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Laguerre Calculus and Its Applications on the Heisenberg Group

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Preface

In the same way that the study of holomorphic functions of one complex variable leads immediately to the recognition that there is a relation between the study of power series and of Fourier series, the corresponding study of holomorphic functions of several complex variables leads to the study boundary value problems for the Cauchy-Riemann operator $\bar{\partial}$ in the typically strongly pesudoconvex domain, the unit ball $\mathbf{B}_{n+1}$ in $\mathbf{C}^{n+1}$. As it turns out, $\partial\mathbf{B}_{n+1}$ is the one point compactification of the Heisenberg group H_n. (For $n = 0$, the unit circle can be identified to the one-point compactification of the real line $\mathbf{R}$.) This group not only acts on $\partial\mathbf{B}_{n+1}$, but also leaves the boundary operator $\bar{\partial}_b$ invariant in the sense of distributions. These analogies drive the analysis of the boundary value problem for $\bar{\partial}$ and makes this analysis dependent on the Fourier analysis of the nilpotent Lie group H_n. This has been the approach taken in the significant, insightful work of E.M. Stein and his collaborators (see *e.g.,* Folland and Stein [1], Müller, Ricci and Stein [2], [3]).

In this short monograph we take a slightly more direct path, inspired by the work of R. Beals, B. Gaveau, D. Geller, P. Greiner, J. Peetre, E.M. Stein and R. Strichartz, reducing the analysis to the symbolic calculus associated to the operator $\bar{\partial}_b$, the Laguerre calculus. The advantage of this approach is that it leads to an "elementary" introduction to the study of the Cauchy-Riemann operator and its fundamental questions for graduate students whose background knowledge includes only one year in complex analysis, as those at the University of Maryland, College Park. One of the additional payoffs of this approach is that the students are also introduced to the study of Lie groups via the study of the concrete compact Lie group $SO(n)$ and the simplest non-commutative and non-compact Lie group, the Heisenberg group H_n. This group serves as a model in many ways: as a contact manifold, a strongly pseudoconvex boundary in $\mathbf{C}^{n+1}$, and a subriemannian or Carnot-Carathéodory manifold. Note that although the concept of pseudoconvexity is essential to a complete study of the theory of functions of several complex variables, it does not play an explicit role here, thus accomodating the reduced background knowledge of the students rather small, while motivating them to continue their study of several complex variables on their own or through more advanced courses.

Another objective we had in mind was, in the tradition of the Calderón-Stein-Zygmund school, to introduce the student to non-conventional but elementary problems in harmonic analysis that arise naturally out the basic ideas of real and complex analysis; differentiation of integrals and Cauchy criterion for holomorphicity, namely, the Pompeiu and Morera problems in the unit ball and the Heisenberg group. After all, it is to this school of thought that we owe what now seems obvious, the seemless integration of harmonic analysis, complex analysis, and partial differential equations. A glance at the table of contents should reassure the prospective reader and, hopefully teacher, that this book may provide a natural introduction to wider classical analysis.

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Bibliography

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