General Introduction

1. Overview

This book consists of two independent works that prove different extensions of D. Christodoulou and S. Klainerman's stability theorem of the Minkowski space in General Relativity. The first part, by Lydia Bieri, discusses solutions of the Einstein vacuum equations (obtained in her Ph.D. thesis [**2**] in 2007), and the second part, by Nina Zipser, discusses solutions of the Einstein-Maxwell equations (obtained in her Ph.D. thesis [**14**] in 2000). To the authors' present knowledge, these are the only extensions of the celebrated results in 'The global nonlinear stability of the Minkowski space' [**8**].

In the first part of the book, Lydia Bieri solves the Cauchy problem for the Einstein vacuum (EV) equations with more general, asymptotically flat initial data, and describes precisely the asymptotic behaviour. In particular, she assumes one less decay in the power of r and one less derivative than in [**8**]. She proves that also in this case, the initial data, being globally close to the trivial data, yields a solution which is a complete spacetime, tending to the Minkowski spacetime at infinity along any geodesic. Contrary to the situation in [**8**], certain estimates in this proof are borderline in view of decay, indicating that the conditions in the main theorem on the decay at infinity on the initial data are sharp. The main results of this work are stated in the 'Introduction', section 1.3, in Theorem 2, and in the chapter 'Main Theorem' in Theorem 3.

In 'The global nonlinear stability of the Minkowski space' [**8**], D. Christodoulou and S. Klainerman proved the following result: 'Every strongly asymptotically flat, maximal, initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime at infinity along any geodesic.'

It is still an open question, what the optimal conditions are for non-trivial asymptotically flat initial data sets to give rise to a maximal complete development. L. Bieri's work contributes to answering this question, stating sharp conditions on the decay at spatial infinity. It addresses the global, nonlinear stability of solutions of the Einstein vacuum (EV) equations in General Relativity. Solutions of the EV equations

$$R_{\mu\nu} = 0$$

are spacetimes (M, g), where M is a four-dimensional, oriented, differentiable manifold and g is a Lorentzian metric obeying the EV equations.

As a consequence of imposing fewer conditions on her data, the spacetime curvature in Bieri's case is no longer in $L^\infty(M)$. She only controls one derivative of the curvature (Ricci) in $L^2(H)$. By the trace lemma, the Gauss curvature K in the leaves of the u-foliation S is only in L^4. Contrary to that, in [**8**], the Ricci curvature is in $L^\infty(H)$, and in $L^\infty(S)$. Christodoulou and Klainerman control two derivatives of the curvature (Ricci) in $L^2(H)$.

The situation in Bieri's case is both a disadvantage and an advantage. First, as she does not have the curvature bounded in L^∞, certain steps of the proof become more subtle. On the other hand, she does not have to control the second derivatives of the curvature, which simplifies the proof. The fact that she does not use any rotational vectorfields in her proof is a major simplification. She gains control of the angular derivatives of the curvature directly from the Bianchi equations, whereas in [**8**], a difficult construction of rotational vectorfields was necessary. Another major difference to the situation studied in [**8**] by Christodoulou and Klainerman, and which arises from Bieri's relaxed assumptions, is the fact that she encounters borderline cases in view of decay in the power of r, indicating that the conditions in her main theorem on the decay at infinity on the initial data are sharp. Any further relaxation would make the corresponding integrals diverge and the argument would not close any more. Also in Bieri's situation, energy and linear momentum are well-defined and conserved, whereas the (ADM) angular momentum is not defined. This is different to the situation investigated in [**8**], where all these quantities are well-defined and conserved.

In the second part of this book, Nina Zipser proves the existence of smooth, global solutions to the Einstein-Maxwell (EM) equations. A nontrivial solution of the EM equations is a nontrivial Lorentzian manifold – or curved spacetime – with an electromagnetic field.

To prove the existence of solutions to the Einstein-Maxwell equations, Zipser follows the argument and methodology introduced in [**8**] and outlined below. To generalize Christodoulou and Klainerman's results, she needs to contend with the additional curvature terms that arise due to the presence of the electromagnetic field F; in her case the Ricci curvature of the spacetime is not identically zero but rather represented by a quadratic in the components of F. In particular the Ricci curvature is a constant multiple of the stress-energy tensor for F. Furthermore, the traceless part of the Riemann curvature tensor no longer satisfies the homogeneous Bianchi equations but rather inhomogeneous equations including components of the spacetime Ricci curvature.

Therefore, the second part of this book focuses primarily on the derivation of estimates for the new terms that arise due to the presence of the electromagnetic field. To produce estimates for the electromagnetic field, Zipser uses the Maxwell equations together with the stress-energy tensor much like Christodoulou and Klainerman use the Bianchi equations and

Bel-Robinson tensor to produce global energy estimates. Also as in [**8**], she uses modified Lie derivatives of the electromagnetic field to obtain higher-order estimates. Once she produces good estimates for the electromagnetic field, she can bound the extra terms that appear in (i) the inhomogeneous equations for the Weyl tensor, (ii) the elliptic system for the parameters of the time foliation, and (iii) the Hessian of the optical function. After the extra terms are controlled, the results follow from generalizations of the proofs in [**8**].

In proving the stability of Minkowski space, Christodoulou and Klainerman rely on the invariant formulation of the EV equations. They cite two primary difficulties to overcome. The first difficulty is that a general spacetime has no symmetries, and therefore the conformal isometry group is trivial. Hence, the vectorfields needed to construct conserved quantities do not exist. The second is the highly non-linear nature of the hyperbolic equations, which makes it difficult to bound the asymptotic behavior of solutions.

To tackle these issues, Christodoulou and Klainerman rely on geometric constructions that are analogous to structures existing on Minkowski spacetime. These include the following: a time-foliation, whose hypersurfaces are defined as the level-sets of a time function; an optical function, whose level sets define the null-structure of the spacetime; and the definition of the action of the subgroup of the conformal isometry group of Minkowski spacetime corresponding to time translations, scale transformations, inverted time translations and the spatial rotation group $O(3)$. Whereas in the second part of the book, all these vectorfields play a crucial role, the first part relies on the first three but does not use rotational vectorfields.

These geometric constructions have three key applications. First, the structure equations of the time-foliation together with the condition that the surfaces are maximal give rise to an elliptic system of equations for the parameters of the foliation. Once Christodoulou and Klainerman produce good estimates for the spacetime Riemann curvature tensor, these parameters are fully determined by this elliptic system. Second, Christodoulou and Klainerman use the optical function to construct quasi-conformal Killing fields. Third, these vectorfields can be used to produce higher-order energy estimates of the curvature. In particular, instead of estimating derivatives of the curvature directly from the Bianchi identities, they apply modified Lie derivatives with respect to the quasi-conformal Killing fields.

The modified Lie derivatives applied to the traceless part of the Riemann curvature tensor retain the attributes of a Weyl tensor and satisfy inhomogeneous equations derived from the Bianchi equations. The norms for the global energy estimates are then constructed from the Bel-Robinson tensor, which is defined as a quadratic of a Weyl tensor.

To prove the main theorem in their work, Christodoulou and Klainerman employ a continuity argument. In essence, once they show the existence of a spacetime slab with appropriate conditions on the initial slice, they use the global energy estimates discussed above to control the geometry of the spacetime slab. This allows them to prove that the geometry on the last slice can be bounded by the initial data and thus extend their solution.

These concepts are explained in detail in the introduction to the first part of this book.

Both works use the main approach of Christodoulou and Klainerman from [**8**]. This method has been elaborated and combined with new ideas by Christodoulou in order to study the formation of black holes in GR [**6**]. Our notation stays close to the notation in [**8**]. However, the two cases investigated here differ fundamentally from the original one and so do the details of the proofs. The differences are pointed out in the two introductions. As for the generalization in the EV case, the details of the proof are very different from the original result and require new ideas. In the EM case, the energy-momentum tensor is equal to the stress-energy tensor of an electro-magnetic field, and therefore, additional curvature terms have to be controlled.

2. Former Work

In the framework of the Cauchy problem for the EV equations $R_{\mu\nu} = 0$ the following question had been the subject of investigations by many authors for a long time: Is there any non-trivial, asymptotically flat initial data whose maximal development is complete?

In 1952, Y. Choquet-Bruhat focussed the question of local existence and uniqueness of solutions, in GR. In [**3**] she treated the Cauchy problem for the Einstein equations, locally in time, she showed existence and uniqueness of solutions, reducing the Einstein equations to wave equations, introducing harmonic (or wave) coordinates. We recall that for a Riemannian manifold (M, g) a function Φ is called harmonic if $\triangle_g \Phi = 0$ with $\triangle_g \Phi = g^{\mu\nu} \nabla_\mu(\partial_\nu \Phi)$, where ∇ is the covariant derivative on M associated to g. If the metric g is Lorentzian, then the equation $\triangle_g \Phi = 0$ is the *wave equation.* She proved the well-posedeness of the local Cauchy problem in these coordinates. The local result led to a global theorem proved by Y. Choquet-Bruhat and R. Geroch in [**4**], stating the existence of a unique maximal future development for each given initial data set. In a next step, it is natural to ask whether this maximal future development is complete. R. Penrose gave a negative answer in his incompleteness theorem [**13**]. See also [**5**].

The said theorem tells us that, if in the initial data set $(H, \bar{g}, k)$, H is non-compact but complete, if the positivity condition on the energy holds, and H contains a closed trapped surface S, the boundary of a compact domain in H, then the corresponding maximal future development is incomplete. An exposition of this theorem is given in [**5**].

Definition 1. *A closed trapped surface S in a non-compact Cauchy hypersurface H is a two-dimensional surface in H, bounding a compact domain such that*

$$tr\chi < 0 \quad on\ S.$$

Note that an infinitesimal displacement of S in M towards the future along the outgoing null geodesic congruence results in a pointwise decrease of the area element.

The theorem of Penrose and its extensions by S. Hawking and R. Penrose led directly to the question formulated above: Is there any non-trivial asymptotically flat initial data whose maximal development is complete?

The answer was given in the joint work of D. Christodoulou and S. Klainerman [**8**], 'The global nonlinear stability of the Minkowski space'. A rough version of the theorem is stated in the first monograph at the end of subsection 1.2, whereas a more precise version is given in theorem 1. The problem studied by Christodoulou and Klainerman in [**8**] was suggested by S.-T. Yau to Klainerman in 1978. (Personal communication Yau, Christodoulou, 2008.)

Lately, a proof under stronger conditions for the global stability of Minkowski space for the EV equations and asymptotically flat Schwarzschild initial data was given by H. Lindblad and I. Rodnianski [**11, 12**], the latter for EV (scalar field) equations. They worked with a wave coordinate gauge, showing the wave coordinates to be stable globally. Concerning the asymptotic behaviour, the results are less precise than the ones of Christodoulou and Klainerman in [**8**]. Moreover, there are more conditions to be imposed on the data than in [**8**]. There is a variant for the exterior part of the proof from [**8**] using a double-null foliation by S. Klainerman and F. Nicolò in [**10**]. Also a semiglobal result was given by H. Friedrich [**9**] with initial data on a spacelike hyperboloid.

A still open question is: What is the sharp criteria for non-trivial asymptotically flat initial data sets to give rise to a maximal development that is complete? Or, to what extent can the result of [**8**] be generalized?

The results of [**8**] as well as of [**14**] and [**2**] are much more general than the others cited above, as all the other works place stronger conditions on the data. In this book, we give the results of [**2**] and [**14**], proving two different generalizations of [**8**].

3. Mathematical and Physical Structures

In this section, we expose some fundamental mathematical and physical structures.

3.1. Spacetime and Curvature. Let (M, g) denote our spacetime, represented by a $3+1$ dimensional manifold M with Lorentzian metric g. The tangent space at each point of (M, g) is isomorphic to Minkowski spacetime.

We recall the following facts about the Riemannian curvature tensor R. For any given vectorfields X, Y, Z on (M, g), it is

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \tag{1}$$

Or in arbitrary local coordinates:

$$R^{\alpha}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\lambda} + \Gamma^{\alpha}_{\beta\lambda}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\lambda}. \tag{2}$$

Also, recall the Christoffel symbols to be:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}).$$

The Ricci tensor then reads:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}. \tag{3}$$

The Riemannian curvature tensor R of the spacetime M fulfills the following *Bianchi identities:*

$$D_{[\alpha} R_{\beta\gamma]\delta\epsilon} := D_\alpha R_{\beta\gamma\delta\epsilon} + D_\beta R_{\gamma\alpha\delta\epsilon} + D_\gamma R_{\alpha\beta\delta\epsilon} = 0. \tag{4}$$

Now, the traceless part of the curvature tensor reads:

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} &- \frac{1}{2}(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}) \\ &+ \frac{1}{6}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R, \end{aligned} \tag{5}$$

with R denoting the scalar curvature. Actually, C is called the *conformal curvature tensor of the spacetime.* This is a particular example of a Weyl tensor. In general, these tensors are defined as follows.

Definition 2. *A Weyl tensor W is a 4-tensor satisfying all the symmetry properties of the curvature tensor and in addition being traceless.*

W is said to fulfill the *Bianchi equation*, if it is:

$$D_{[\alpha} W_{\beta\gamma]\delta\epsilon} = 0. \tag{6}$$

We mainly work with the Weyl tensor and call it W. Note that, in general, a Weyl field is not required to satisfy the Bianchi equation. However, in our situation, W does indeed obey it, and it even plays a major role in the proof of our main result. To see it, we remark that splitting the Riemannian curvature tensor into a part given by the Ricci tensor and a part represented by the Weyl tensor, the Bianchi identities (4) then yield differential relations between the Ricci and the Weyl tensor. One takes the first and second contractions of (4) and rewrites this first contraction for the Weyl tensor, obtaining an equation which in dimension $n = 4$ is equivalent to the Bianchi identities. In the EV case, the Weyl tensor satisfies the homogeneous equations

$$D^\alpha W_{\alpha\beta\gamma\delta} = 0,$$

and in the EM case the inhomogeneous equations

$$D^\alpha W_{\alpha\beta\gamma\delta} = \frac{1}{2}(D_\gamma R_{\beta\delta} - D_\delta R_{\beta\gamma}).$$

Given a Weyl field W, we can define the *left* ${}^*W$ and *right* W^* *Hodge duals* to be:

$$ {}^*W_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta} \tag{7} $$

$$ W^*{}_{\alpha\beta\gamma\delta} = \frac{1}{2}W_{\alpha\beta}{}^{\mu\nu}\epsilon_{\mu\nu\gamma\delta}, \tag{8} $$

where $\epsilon^{\alpha\beta\gamma\delta}$ are the components of the volume element of M. One can think of (7) as freezing the second pair of indices and considering W as a 2-form relative to the first pair, correspondingly of (8) as freezing the first pair of indices and considering W as a 2-form in the second pair. Note that these definitions of left and right Hodge duals are equivalent. It can easily be checked that ${}^*W = W^*$ is also a Weyl tensorfield. Further, it is

$$ {}^*({}^*W) = -W. $$

As the volume element of M comes into play in defining the Hodge duals right above, and as it will be involved in future parts of this work, let us write down the multiplication properties of the coefficients. The second up to the fifth of the subsequent equations are obtained by corresponding contractions.

$$ \begin{aligned} \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4} &= -det(\delta^{\mu_i}_{\nu_j})_{i,j=1,\dots,4} \\ \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1\nu_2\nu_3\nu_4} &= -det(\delta^{\mu_i}_{\nu_j})_{i,j=2,3,4} \\ \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1\mu_2\nu_3\nu_4} &= -2det(\delta^{\mu_i}_{\nu_j})_{i,j=3,4} \\ \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1\mu_2\mu_3\nu_4} &= -6\delta^{\mu_4}_{\nu_4} \\ \epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\mu_1\mu_2\mu_3\mu_4} &= -24. \end{aligned} $$

Next, we define the *electric-magnetic decomposition of* W to be the following contractions with X, where X is an arbitrary given vectorfield. The decomposition consists of the two 2-tensors:

$$ ii_X(W)_{\alpha\beta} = W_{\mu\alpha\nu\beta}X^\mu X^\nu \tag{9} $$

$$ ii_X({}^*W)_{\alpha\beta} = {}^*W_{\mu\alpha\nu\beta}X^\mu X^\nu. \tag{10} $$

These tensors are symmetric, traceless and orthogonal to X. It can be shown that they completely determine W, if X is not null. (See also [**7**].)

As in our spacetime manifold (M, g), the metric g is Lorentzian, there exists a vector V in T_pM such that $g_p(V, V) < 0$. Then, its g_p-orthogonal complement is defined as $\Sigma_V = \{X : g_p(X, V) = 0\}$ and g_p restricted to Σ_V is positive definite.

Then we can choose a positive orthonormal frame $(e_0, e_1, e_2, e_3)_p$ at each p in M continuously. That is, we obtain the positive orthonormal frame field consisting of e_0, e_1, e_2, e_3 with:

$$ e_0 = \frac{V}{\sqrt{-g(V, V)}} \tag{11} $$

and e_1, e_2, e_3 being an orthonormal basis for Σ_V.

Any given vector X in T_pM can be expanded as

$$X = X^0 e_0 + X^1 e_1 + X^2 e_2 + X^3 e_3$$
$$= \sum_i X^i e_i \quad (i = 0, 1, 2, 3).$$

Consequently, one has

$$g(e_i, e_j) = \eta_{ij} = diag(-1, +1, +1, +1).$$
$$g(X, X) = -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2$$
$$= \sum_{ij} \eta_{ij} X^i X^j$$

At a point p in M, we can distinguish three types of vectors. Namely, null, timelike and spacelike vectors. The vectors of the first type form a double cone at p, while the vectors of the second type form an open set of two connected components, that is, the interior of this cone, and the vectors of the third type a connected open set being the exterior of the cone. They are defined as follows.

Definition 3. *The null cone (or light cone) at p in M is*

$$N_p = \{X \neq 0 \in T_pM : g_p(X, X) = 0\}.$$

The double cone consists of N_p^+ and N_p^-: $N_p = N_p^+ \cup N_p^-$.

Denote by I_p^+ the interior of N_p^+ and by I_p^- the interior of N_p^-.

Definition 4. *The set of timelike vectors at p in M is given by*

$$I_p := I_p^+ \cup I_p^- = \{X \in T_pM : g_p(X, X) < 0\}.$$

Definition 5. *The set of spacelike vectors at p in M is defined to be*

$$S_p := \{X \in T_pM : g_p(X, X) > 0\}.$$

Thus, S_p is the exterior of N_p.

For physical reasons, the spacetime should be time-orientable. Therefore, one assumes that it is possible to choose continuously a vector $V \in I_p^+$ at each point p in M. That is, one has a continuous timelike future directed vectorfield. Denote by e_0 a continuous future-directed timelike vectorfield on M at unit magnitude $\sqrt{-g(e_0, e_0)} = 1$. Thus, one is able to say what the causal future and past of any event (point) in spacetime means. To do so, we first give the definition of a causal curve.

Definition 6. *A causal curve in M is a differentiable curve γ whose tangent vector $\dot{\gamma}$ at each point p in M belongs to $I_p \cup N_p$, i.e. is either timelike or null.*

Remark: This means that either $\dot{\gamma}_p \in I_p^+ \cup N_p^+$ at each p along γ in which case γ is called future-directed, or $\dot{\gamma}_p \in I_p^- \cup N_p^-$ at each p along γ in which case γ is called past-directed.

Definition 7. *The causal future of a point p in M, denoted by $J^+(p)$, is the set of all points $q \in M$ for which there exists a future-directed causal curve initiating at p and ending at q.*

Correspondingly, we can define $J^-(p)$, the causal past of p. To be more general, we also need the causal future of a set S in M:

Definition 8. *The causal future $J^+(S)$ of any set $S \subset M$, in particular in the case that S is a closed set, is*

$$J^+(S) = \{q \in M : q \in J^+(p) \textit{for some } p \in S\}.$$

Similarly, the definition is given for $J^-(S)$. The boundaries $\partial J^+(S)$ and $\partial J^-(S)$ of $J^+(S)$ and $J^-(S)$, respectively, for closed sets S are null hypersurfaces. They are generated by null geodesic segments. The null geodesics generating $J^+(S)$ have past end points only on S. These null hypersurfaces $\partial J^+(S)$ and $\partial J^-(S)$ are realized as level sets of functions u satisfying the eikonal equation $g^{\mu\nu}\partial_\mu u \partial_\nu u = 0$.

Let us come back to a causal curve (definition 6) and say how distances are measured. For this, we define the arc length of this curve and the temporal distance of two points as follows:

Definition 9. *The arc length of a causal curve γ between the points corresponding to the parameter values $\lambda = a, \lambda = b$ is*

$$L[\gamma](a,b) = \int_a^b \sqrt{-g(\dot\gamma(\lambda), \dot\gamma(\lambda))} d\lambda.$$

If $q \in J^+(p)$, we define the temporal distance of q from p by

$$\tau(q,p) = \sup_{\textit{all future-directed causal curves from } p \textit{ to } q} L[\gamma].$$

Note that the arc length is independent of the parametrization.

Recall that in Riemannian geometry the following statement about minimizing geodesics holds.

Theorem 1. *(Hopf-Rinow): For a complete Riemannian manifold any* 2 *points can be joined by a minimizing geodesic.*

In Lorentzian geometry the analogous statement is, in general, false. However, it is true, if the spacetime admits a Cauchy hypersurface. If the supremum is achieved and the metric is C^1, the maximizing curve is a causal geodesic; after suitable reparametrization the tangent vector is parallelly transported along the curve.

For the next statement, let us first introduce another important quantity: The *deformation tensor* of X, namely ${}^{(X)}\pi$, is given as

$$ {}^{(X)}\pi_{\alpha\beta} = (\mathcal{L}_X g)_{\alpha\beta} \tag{12}$$

$$ -{}^{(X)}\pi^{\alpha\beta} = (\mathcal{L}_X g^{-1})^{\alpha\beta}. \tag{13}$$

Given a Weyl field W and a vectorfield X, the *Lie derivative* of W with respect to X is not, in general, a Weyl field. For, it has trace. In fact, it is:

$$g^{\alpha\gamma}(\mathcal{L}_X W_{\alpha\beta\gamma\delta}) = \pi^{\alpha\gamma} W_{\alpha\beta\gamma\delta}. \tag{14}$$

In view of this, we define the following *modified Lie derivative:*

$$\hat{\mathcal{L}}_X W := \mathcal{L}_X W - \frac{1}{2}{}^{(X)}[W] + \frac{3}{8} tr^{(X)}\pi W \tag{15}$$

with

$$^{(X)}[W]_{\alpha\beta\gamma\delta} := \pi_\alpha^\mu W_{\mu\beta\gamma\delta} + \pi_\beta^\mu W_{\alpha\mu\gamma\delta} + \pi_\gamma^\mu W_{\alpha\beta\mu\delta} + \pi_\delta^\mu W_{\alpha\beta\gamma\mu}. \tag{16}$$

To a Weyl field one can associate a tensorial quadratic form, a 4-covariant tensorfield which is fully symmetric and trace-free; a generalization of one found previously by Bel and Robinson [**1**]. As in [**8**] it is called the Bel-Robinson tensor:

$$Q_{\alpha\beta\gamma\delta} = \frac{1}{2}(W_{\alpha\rho\gamma\sigma} W_\beta{}^\rho{}_\delta{}^\sigma + {}^*W_{\alpha\rho\gamma\sigma} {}^*W_\beta{}^\rho{}_\delta{}^\sigma). \tag{17}$$

It satisfies the following positivity condition:

$$Q(X_1, X_2, X_3, X_4) \geq 0 \tag{18}$$

where X_1, X_2, X_3 and X_4 are future-directed timelike vectors. Moreover, if W satisfies the Bianchi equations then Q is divergence-free:

$$D^\alpha Q_{\alpha\beta\gamma\delta} = 0. \tag{19}$$

Equation (19) is a certain property of the Bianchi equations. It is connected with their conformal behaviour. In fact, they are covariant under conformal isometries. To be precise, let Ω be a scalar. Then, if $\Phi : M \to M$ is a conformal isometry of the spacetime, i.e.,

$$\Phi_* g = \Omega^2 g,$$

and if W is a solution, also $\Omega^{-1}\Phi_* W$ is a solution.

The Bel-Robinson tensor Q is an important tool in our work. We shall come back to it later in the two parts.

In view of the principal part of the Ricci curvature, let us say a few words about the symbol. The principal part of the Ricci curvature is

$$\frac{1}{2} g^{\alpha\beta}\{\partial_\mu\partial_\alpha g_{\beta\nu} + \partial_\nu\partial_\alpha g_{\beta\mu} - \partial_\mu\partial_\nu g_{\alpha\beta} - \partial_\alpha\partial_\beta g_{\mu\nu}\}.$$

If we substitute in the principal part $\partial_\mu\partial_\nu g_{\alpha\beta}$ by $\xi_\mu\xi_\nu\dot{g}_{\alpha\beta}$, where ξ_μ are the components of a covector and $\dot{g}_{\alpha\beta}$ the components of a possible variation of g, then we obtain the symbol σ_ξ at a point $p \in M$ and a covector $\xi \in T_p^* M$ for a given metric g:

$$(\sigma_\xi \cdot \dot{g})_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\xi_\mu\xi_\alpha\dot{g}_{\beta\nu} + \xi_\nu\xi_\alpha\dot{g}_{\beta\mu} - \xi_\mu\xi_\nu\dot{g}_{\alpha\beta} - \xi_\alpha\xi_\beta\dot{g}_{\mu\nu}).$$

More generally, for a given metric g the *symbol* σ_ξ at a point p in M and a covector ξ at p is the linear operator on the space of 2-covariant, symmetric tensors h at p, defined by:

$$(\sigma_\xi \cdot h) = \frac{1}{2}\{\xi \otimes i_\xi h + i_\xi h \otimes \xi - trh\xi \otimes \xi - (\xi,\xi)h\}. \tag{20}$$

We use the following notation:

$$\begin{aligned}(i_\xi h)_\nu &= g^{\alpha\beta}\xi_\alpha h_{\beta\nu},\\ (\xi,\xi) &= g^{\alpha\beta}\xi_\alpha\xi_\beta,\\ (\xi\otimes l)_{\mu\nu} &= \xi_\mu l_\nu,\\ g^{\alpha\beta}h_{\alpha\beta} &= trh.\end{aligned}$$

One observes that for any given covector ξ and any $l \in T^*_pM$,

$$h = \xi \otimes l + l \otimes \xi$$

belongs to the null space.

$$\sigma_\xi \cdot h = 0.$$

This mirrors the general covariance of the EV equations. Whenever g is a solution of the EV equations, then the pullback of a diffeomorphism of the manifold onto itself of g is also a solution. For X being a vectorfield on M generating a 1-parameter group of diffeomorphisms on M, the symbol for the Lie derivative $(\mathcal{L}_X g)_{\mu\nu} = D_\mu X_\nu + D_\nu X_\mu$ is

$$\xi_\mu \dot{X}_\nu + \xi_\nu \dot{X}_\mu,$$

whith $\dot{X}_\mu$ being the components of an arbitrary covector. Consider the following equivalence relation

$$h_1 \sim h_2 \Leftrightarrow h_2 - h_1 = l \otimes \xi + \xi \otimes l$$

for l in T^*_pM, which gives a quotient space Q. There are two possibilities in view of the null space of the symbol σ_ξ. It depends on the choice of the covector ξ with $(\xi,\xi) \neq 0$ or $(\xi,\xi) = 0$, whether it is trivial or nontrivial. Let us consider the two situations:

First, $(\xi,\xi) \neq 0$: If ξ is *not* null, then σ_ξ has only trivial null space on Q.

Second, $(\xi,\xi) = 0$: If ξ *is* null, we can choose $\underline{\xi}$ in the same component of the null cone N^*_p in T^*_pM such that $(\xi,\underline{\xi}) = -2$. Then select a unique representative h out of each equivalence class $\{h\} \in Q$ such that

$$i_{\underline{\xi}}h = 0.$$

Then it follows that the null space of σ_ξ can be identified with the space of trace-free quadratic forms on the 2-dimensional spacelike plane Π, the g-orthogonal complement of the linear span of ξ and $\underline{\xi}$. This is the space of gravitational degrees of freedom at a point.

If we suppose to be given a t-foliation and the EV equations, as explained in the introduction of the first monograph. Then the *electric-magnetic decomposition* of the curvature tensor R with respect to the future-oriented unit normal to the time foliation is denoted by E, H. They are symmetric, traceless 2-tensors tangent to the foliation. In terms of these quantities the equations (1.19) and (1.20) read:

$$\nabla_i k_{jm} - \nabla_j k_{im} = \epsilon_{ij}{}^l H_{lm} \tag{21}$$

$$\bar{R}_{ij} + trk k_{ij} - k_{im} k_j^m = E_{ij}. \tag{22}$$

4. Preliminary Tools

4.1. Hodge Theory. Throughout this book, we use many analytic and geometric tools. A major one is the Hodge theory. We mainly apply it to obtain estimates in 2 and 3 dimensions. Therefore, let us introduce Hodge systems and derive estimates for them. Most of the presented results have been proven by D. Christodoulou and S. Klainerman in [**8**] for their setting. We give the proofs for certain of the following results, and refer to [**8**] for the others stating the difference of our proofs from theirs.

4.1.1. *Hodge Systems on S.* In the sequel we assume (S, γ) to be a compact, 2-dimensional Riemannian manifold. In the first monograph, L. Bieri shows that the Gauss-Bonnet theorem as well as the uniformization theorem hold in this case with L^4 bounds on the curvature in S. In the second monograph, N. Zipser works with the correspondingly same assumptions on the curvature as D. Christodoulou and S. Klainerman in [**8**]. By K we denote the Gauss curvature of S. Also, let S have strictly positive curvature, that is $k_m > 0$ with $k_m = \min_S r^2 K$.

We shall first introduce different types of Hodge systems, and afterwards we will state the corresponding theorems for these systems. So, recall now the following definition of a Hodge dual.

Definition 10. *Let ξ be a given vectorfield on S, then its Hodge dual ${}^*\xi$ is defined by*

$${}^*\xi_A = \epsilon_{AB}\xi^B,$$

where ϵ_{AB} denote the components of the area element relative to an arbitrary frame e_A with $A = 1, 2$.

If ξ is a symmetric, traceless 2-tensor, its left, ${}^*\xi$, and right, ξ^*, Hodge duals are defined as

$${}^*\xi_{AB} = \epsilon_{AC}\xi^C_B, \quad \xi^*_{AB} = \xi^C_A \epsilon_{CB}.$$

Remark: Note that the tensors ${}^*\xi$ and ξ^* are also symmetric and traceless. Also, one easily verifies that

$${}^*\xi = -\xi^*.$$

We denote by $\text{d}\!\!/\text{iv}$ the divergence operator on S and by $\text{c}\!\!/\text{url}$ the curl operator on S. For any $(k+1)$-dimensional tensor ξ these are given by

$$\text{d}\!\!/\text{iv}\ \xi_{A_1\cdots A_k} = \nabla\!\!\!\!/^{\,B}\xi_{A_1\cdots A_k B} \tag{23}$$

$$\text{c}\!\!/\text{url}\ \xi_{A_1\cdots A_k} = \epsilon^{BC}\nabla\!\!\!\!/_B\xi_{A_1\cdots A_k C}. \tag{24}$$

Also, recall that the trace operator on S is

$$tr\xi_{A_1\cdots A_{k-1}} = \gamma^{BC}\xi_{A_1\cdots A_{k-1}BC}. \tag{25}$$

Next, we are going to present the types of *Hodge systems*, that we will always use throughout this work.

H1 Let ξ be a vector on S that verifies

$$\text{d}\!\!/\text{iv}\ \xi = f \tag{26}$$

$$\text{c}\!\!/\text{url}\ \xi = g \tag{27}$$

with f and g being given scalar functions on S.

H2 Let ξ be a symmetric, traceless 2-tensor on S that verifies

$$\text{d}\!\!/\text{iv}\,\xi = f \tag{28}$$

with f being a given vector.

H0 This is a special case of **H1**. Namely, we consider the scalar Poisson equation on S. So, let Φ be a scalar function on S that verifies

$$\triangle\!\!\!\!/\ \Phi = f \tag{29}$$

with f being an arbitrary scalar function on S.

H(k+1) Let ξ be a symmetric, traceless (k+1)-tensor that verifies

$$\text{d}\!\!/\text{iv}\,\xi = f \tag{30}$$

$$\text{c}\!\!/\text{url}\ \xi = g \tag{31}$$

with f and g being given k covariant, symmetric tensors on S.

Further on, we will study properties of results of the Hodge systems above, and derive corresponding estimates. First, we state the following:

Proposition 1. *Let (S,γ) be a 2-dimensional, compact Riemannian manifold.*

1. *Assume that the vectorfield ξ is a solution of* **H1**. *Then it is*

$$\int_S |\nabla\!\!\!\!/\ \xi|^2 + K|\xi|^2 = \int_S |f|^2 + |g|^2. \tag{32}$$

2. *Assume that the symmetric, traceless 2-tensor ξ is a solution of* **H2**. *Then it is*

$$\int_S |\nabla\!\!\!\!/\ \xi|^2 + 2K|\xi|^2 = 2\int_S |f|^2. \tag{33}$$

3. *This is a particular case of the first item of this proposition. Assume that* Φ *is a solution of* **H0**. *Then it is*

$$\int_S |\nabla\!\!\!/^2 \Phi|^2 + K|\Phi|^2 = \int_S |f|^2 . \tag{34}$$

Proof. This proof follows from the proof of proposition 2.

Proposition 2. *Assume that* ξ *is an arbitrary, (k+1) covariant, totally symmetric tensor that verifies the following generalized Hodge system:*
H'(k+1):

$$d\!\!/iv\, \xi = f \tag{35}$$

$$c\!\!/url\, \xi = g \tag{36}$$

$$tr\xi = h \tag{37}$$

with f *and* g *given* k *covariant, symmetric tensors and* h *a covariant symmetric tensor of rank* $(k-1)$. *Then it is*

$$\int_S |\nabla\!\!\!/\xi|^2 + (k+1)K|\xi|^2 = \int_S |f|^2 + |g|^2 + kK|h|^2 . \tag{38}$$

Note that if $k = 0$, then we take $tr\xi = 0$. That is

$$\int_S |\nabla\!\!\!/\xi|^2 + K|\xi|^2 = \int_S |f|^2 + |g|^2 . \tag{39}$$

Proof of Proposition 2: Writing out the curl equation (36), we obtain

$$\nabla\!\!\!/_C \xi_{A_1 \cdots A_k B} = \nabla\!\!\!/_B \xi_{A_1 \cdots A_k C} - \epsilon_{BC} g_{A_1 \cdots A_k} . \tag{40}$$

Now, differentiate (40) and recall the fact that

$$\begin{aligned} &\nabla\!\!\!/_B \nabla\!\!\!/_D \xi_{A_1 \cdots A_k C} - \nabla\!\!\!/_D \nabla\!\!\!/_B \xi_{A_1 \cdots A_k C} \\ &\qquad = \sum_{j=1}^{k} R_{A_j MBD} \xi^M_{A_1 \cdots \hat{A}_j \cdots A_k C} + R_{CMBD} \xi^M_{A_1 \cdots A_k} , \end{aligned}$$

the commutator for the corresponding tensors ξ being zero. We obtain

$$\begin{aligned} \nabla\!\!\!/_D \nabla\!\!\!/_C \xi_{A_1 \cdots A_k B} &= \nabla\!\!\!/_D \nabla\!\!\!/_B \xi_{A_1 \cdots A_k C} - \epsilon_{BC} \nabla\!\!\!/_D g_{A_1 \cdots A_k} \\ &= \nabla\!\!\!/_B \nabla\!\!\!/_D \xi_{A_1 \cdots A_k C} + \sum_{j=1}^{k} R_{A_j MBD} \xi^M_{A_1 \cdots \hat{A}_j \cdots A_k C} \\ &\quad + R_{CMBD} \xi^M_{A_1 \cdots A_k} - \epsilon_{BC} \nabla\!\!\!/_D g_{A_1 \cdots A_k} . \end{aligned}$$

Taking the trace relative to C and D yields

$$\begin{aligned}
\not\triangle \xi_{A_1\cdots A_kB} &= \not\nabla_B(\not\nabla^C \xi_{A_1\cdots A_kC}) - \epsilon_{BC}\not\nabla^C g_{A_1\cdots A_k} \\
&\quad + \sum_{j=1}^{k} R_{A_jMB}{}^C \xi^M_{A_1\cdots \hat{A}_j\cdots A_kC} - R_{MB}\xi^M_{A_1\cdots A_k} \\
&= \not\nabla_B f_{A_1\cdots A_k} - \epsilon_{BC}\not\nabla^C g_{A_1\cdots A_k} \\
&\quad + \sum_{j=1}^{k} \gamma_M{}^C \gamma_{A_jB}\gamma^{A_jB}\ R_{A_jCB}{}^C \xi^M{}_{A_1\cdots \hat{A}_j\cdots A_kC} - R\xi_{A_1\cdots A_kB} \\
&= \not\nabla_B f_{A_1\cdots A_k} - \epsilon_{BC}\not\nabla^C g_{A_1\cdots A_k} \\
&\quad + \sum_{j=1}^{k} \gamma_{A_jB} R h_{A_1\cdots \hat{A}_j\cdots A_k} - R\xi_{A_1\cdots A_kB}.
\end{aligned}$$

In view of the formula relating the Gauss curvature K to the Riemannian curvature R_{abcd}, multiplying the last equation by $\xi^{A_1\cdots A_kB}$ and integrating on S, we deduce that

$$\begin{aligned}
\int_S \xi^{A_1\cdots A_kB}\not\triangle \xi_{A_1\cdots A_kB} &= \int_S \xi^{A_1\cdots A_kB}\not\nabla_B f_{A_1\cdots A_k} - \int_S \xi^{A_1\cdots A_kB}\epsilon_{BC}\not\nabla^C g_{A_1\cdots A_k} \\
&\quad + \int_S (k+1)K\xi^{A_1\cdots A_kB}\xi_{A_1\cdots A_kB} \\
&\quad - \int_S \sum_{j=1}^{k} \gamma_{A_jB}K\xi^{A_1\cdots A_kB}\ h_{A_1\cdots \hat{A}_i\cdots A_k}.
\end{aligned}$$

That is

$$\int_S |\not\nabla \xi|^2 = \int_S |f|^2 + \int_S |g|^2 - \int_S (k+1)K|\xi|^2 + \int_S kK|h|^2.$$

This closes the proof.

Then the next proposition follows directly.

Proposition 3. *Under the same assumptions as in proposition 2, there exists a constant c such that*

$$\int_S |\not\nabla \xi|^2 + K|\xi|^2 \le c\int_S |f|^2 + |g|^2 + K|h|^2. \tag{41}$$

We immediately deduce the following proposition.

Proposition 4. *Let ξ be as in proposition 2 and traceless in addition. Then there are constants C_1 and C_2 such that*

$$\begin{aligned}
\int_S |\not\nabla \xi|^2 &\le C_1\int_S |f|^2 + |g|^2 \\
\int_S K|\xi|^2 &\le C_2\int_S |f|^2 + |g|^2.
\end{aligned}$$

As a consequence of these, one has the following.

Proposition 5. *Assume that $k_m > 0$. Let ξ be a solution of either* **H1** *or* **H2**. *Then the next statements are valid.*

1. *There exists a constant $c_1(K)$, such that*

$$\int_S \{ |\nabla\!\!\!/ \xi|^2 + r^{-2} |\xi|^2 \} \leq c_1 \int_S |f|^2. \tag{42}$$

2. *There exists a constant $c_2(K)$, such that*

$$\int_S \{ |\nabla\!\!\!/^2 \xi|^2 + r^{-2} |\xi|^2 \} \leq c_2 \int_S \{ |\nabla\!\!\!/ f|^2 + r^{-2} |f|^2 \}. \tag{43}$$

The analogue, namely L^p estimates, can be shown for any $2 \leq p < \infty$. We will make use of them as well. The corresponding proof applies the classical Calderon-Zygmund inequalities and the uniformization theorem. As shown in a separate chapter of the first monograph, the uniformization theorem is also valid for the case with $K \in L^p$ for any $2 \leq p < \infty$. Now, we have the following result.

Proposition 6. *Let (S, γ) be a 2-dimensional, compact Riemannian manifold. Assume that $k_m > 0$, and let ξ be a solution of either* **H1** *or* **H2**. *Then the following estimates hold:*

1. *There exists a constant $c(K, p)$ such that for all $2 \leq p < \infty$, it is*

$$\int_S \{ |\nabla\!\!\!/ \xi|^p + r^{-p} |\xi|^p \} \leq c \int_S \{ |f|^p + |g|^p \}. \tag{44}$$

2. *There exists a constant $c(K, p)$ such that for all $2 \leq p < \infty$, it is*

$$\int_S |\nabla\!\!\!/^2 \xi|^p \leq c \int_S \{ |\nabla\!\!\!/ f|^p + r^{-p} |f|^p + |\nabla\!\!\!/ g|^p + r^{-p} |g|^p \}. \tag{45}$$

4.1.2. ***Hodge Systems on H.*** Throughout this chapter, we will denote by (H, g) a 3-dimensional Riemannian manifold diffeomorphic to $\mathbb{R}^3$, on which there exists a generalized radial function u with second fundamental form θ and Gaussian curvature K. We will require that u is quasiconvex, which means that

$$tr\theta > 0, \quad K > 0.$$

We shall work with Hodge systems on H_t in the first monograph in the Chapter 6.2 'Estimating the Components $\delta, \epsilon, \hat{\eta}$ of the Second Fundamental Form', where we will state and prove the estimates for the considered Hodge systems in detail. In the second monograph, we shall use the L^p theory for 2-d Hodge systems developed in [**8**] to produce estimates for the second fundamental form of the time foliation.

Here, let us consider the following: Starting from a Hodge system for a 1-form A on H, we shall now give results for A and its corresponding derivatives. So, let us introduce the following Hodge system for a smooth, compactly supported symmetric 1-tensor A on H:

$$div\, A = \nabla^j A_j = f \tag{46}$$

$$(curl A)_i = \frac{1}{2}\epsilon_i^{jk}\nabla_j A_k = g. \tag{47}$$

Now, we state the following proposition.

Proposition 7. *Let A be a smooth, compactly supported, symmetric 1-tensor on H. Then the following identity holds:*

$$\int_H \mid \nabla A \mid^2 = \int_H \mid f \mid^2 + \int_H \mid g \mid^2 - \int_H R \mid A \mid^2, \tag{48}$$

where R is the magnitude of the Ricci curvature of H.

Proof. We have

$$(curl A)_i = \frac{1}{2}\epsilon_i^{jk}\nabla_j A_k.$$

Then, we calculate

$$\begin{aligned} \mid curl A \mid^2 &= \frac{1}{2}(\nabla_j A_k - \nabla_k A_j)(\nabla^j A^k - \nabla^k A^j) \\ &= \mid \nabla A \mid^2 - (\nabla_j A_k)(\nabla^k A^j). \end{aligned} \tag{49}$$

Integrating this on H, we obtain for the last term, using integration by parts and commuting derivatives:

$$\begin{aligned} \int_H (\nabla_j A_k)(\nabla^k A^j) &= -\int_H A_k \nabla_j \nabla^k A^j = -\int_H A^j \nabla^k \nabla_j A_k \\ &= -\int_H \left(A^j \nabla_j div A + A^j R^k{}_{jkl} A^l\right) \\ &= -\int_H A^j \nabla_j div A - \int_H R_{jl} A^j A^l \\ &= \int_H \mid div A \mid^2 - \int_H R \mid A \mid^2 . \end{aligned} \tag{50}$$

Thus, we now see that

$$\int_H \mid curl A \mid^2 = \int_H \mid \nabla A \mid^2 - \int_H \mid div A \mid^2 + \int_H R \mid A \mid^2, \tag{51}$$

which proves the proposition.

Remark: Take equation (48) and write it as follows:

$$\int_H \mid \nabla A \mid^2 + \int_H R \mid A \mid^2 = \int_H \mid f \mid^2 + \int_H \mid g \mid^2 . \tag{52}$$

Note that, if the right-hand side of (1.52) is finite, that is if $\| curl A \|_{L^2} < \infty$ and $\| div A \|_{L^2} < \infty$, then it is $\int_H \mid \nabla A \mid^2 + \int_H R \mid A \mid^2 < \infty$.

Next, we will state two propositions concerning the first and second derivatives of a symmetric, 2-tensor on H. So, let V be a symmetric 2-tensor on H. Consider the Hodge system

$$div\ V = \rho \tag{53}$$

$$curl\ V = \sigma \tag{54}$$

$$tr\ V = 0, \tag{55}$$

where ρ is a given 1-form and σ a given symmetric, traceless, 2-covariant tensor. We can now formulate the following.

Proposition 8. *Let V be a smooth, compactly supported 2-symmetric tensor on H, that verifies the Hodge system ((53)–(55)).*

Then it is

$$\int_H \left(\mid \nabla V \mid^2 + 3R_{mn}V^{im}V_i^n - \frac{1}{2}R \mid V \mid^2 \right) = \int_H \left(\mid \sigma \mid^2 + \frac{1}{2} \mid \rho \mid^2 \right). \tag{56}$$

We can also derive the next statement, estimating the second derivative of V.

Proposition 9. *Let the assumptions of proposition 8 hold.*

Then there exists a constant c such that

$$\int_H \mid \nabla^2 V \mid^2 \leq c \int_H \left(\mid \nabla\sigma \mid^2 + \mid \nabla\rho \mid^2 + \mid Ric \mid\mid \nabla V \mid^2 + \mid Ric \mid^2 \mid V \mid^2 \right). \tag{57}$$

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