four memorable summer evenings in Tehran, my dad told me about the history of calculus, challenged me with problems, and taught me enough calculus to be able to answer the challenge correctly. My second pass was as a freshman at Oberlin College where—at the time—students learned calculus through a self-paced method. I carefully read Thomas [36] and learned most by doing problems. Finally, as a teaching assistant and as an apprentice, I learned much from watching master teacher Marty Isaacs teach calculus.

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