

Preface

Even to this day, I recall my first algebra course as wonderfully delightful and exciting. In my youthful pride, I also recall secretly fearing that when I told people I was studying algebra, they would think I was doing the type of mathematics taught to twelve-year-olds. After all, once you can solve $2x = 4$ and know how to use the quadratic equation, what else is there?

In fact, algebra is a large subfield of modern mathematics that has been intensely researched over the last few hundred years. It is very different from the type of mathematics that most students see during the bulk of their college years. Usually the majority of undergraduate classes are devoted to studying the subfield of mathematics known as analysis, which deals with limits, derivatives, integrals, and all sorts of related concepts.

In contrast, algebra is the study of mathematical systems that, in some sense, generalize the behavior of numbers. For example, the set of integers, the set of polynomials, and the set of $n \times n$ matrices are all very different types of objects. Nevertheless, it makes sense to “add” and “multiply” in each of these cases. Moreover in each of these settings, the formal behavior of addition and multiplication is strikingly similar. As a result, it is possible to abstract the properties of addition and multiplication in order to prove a single theorem applying simultaneously to each different case. This allows for a great deal of efficiency.

More importantly, these “algebraic” techniques frequently make it possible to tackle problems that are otherwise completely inaccessible. As a remarkable example, the ancient Greeks were interested in constructions using only a compass and straightedge. For instance, they could construct a regular 3-gon, 4-gon, 5-gon, 6-gon, 8-gon, and 10-gon, but they never succeeded in constructing a regular 7-gon, 9-gon, or 11-gon. For centuries afterward, geometers tried and failed to repair this gap. In fact, it took a few thousand years and the advent of algebraic techniques for people to figure out what was going on. In 1837, Pierre Wantzel succeeded in showing that the difficulty did not lie with a failure of human ingenuity, but rather that it was *impossible* to construct a regular 7-gon, 9-gon, or 11-gon (Exercise 4.179).

This book is intended as a first course in algebra. For the most part, the only formal prerequisite is the ability to add and multiply integers. In principal, my nine-year-old daughter is therefore more than prepared to study algebra and ought to have no problem jumping right in on page one. Excepting the rare genius, of course, this is a big fallacy. Generally speaking, algebra is considered one of the most difficult courses taken by an undergraduate mathematics major. While it is technically true that the greatest prerequisite for this material is familiarity with the integers, the true prerequisite is mathematical maturity. For many students, a “how to prove things” course is a good way to obtain this experience, though this

is certainly not the only way. Besides maturity, it is also assumed that students are familiar with the basics of linear algebra such as matrix multiplication and that they know how to multiply complex numbers.

When I teach this course, I begin with the integers in order to start off gently and to lay a foundation for future work. The integers and modular arithmetic can be polished off fairly quickly and then I move on to group theory, ring theory, and finally field theory. In terms of big goals, I like to get my students all the way through Sylow's theorems and Galois theory. The text is therefore partly designed with efficiency in mind so that these milestones can be reached by the adventurous or the insane in one semester (perhaps omitting a few sections such as §2.5.3 of Chapter 1, §7.3 and §10 of Chapter 2, and §5.3 of Chapter 3). Of course there is much to be said for adopting a more leisurely pace and the text is equally appropriate for such a relaxed style spanning a year-long course. In fact, there are over 750 exercises in this book and spending plenty of class time on these problems is an excellent investment. In either case, I highly recommend regularly meeting with the students during or outside of class and having them present proofs and exercises on the board. In my experience, this is one of the best ways to help students absorb and master the material.

In terms of the arrangement of the book, Chapter 1 deals with arithmetic. Section 1 examines the integers, mainly presenting the division algorithm and the notion of divisors. This naturally leads to modular arithmetic in Section 2 where congruence classes and their structure are developed. As applications, the Chinese Remainder Theorem, Euler's Phi Function, Fermat's Little Theorem, and Public-Key Cryptography are studied. Finally, the definition of a general equivalence relation is given at this juncture since it fits nicely with the notion of congruence.

Spingboarding off modular arithmetic, Chapter 2 is devoted to the study of groups. Section 1 gives the basic definition and examples of groups. Both multiplicative and additive notation are simultaneously developed and examples are drawn from arithmetic, matrix groups, symmetric groups, and dihedral groups. Section 2 studies the most basic properties of groups, especially the notion of order. Section 3 examines how to construct new groups from already existing groups by means of subgroups or direct products. Section 4 develops the notion and basic properties of morphisms between groups. Section 5 is devoted to the study of quotients of groups. The definition of a coset is given in §5.1, Lagrange's Theorem is in §5.2, the notion of normality is in §5.3, the Correspondence Theorem is in §5.4, and the First Isomorphism Theorem is in §5.5. Section 6 presents the Fundamental Theorem of Finite Abelian Groups. Section 7 studies the symmetric group. Cayley's Theorem is in §7.1, the Cyclic Decomposition Theorem is in §7.2, partitions and the classification of conjugacy classes are in §7.3, and parity and the alternating group are in §7.4. Section 8 studies group actions and includes the Orbit Stabilizer Theorem, the Orbit Decomposition Theorem, Burnside's Counting Theorem, and the Class Equation. Section 9 contains Cauchy's Theorem and the three Sylow theorems. Section 10 studies simple groups and composition series and proves the Jordan-Hölder Theorem for finite groups.

Further generalizing modular arithmetic, Chapter 3 is devoted to the study of rings with a special emphasis on polynomial rings over fields. Because a number of

early concepts in ring theory are completely analogous to ones in group theory with virtually identical proofs, the rate at which these early concepts are introduced is greatly accelerated. Section 1 presents the basic definition and properties of a ring. The definition is given in §1.1, examples drawn from arithmetic, matrix rings, and polynomial rings are in §1.2, special properties relating to units, zero divisors, and domains are in §1.3, subrings are in §1.4, and direct products are in §1.5. Section 2 is devoted to the study of morphisms and quotients. Morphisms are developed in §2.1, ideals are introduced in §2.2, quotients, the Correspondence Theorem, prime ideals, and maximal ideals are in §2.3, and the First Isomorphism Theorem is in §2.4. Section 3 begins the study of polynomial rings and roots. The Division Algorithm is in §3.1, roots and the fact that a finite subgroup of a field is cyclic are contained in §3.2, the Rational Root Test is in §3.3, and the Fundamental Theorem of Algebra is in §3.4. Section 4 studies the irreducibility of polynomials including Gauss's Lemma, Eisenstein's Criterion, and Modular Reduction. Section 5 develops the theory of unique factorization domains (UFD's). The definition and two equivalent conditions in terms of the ascending chain condition on principle ideals (ACCP), primes, and gcd's are given in §5.1, quotient fields are in §5.2, and the fact that a polynomial ring over a UFD is a UFD is in §5.3. Finally, Section 6 gives the basic theory of principle ideal domains (PID's) and Euclidean domains including Fermat's Theorem on the Sum of Two Squares and factorization of the Gaussian integers.

Making heavy use of the theory of polynomial rings, Chapter 4 is devoted to the study of fields. Section 1 gives the basic theory of finite and algebraic extensions. Section 2 develops the theory of splitting fields. Existence, isomorphic extension theorems, and normality are in §2.1 and the existence of algebraic closures is in §2.2. Section 3 gives the theory of finite fields. Section 4 contains Galois theory. The notion of the Galois group is in §4.1, separability is in §4.2, and the Fundamental Theorem of Galois Theory on intermediate fields is in §4.3. Section 5 gives proofs of some famous impossibilities involving ruler and compass constructions and factoring the quintic. Section 6 gives the theory of cyclotomic fields over \mathbb{Q} .

Throughout the text of the book, any text cross-reference to a section (subsection or subsubsection) refers to that section (subsection or subsubsection) in the current chapter unless otherwise specified.

There are a number of resources that had a powerful impact on this work and to which I am greatly indebted. I was first introduced to the subject in the summer of 1988 while reading *Abstract Algebra: A First Course* by L. Goldstein (Prentice-Hall, Inc., 1973). It was a remarkably enjoyable experience and has shaped what material I use and how I teach algebra to this day. More recently, I have used T. Hungerford's excellent book *Abstract Algebra: An Introduction* (Thomson Brooks/Cole, 2nd ed., 1996) when teaching algebra and the text has had a strong impact on my presentation of the subject. There are also two books that deserve special mention due to their outstanding quality: K. Nicholson's *Introduction to Abstract Algebra* (Wiley-Interscience, 3rd ed., 2007) and C. Lanski's *Concepts in Abstract Algebra* (Thomson Brooks/Cole, 2005). Both texts are exemplary introductions to algebra and have been extremely useful and influential to me. In addition, M. Artin's *Algebra* (Prentice Hall, Inc., 1991) and D. Dummit and R. Foote's *Abstract Algebra* (John Wiley & Sons, Inc., 3rd ed., 2004) are both excellent works that played a

role in shaping certain sections of this book. Finally, the author is grateful to the Baylor Sabbatical Committee for its support during part of the preparation of this text and to M. Hunziker for help with a number of diagrams and pictures.