leading to equations of the form (0.7), namely spring motion problems and models of certain simple electrical circuits, called RLC circuits. In §14 we bring up another method, "variation of parameters," which applies to general functions f in (0.7).

Section 15 gives some results on variable coefficient second order linear differential equations. Exercises cover specific results applicable to two particular equations, Airy's equation and Bessel's equation, and there are references to further material on these equations. Techniques brought to bear on these equations include power series representations, extending the power series attack used on (0.3), and the Wronskian, first introduced in the constant-coefficient context in §12. We conclude with a very brief discussion of differential equations of order ≥ 3, in §16. Material introduced in §§15–16 will be covered, on a much more general level, in Chapter 3.

Bessel functions, introduced in the exercise set for §15, play a prominent role in these exercises. Appendix A explains how Bessel functions arise in the search for solutions to some basic partial differential equations.

1. The exponential and trigonometric functions

We construct the exponential function to solve the differential equation

$$\frac{dx}{dt} = x, \quad x(0) = 1. \tag{1.1}$$

We seek a solution as a power series

$$x(t) = \sum_{k=0}^{\infty} a_k t^k. \tag{1.2}$$

In such a case,

$$\begin{aligned} x'(t) &= \sum_{k=1}^{\infty} k a_k t^{k-1} \\ &= \sum_{\ell=0}^{\infty} (\ell+1) a_{\ell+1} t^{\ell}, \end{aligned} \tag{1.3}$$

so for (1.1) to hold we need

$$a_0 = 1, \quad a_{k+1} = \frac{a_k}{k+1}, \tag{1.4}$$

i.e., $a_k = 1/k!$, where $k! = k(k-1)\cdots 2\cdot 1$. Thus (1.1) is solved by

$$x(t) = e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k, \quad t \in \mathbb{R}. \tag{1.5}$$

This defines the exponential function e^t. See (1.45)–(1.50) below for further comments on this calculation.

More generally, we can define

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k, \quad z \in \mathbb{C}. \tag{1.6}$$

The issue of convergence for complex power series is essentially the same as for real power series. Given $z = x + iy$, $x, y \in \mathbb{R}$, we have $|z| = \sqrt{x^2 + y^2}$. If also $w \in \mathbb{C}$, then $|z + w| \le |z| + |w|$ and $|zw| = |z| \cdot |w|$. Hence

$$\Bigl|\sum_{k=m}^{m+n} \frac{1}{k!} z^k\Bigr| \le \sum_{k=m}^{m+n} \frac{1}{k!} |z|^k.$$

The ratio test then shows that the series (1.6) is absolutely convergent for all $z \in \mathbb{C}$, and uniformly convergent for $|z| \le R$, for each $R < \infty$. Note that

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k}{k!} t^k \tag{1.7}$$

solves

$$\frac{d}{dt} e^{at} = a e^{at}, \tag{1.8}$$

and this works for each $a \in \mathbb{C}$.

We claim that e^{at} is the only solution to

$$\frac{dy}{dt} = ay, \quad y(0) = 1. \tag{1.9}$$

To see this, compute the derivative of $e^{-at} y(t)$:

$$\frac{d}{dt}\bigl(e^{-at} y(t)\bigr) = -a e^{-at} y(t) + e^{-at} a y(t) = 0, \tag{1.10}$$

where we use the product rule, (1.8) (with a replaced by $-a$) and (1.9). Thus $e^{-at} y(t)$ is independent of t. Evaluating at $t = 0$ gives

$$e^{-at} y(t) = 1, \quad \forall\, t \in \mathbb{R}, \tag{1.11}$$

whenever $y(t)$ solves (1.9). Since e^{at} solves (1.9), we have $e^{-at} e^{at} = 1$, hence

$$e^{-at} = \frac{1}{e^{at}}, \quad \forall\, t \in \mathbb{R},\ a \in \mathbb{C}. \tag{1.12}$$

Thus multiplying both sides of (1.11) by e^{at} gives the asserted uniqueness:

$$y(t) = e^{at}, \quad \forall\, t \in \mathbb{R}. \tag{1.13}$$

We can draw further useful conclusions from applying d/dt to products of exponential functions. In fact, let $a, b \in \mathbb{C}$; then

$$\begin{aligned} &\frac{d}{dt}\Big(e^{-at}e^{-bt}e^{(a+b)t}\Big) \\ &\quad = -ae^{-at}e^{-bt}e^{(a+b)t} - be^{-at}e^{-bt}e^{(a+b)t} + (a+b)e^{-at}e^{-bt}e^{(a+b)t} \\ &\quad = 0, \end{aligned} \tag{1.14}$$

so again we are differentiating a function that is independent of t. Evaluation at $t = 0$ gives

$$e^{-at}e^{-bt}e^{(a+b)t} = 1, \quad \forall\, t \in \mathbb{R}. \tag{1.15}$$

Using (1.12), we get

$$e^{(a+b)t} = e^{at}e^{bt}, \quad \forall\, t \in \mathbb{R},\ a, b \in \mathbb{C}, \tag{1.16}$$

or, setting $t = 1$,

$$e^{a+b} = e^a e^b, \quad \forall\, a, b \in \mathbb{C}. \tag{1.17}$$

We next record some properties of $\exp(t) = e^t$ for real t. The power series (1.5) clearly gives $e^t > 0$ for $t \geq 0$. Since $e^{-t} = 1/e^t$, we see that $e^t > 0$ for all $t \in \mathbb{R}$. Since $de^t/dt = e^t > 0$, the function is monotone increasing in t, and since $d^2e^t/dt^2 = e^t > 0$, this function is convex. Note that

$$e^1 = 1 + 1 + \frac{1}{2} + \cdots > 2, \tag{1.18}$$

so $e^k > 2^k \nearrow +\infty$ as $k \to +\infty$. Hence

$$\lim_{t\to+\infty} e^t = +\infty. \tag{1.19}$$

Since $e^{-t} = 1/e^t$,

$$\lim_{t\to-\infty} e^t = 0. \tag{1.20}$$

As a consequence,

$$\exp : \mathbb{R} \longrightarrow (0, \infty) \tag{1.21}$$

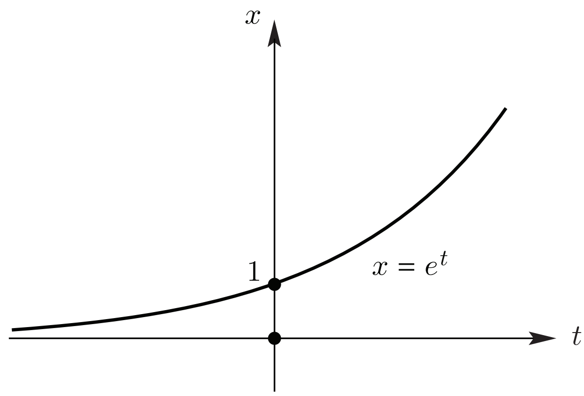

Figure 1.1

is smooth and one-to-one and onto, with positive derivative, so the inverse function theorem of one-variable calculus applies. There is a smooth inverse

$$L : (0, \infty) \longrightarrow \mathbb{R}. \tag{1.22}$$

We call this inverse the natural logarithm:

$$\log x = L(x). \tag{1.23}$$

See Figures 1.1 and 1.2 for graphs of $x = e^t$ and $t = \log x$.

Applying d/dt to

$$L(e^t) = t \tag{1.24}$$

gives

$$L'(e^t)e^t = 1, \quad \text{hence} \quad L'(e^t) = \frac{1}{e^t}, \tag{1.25}$$

i.e.,

$$\frac{d}{dx} \log x = \frac{1}{x}. \tag{1.26}$$

Since $\log 1 = 0$, we get

$$\log x = \int_1^x \frac{dy}{y}. \tag{1.27}$$

An immediate consequence of (1.17) (for $a, b \in \mathbb{R}$) is the identity

$$\log xy = \log x + \log y, \quad x, y \in (0, \infty). \tag{1.28}$$

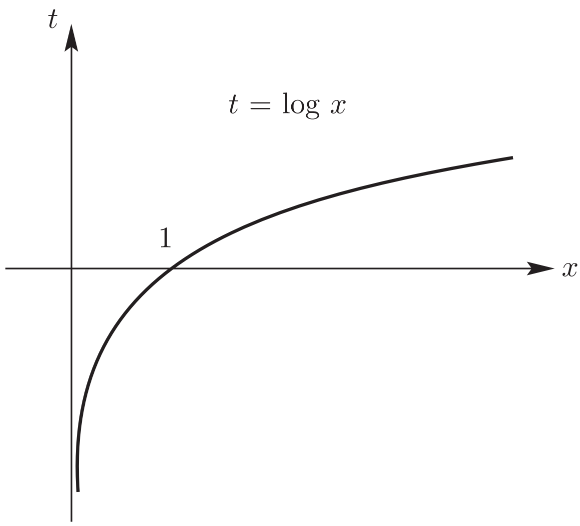

Figure 1.2

We move on to a study of e^z for purely imaginary z, i.e., of

$$\gamma(t) = e^{it}, \quad t \in \mathbb{R}. \tag{1.29}$$

This traces out a curve in the complex plane, and we want to understand which curve it is. Let us set

$$e^{it} = c(t) + is(t), \tag{1.30}$$

with $c(t)$ and $s(t)$ real valued. First we calculate $|e^{it}|^2 = c(t)^2 + s(t)^2$. For $x, y \in \mathbb{R}$,

$$z = x + iy \Longrightarrow \overline{z} = x - iy \Longrightarrow z\overline{z} = x^2 + y^2 = |z|^2. \tag{1.31}$$

It is elementary that

$$\begin{aligned} z, w \in \mathbb{C} \Longrightarrow \overline{zw} = \overline{z}\,\overline{w} \Longrightarrow \overline{z^n} = \overline{z}^n, \\ \text{and} \quad \overline{z+w} = \overline{z} + \overline{w}. \end{aligned} \tag{1.32}$$

Hence

$$\overline{e^z} = \sum_{k=0}^{\infty} \frac{\overline{z}^k}{k!} = e^{\overline{z}}. \tag{1.33}$$

In particular,

$$t \in \mathbb{R} \Longrightarrow |e^{it}|^2 = e^{it}e^{-it} = 1. \tag{1.34}$$

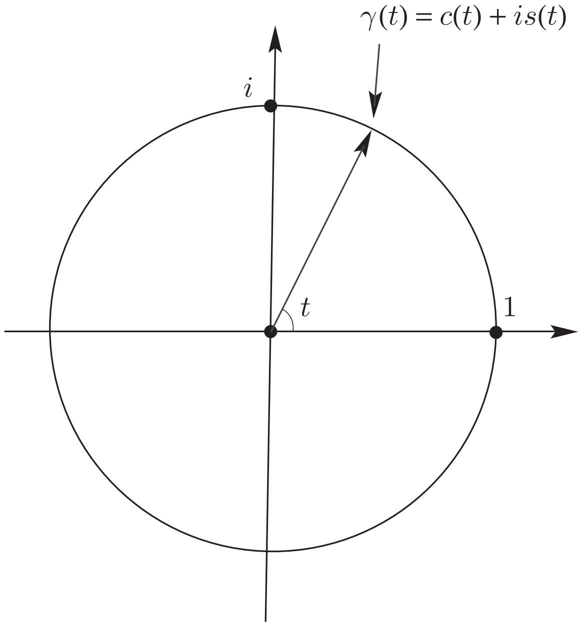

Figure 1.3

Hence $t \mapsto \gamma(t) = e^{it}$ has image in the unit circle centered at the origin in $\mathbb{C}$. Also

$$\gamma'(t) = ie^{it} \Longrightarrow |\gamma'(t)| \equiv 1, \tag{1.35}$$

so $\gamma(t)$ moves at unit speed on the unit circle. We have

$$\gamma(0) = 1, \quad \gamma'(0) = i. \tag{1.36}$$

Thus, for t between 0 and the circumference of the unit circle, the arc from $\gamma(0)$ to $\gamma(t)$ is an arc on the unit circle, pictured in Figure 1.3, of length

$$\ell(t) = \int_0^t |\gamma'(s)|\, ds = t. \tag{1.37}$$

Standard definitions from trigonometry say that the line segments from 0 to 1 and from 0 to $\gamma(t)$ meet at angle whose measurement in radians is equal to the length of the arc of the unit circle from 1 to $\gamma(t)$, i.e., to $\ell(t)$. The cosine of this angle is defined to be the x-coordinate of $\gamma(t)$ and the sine of the angle is defined to be the y-coordinate of $\gamma(t)$. Hence the computation (1.37) gives

$$c(t) = \cos t, \quad s(t) = \sin t. \tag{1.38}$$

Thus (1.30) becomes

$$e^{it} = \cos t + i \sin t, \tag{1.39}$$

which is Euler's formula. The identity

$$\frac{d}{dt} e^{it} = i e^{it}, \tag{1.40}$$

applied to (1.39), yields

$$\frac{d}{dt} \cos t = -\sin t, \quad \frac{d}{dt} \sin t = \cos t. \tag{1.41}$$

We can use (1.17) to derive formulas for sin and cos of the sum of two angles. Indeed, comparing

$$e^{i(s+t)} = \cos(s+t) + i \sin(s+t) \tag{1.42}$$

with

$$e^{is} e^{it} = (\cos s + i \sin s)(\cos t + i \sin t) \tag{1.43}$$

gives

$$\begin{aligned} \cos(s+t) &= (\cos s)(\cos t) - (\sin s)(\sin t), \\ \sin(s+t) &= (\sin s)(\cos t) + (\cos s)(\sin t). \end{aligned} \tag{1.44}$$

Returning to basics, we recall that the calculations done so far in this section were all predicated on the fact that the power series (1.7) can be differentiated term by term. The validity of this operation is established in many calculus texts, but for the sake of completeness we include a direct demonstration. To begin, look at

$$E_n^a(t) = \sum_{k=0}^{n} \frac{a^k}{k!} t^k, \tag{1.45}$$

which satisfies

$$\begin{aligned} \frac{d}{dt} E_n^a(t) &= \sum_{k=1}^{n} \frac{a^k}{(k-1)!} t^{k-1} \\ &= \sum_{\ell=0}^{n-1} \frac{a^{\ell+1}}{\ell!} t^{\ell} \\ &= a E_{n-1}^a(t). \end{aligned} \tag{1.46}$$

Integration gives

$$a \int_0^t E_{n-1}^a(s)\,ds = E_n^a(t) - 1. \tag{1.47}$$

Now we have

$$E_{n-1}^a(s) \longrightarrow e^{as}, \quad E_n^a(t) \longrightarrow e^{at}, \tag{1.48}$$

uniformly on finite intervals, as $n \to \infty$, and then the integral estimate

$$\Bigl| \int_0^t (E(s) - F(s))\,ds \Bigr| \le |t| \max_{0\le s\le t} |E(s) - F(s)|$$

implies

$$\int_0^t E_{n-1}^a(s)\,ds \longrightarrow \int_0^t e^{as}\,ds, \tag{1.49}$$

as $n \to \infty$. Consequently, we can pass to the limit $n \to \infty$ in (1.47) and get

$$a \int_0^t e^{as}\,ds = e^{at} - 1. \tag{1.50}$$

Applying d/dt to the left side of (1.50) gives ae^{at}, by the fundamental theorem of calculus. Hence this must be the derivative of the right side of (1.50), and this gives (1.8).

Having the integral formula (1.50), we proceed to obtain formulas for $\int t^n e^{at}\,dt$. In fact, from (1.46), (1.8), and the product rule, we obtain

$$\begin{aligned} \frac{d}{dt}\bigl(e^{-at}E_n^a(t)\bigr) &= -ae^{-at}E_n^a(t) + ae^{-at}E_{n-1}^a(t) \\ &= -\frac{a^{n+1}}{n!} t^n e^{-at}. \end{aligned} \tag{1.51}$$

Then the fundamental theorem of calculus gives

$$\begin{aligned} \int t^n e^{-at}\,dt &= -\frac{n!}{a^{n+1}} E_n^a(t) e^{-at} + C \\ &= -\frac{n!}{a^{n+1}} \Bigl(1 + at + \frac{a^2t^2}{2!} + \cdots + \frac{a^nt^n}{n!}\Bigr) e^{-at} + C. \end{aligned} \tag{1.52}$$

We have an analogous formula for $\int t^n e^{at}\,dt$, by replacing a with $-a$.

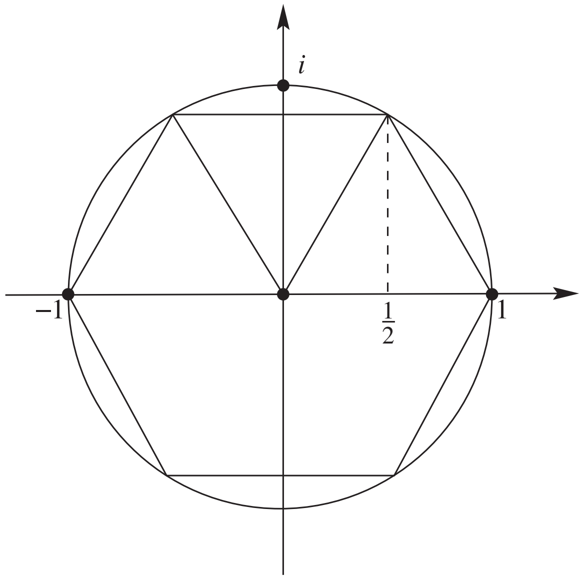

Figure 1.4

Exercises

1. As noted, if $z = x + iy,\ x, y \in \mathbb{R}$, then $|z| = \sqrt{x^2 + y^2}$ is equivalent to $|z|^2 = z\,\overline{z}$. Use this to show that if also $w \in \mathbb{C}$,

$$|zw| = |z| \cdot |w|.$$

   Note that

$$\begin{aligned}|z + w|^2 &= (z + w)(\overline{z} + \overline{w}) \\ &= |z|^2 + |w|^2 + w\overline{z} + z\overline{w} \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}\,zw.\end{aligned}$$

   Show that $\mathrm{Re}(zw) \le |zw|$ and use this in concert with an expansion of $(|z| + |w|)^2$ and the first identity above to deduce that

$$|z + w| \le |z| + |w|.$$

2. Define π to be the smallest positive number such that $e^{\pi i} = -1$. Show that

$$e^{\pi i/2} = i, \quad e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

   Hint. See Figure 1.4.

3. Show that

$$\cos^2 t + \sin^2 t = 1,$$

and

$$1 + \tan^2 t = \sec^2 t,$$

where

$$\tan t = \frac{\sin t}{\cos t}, \quad \sec t = \frac{1}{\cos t}.$$

4. Show that

$$\frac{d}{dt} \tan t = \sec^2 t = 1 + \tan^2 t,$$
$$\frac{d}{dt} \sec t = \sec t \ \tan t.$$

5. Evaluate

$$\int_0^y \frac{dx}{1 + x^2}.$$

Hint. Set $x = \tan t$.

6. Evaluate

$$\int_0^y \frac{dx}{\sqrt{1 - x^2}}.$$

Hint. Set $x = \sin t$.

7. Show that

$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}.$$

Hint. Show that $\sin \pi/6 = 1/2$. Use Exercise 2 and the identity $e^{\pi i/6} = e^{\pi i/2} e^{-\pi i/3}$.

8. Set

$$\cosh t = \frac{1}{2}(e^t + e^{-t}), \quad \sinh t = \frac{1}{2}(e^t - e^{-t}).$$

Show that

$$\frac{d}{dt} \cosh t = \sinh t, \quad \frac{d}{dt} \sinh t = \cosh t,$$

and

$$\cosh^2 t - \sinh^2 t = 1.$$

9. Evaluate

$$\int_0^y \frac{dx}{\sqrt{1 + x^2}}.$$

Hint. Set $x = \sinh t$.

10. Evaluate

$$\int_0^y \sqrt{1+x^2}\,dx.$$

11. Using Exercise 4, verify that

$$\begin{aligned}\frac{d}{dt}(\sec t + \tan t) &= \sec t(\sec t + \tan t),\\ \frac{d}{dt}(\sec t\ \tan t) &= \sec^3 t + \sec t\ \tan^2 t,\\ &= 2\sec^3 t - \sec t.\end{aligned}$$

12. Next verify that

$$\begin{aligned}\frac{d}{dt}\log|\sec t| &= \tan t,\\ \frac{d}{dt}\log|\sec t + \tan t| &= \sec t.\end{aligned}$$

13. Now verify that

$$\begin{aligned}\int \tan t\,dt &= \log|\sec t|,\\ \int \sec t\,dt &= \log|\sec t + \tan t|,\\ 2\int \sec^3 t\,dt &= \sec t\ \tan t + \int \sec t\,dt.\end{aligned}$$

(Here and below, we omit the arbitrary additive constants.)

14. Here is another approach to the evaluation of $\int \sec t\,dt$. Using Exercise 8 and the chain rule, show that

$$\frac{d}{du}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}.$$

Take $u = \sec t$ and use Exercises 3–4 to get

$$\frac{d}{dt}\cosh^{-1}(\sec t) = \frac{\sec t\ \tan t}{\tan t} = \sec t,$$

hence

$$\int \sec t\,dt = \cosh^{-1}(\sec t).$$

Compare this with the analogue in Exercise 13.

15. For $E_n^a(t)$ as in (1.45), $k \geq 1$, $0 < T < \infty$, show that

$$\max_{|t|\leq T} |E_{n+k}^a(t) - E_n^a(t)| \leq \frac{|aT|^{n+1}}{(n+1)!}\Big(1 + \frac{|aT|}{n+2} + \frac{|aT|^2}{(n+2)(n+3)} + \cdots\Big), \tag{1.53}$$

and that this is

$$\leq 2\frac{|aT|^{n+1}}{(n+1)!}, \quad \text{for} \ \ n+2 > 2|aT|. \tag{1.54}$$

Deduce that

$$\max_{|t|\leq T} |e^{at} - E_n^a(t)| \tag{1.55}$$

satisfies (1.54). Show that, for each a, T, (1.54) tends to 0 as $n \to \infty$, yielding the assertion made about convergence in (1.48).

16. Show that

$$\Big|\int_0^t e^{as}\,ds - \int_0^t E_n^a(s)\,ds\Big| \leq |t| \max_{|s|\leq|t|} |e^{as} - E_n^a(s)|,$$

and observe how this, together with Exercise 15, yields (1.49).

17. Show that

$$|t| < 1 \Rightarrow \log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots. \tag{1.56}$$

Hint. Rewrite (1.27) as

$$\log(1+t) = \int_0^t \frac{ds}{1+s},$$

expand

$$\frac{1}{1+s} = 1 - s + s^2 - s^3 + \cdots, \quad |s| < 1,$$

and integrate term by term.

18. Use (1.52) with $a = -i$ to produce formulas for

$$\int t^n \cos t\,dt \quad \text{and} \quad \int t^n \sin t\,dt.$$

19. Figure 1.5 (a)–(b) shows graphs of the image of

$$\gamma(t) = e^{\alpha t}, \quad 0 \leq t \leq 6\pi,$$

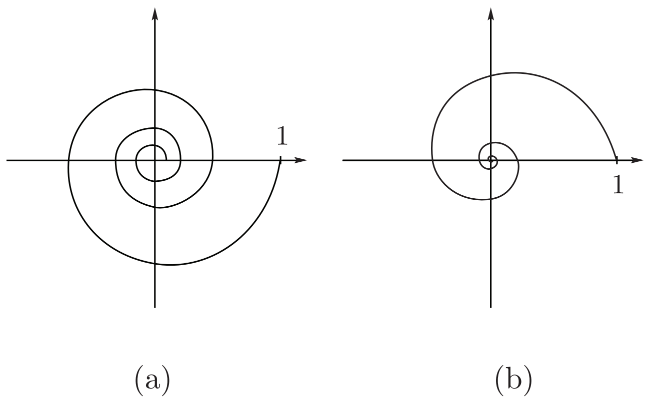

Figure 1.5

for

$$\alpha = -\frac{1}{4} + i,$$
$$\alpha = -\frac{1}{8} - i.$$

Match each value of α to (a) or (b).

2. First order linear equations

Here we tackle first order linear equations. These are equations of the form

$$\frac{dx}{dt} + a(t)x = b(t), \quad x(t_0) = x_0, \tag{2.1}$$

given functions $a(t)$ and $b(t)$, continuous on some interval containing t_0. As a warm-up, we first treat

$$\frac{dx}{dt} + ax = b, \quad x(0) = x_0, \tag{2.2}$$

with a and b constants. One key to solving (2.2) is the identity

$$\frac{d}{dt}(e^{at}x) = e^{at}\Big(\frac{dx}{dt} + ax\Big), \tag{2.3}$$

which follows by applying the product formula and (1.8). Thus, multiplying both sides of (2.2) by e^{at} gives

$$\frac{d}{dt}(e^{at}x) = e^{at}b, \tag{2.4}$$

and then integrating both sides from 0 to t gives

$$e^{at}x(t) = x_0 + \int_0^t e^{as} b\, ds. \tag{2.5}$$

We can carry out the integral, using (1.45), and get

$$e^{at}x(t) = x_0 + \frac{e^{at}-1}{a} b, \tag{2.6}$$

and finally division by e^{at} yields

$$\begin{aligned} x(t) &= e^{-at}x_0 + \frac{b}{a}(1 - e^{-at}) \\ &= \frac{b}{a} + e^{-at}\Big(x_0 - \frac{b}{a}\Big). \end{aligned} \tag{2.7}$$

In order to tackle (2.1), we need a replacement for (2.3). To get it, note that if $A(t)$ is differentiable, the chain rule plus (1.8) gives

$$\frac{d}{dt} e^{A(t)} = e^{A(t)} A'(t). \tag{2.8}$$

Hence

$$\frac{d}{dt}\big(e^{A(t)}x\big) = e^{A(t)}\Big(\frac{dx}{dt} + A'(t)x\Big). \tag{2.9}$$

Thus we can multiply (2.1) by $e^{A(t)}$ and get

$$\frac{d}{dt}\big(e^{A(t)}x\big) = e^{A(t)} b(t), \tag{2.10}$$

provided

$$A'(t) = a(t). \tag{2.11}$$

To arrange this, we can set

$$A(t) = \int_{t_0}^t a(s)\, ds. \tag{2.12}$$

Then we can integrate (2.10) from t_0 to t, to get

$$e^{A(t)}x(t) = x_0 + \int_{t_0}^t e^{A(s)} b(s)\, ds, \tag{2.13}$$

and hence

$$x(t) = e^{-A(t)}x_0 + e^{-A(t)} \int_{t_0}^{t} e^{A(s)} b(s)\, ds. \tag{2.14}$$

For example, consider

$$\frac{dx}{dt} - tx = b(t), \quad x(0) = x_0. \tag{2.15}$$

From (2.12) we get

$$A(t) = -\frac{t^2}{2}, \tag{2.16}$$

and (2.10) becomes

$$\frac{d}{dt}(e^{-t^2/2}x) = e^{-t^2/2}b(t), \tag{2.17}$$

hence

$$e^{-t^2/2}x(t) = x_0 + \int_0^t e^{-s^2/2} b(s)\, ds. \tag{2.18}$$

Let us look at two special cases. First,

$$b(t) = t. \tag{2.19}$$

Then the integral in (7.18) is

$$\int_0^t e^{-s^2/2} s\, ds = \int_0^{t^2/2} e^{-\sigma}\, d\sigma = 1 - e^{-t^2/2}. \tag{2.20}$$

The second case is

$$b(t) = 1. \tag{2.21}$$

Then the integral in (2.18) is

$$\int_0^t e^{-s^2/2}\, ds. \tag{2.22}$$

This is not an elementary function, but it can be related to the special function

$$\mathrm{Erf}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2}\, ds. \tag{2.23}$$

Namely,

$$\frac{1}{\sqrt{2\pi}} \int_0^t e^{-s^2/2}\, ds = \mathrm{Erf}(t) - \mathrm{Erf}(0). \tag{2.24}$$

Note that

$$\mathrm{Erf}(0) = \frac{1}{2}\,\mathrm{Erf}(\infty) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} I, \tag{2.25}$$

where

$$\begin{aligned} I = \int_{-\infty}^{\infty} e^{-s^2/2}\, ds \Rightarrow I^2 &= \int\limits_{\mathbb{R}^2} e^{-|x|^2/2}\, dx \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \\ &= 2\pi \int_0^{\infty} e^{-s}\, ds \\ &= 2\pi. \end{aligned} \tag{2.26}$$

Hence we have

$$\mathrm{Erf}(\infty) = 1, \quad \mathrm{Erf}(0) = \frac{1}{2}. \tag{2.27}$$

Bernoulli equations

Equations of the form

$$\frac{dx}{dt} + a(t)x = b(t)x^n \tag{2.28}$$

are called Bernoulli equations. Such an equation is not linear if $n \neq 1$ or 0, but in these cases one gets a linear equation by the substitution

$$y = x^{1-n}. \tag{2.29}$$

In fact, (2.29) gives $y' = (1-n)x^{-n}x'$, and plugging in (2.28) gives

$$\frac{dy}{dt} = (1-n)[b(t) - a(t)y], \tag{2.30}$$

which is linear.

Exercises

Solve the following initial value problems. Do the integrals if you can.

(1)
$$\frac{dx}{dt} + \frac{1}{t}x = t^2, \quad x(1) = 0.$$

(2)
$$\frac{dx}{dt} + t^2 x = t^2, \quad x(0) = 1.$$

(3)
$$\frac{dx}{dt} + x = \cos t, \quad x(0) = 0.$$

(4)
$$\frac{dx}{dt} + tx = t^3, \quad x(0) = 1.$$

(5)
$$\frac{dx}{dt} + tx = x^3, \quad x(0) = 1.$$

(6)
$$\frac{dx}{dt} + (\tan t)x = \cos t, \quad x(0) = 1.$$

(7)
$$\frac{dx}{dt} + (\sec t)x = \cos t, \quad x(0) = 1.$$

3. Separable equations

A separable differential equation is one for which the method of separation of variables, which we introduce in this section, is applicable. We illustrate this with another approach to the equation (2.2), which we rewrite as

(3.1)
$$\frac{dx}{dt} = b - ax, \quad x(0) = x_0.$$

Separating variables involves moving the x-dependent objects to the left and the t-dependent objects to the right, when possible. In case (3.1), this is possible; we have

(3.2)
$$\frac{dx}{b - ax} = dt.$$

We next integrate both sides. A change of variable allows us to use (1.27), to obtain

$$\int \frac{dx}{b - ax} = -\frac{1}{a}\int \frac{dx}{x - b/a} = -\frac{1}{a}\log\Big|x - \frac{b}{a}\Big| + C. \tag{3.3}$$

Hence (3.2) yields

$$-\frac{1}{a}\log\Big|x - \frac{b}{a}\Big| = t - C, \tag{3.4}$$

and therefore

$$x(t) - \frac{b}{a} = \pm e^{-at+aC} = Ke^{-at}. \tag{3.5}$$

Here K is a constant, which can be found by using the initial condition $x(0) = x_0$. We get $x_0 - b/a = K$, so (3.5) yields

$$x(t) = \frac{b}{a} + e^{-at}\Big(x_0 - \frac{b}{a}\Big), \tag{3.6}$$

consistent with (2.7).

Generally, a separable differential equation is one that can be put in the form

$$\frac{dx}{dt} = f(x)g(t), \tag{3.7}$$

and then separation of variables gives

$$\frac{dx}{f(x)} = g(t)\, dt, \tag{3.8}$$

integrating to

$$\int \frac{dx}{f(x)} = \int g(t)\, dt. \tag{3.9}$$

Here is another basic example:

$$\frac{dx}{dt} = x^2, \quad x(0) = 1. \tag{3.10}$$

We get

$$\frac{dx}{x^2} = dt, \tag{3.11}$$

which integrates to

$$-\frac{1}{x} = t + C, \tag{3.12}$$

hence $x = -1/(t + C)$. The initial condition in (3.10) gives $C = -1$, so the solution to (3.10) is

$$x(t) = \frac{1}{1-t}. \tag{3.13}$$

Note that this solution blows up as $t \nearrow 1$.

The hanging cable

Suppose a length of cable, lying in the (x, y)-plane, is fastened at $(-a, 0)$ and at $(a, 0)$, and hangs down freely, in equilibrium, as pictured in Figure 3.1. The force of gravity acts in the direction of the negative y-axis. We want the equation of the curve traced out by the cable, which we assume to have length $2L$ (not stretchable) and uniform mass density.

To tackle this problem, we introduce $\theta(x)$, the angle the tangent to the curve at $(x, y(x))$ makes with the x-axis, which is given by

$$\tan\theta(x) = y'(x). \tag{3.14}$$

We will derive a differential equation for $\theta(x)$, as follows.

At each point $(x, y(x))$, there is a tension on the cable, of magnitude $T(x)$, and the physical laws governing the behavior of the cable are the following. First, the horizontal component of the tension, given by $T(x)\cos\theta(x)$, is constant. Second, the vertical component of the tension, given by $T(x)\sin\theta(x)$, is proportional to the weight of the cable lying below $y = y(x)$, hence to the length $L(x)$ of the cable, from $(0, y(0))$ to $(x, y(x))$. In other words, we have

$$\begin{aligned} T(x)\cos\theta(x) &= T_0, \\ T(x)\sin\theta(x) &= \kappa L(x), \end{aligned} \tag{3.15}$$

where T_0 and κ are certain constants (whose quotient will be specified below). As for $L(x)$, we have

$$\begin{aligned} L(x) &= \int_0^x \sqrt{1 + y'(t)^2}\, dt \\ &= \int_0^x \sec\theta(t)\, dt, \end{aligned} \tag{3.16}$$

by (3.14) and Exercise 3 of §1.

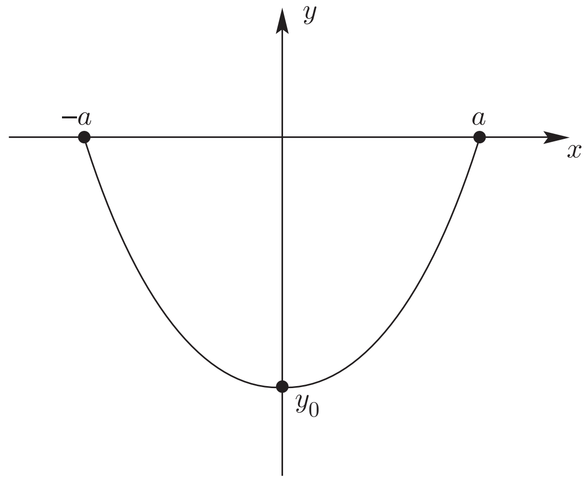

Figure 3.1

Taking the quotient of the two identities in (3.15) yields

$$\tan\theta(x) = \beta \int_0^x \sec\theta(t)\, dt, \quad \beta = \frac{\kappa}{T_0}. \tag{3.17}$$

Differentiating (3.17) with respect to x and using Exercise 4 of §1, we get

$$\sec^2\theta(x)\,\frac{d\theta}{dx} = \beta\sec\theta(x), \tag{3.18}$$

i.e.,

$$\frac{d\theta}{dx} = \beta\cos\theta. \tag{3.19}$$

We can separate variables here, to obtain

$$\int \sec\theta\, d\theta = \int \beta\, dx. \tag{3.20}$$

Exercise 14 of §1 applies to the integral on the left, and we get

$$\sec\theta(x) = \cosh(\beta x + \alpha). \tag{3.21}$$

To yield the expected result $\theta(0) = 0$ (see Figure 3.1 again), we set $\alpha = 0$.

To get a formula for $y(x)$, use (3.14) to write

$$y(x) = y_0 + \int_0^x \tan\theta(t)\, dt, \quad y_0 = y(0). \tag{3.22}$$

Now, by Exercises 3 and 8 of §1, together with (3.21), we have

$$\tan^2\theta(x) = \sec^2\theta(x) - 1 = \cosh^2\beta x - 1 = \sinh^2\beta x, \tag{3.23}$$

so (3.22) gives

$$\begin{aligned} y(x) &= y_0 + \int_0^x \sinh\beta t\,dt \\ &= y_0 - \frac{1}{\beta} + \frac{1}{\beta}\cosh\beta x. \end{aligned} \tag{3.24}$$

The graph of such a curve is called a *catenary.*

If we are given that the endpoints of the cable are at $(\pm a, 0)$ and that the total length is $2L$ (necessarily $L > a$), we can recover β and y_0 in (3.24), as follows. From (3.16) and (3.21),

$$L = \int_0^a \cosh\beta t\,dt = \frac{1}{\beta}\sinh\beta a, \tag{3.25}$$

so β is uniquely determined by the property that

$$\frac{\sinh\tau}{\tau} = \frac{L}{a}, \quad \beta = \frac{\tau}{a} > 0. \tag{3.26}$$

Note that $h(\tau) = (\sinh\tau)/\tau$ is smooth, $h(0) = 1$, $h'(\tau) > 0$ for $\tau > 0$, and $h(\tau) \nearrow +\infty$ as $\tau \nearrow +\infty$. Once one has β, then the identity $y(a) = 0$ gives

$$y_0 = \frac{1}{\beta} - \frac{1}{\beta}\cosh\beta a. \tag{3.27}$$

Homogeneous equations, separable in new variables

One can make a change of variable to convert a differential equation of the form

$$\frac{dx}{dt} = f(t, x) \tag{3.28}$$

to a separable equation when $f(t, x)$ has the following homogeneity property:

$$f(rt, rx) = f(t, x), \quad \forall\, r \in \mathbb{R} \setminus 0. \tag{3.29}$$

In such a case, f has the form

$$f(t, x) = g\Big(\frac{x}{t}\Big). \tag{3.30}$$

We can set

$$y = \frac{x}{t}, \tag{3.31}$$

so $x = ty,\ x' = ty' + y$, and (3.28) turns into

$$\frac{dy}{dt} = \frac{g(y) - y}{t}, \tag{3.32}$$

which is separable.

For example, consider

$$\frac{dx}{dt} = \frac{x^2 - t^2}{x^2 + t^2} + \frac{x}{t}. \tag{3.33}$$

In this case, (3.29) applies, and we can take $g(y) = (y^2 - 1)/(y^2 + 1) + y$ in (3.30), so with y as in (3.31) we have

$$\frac{dy}{dt} = \frac{1}{t}\,\frac{y^2 - 1}{y^2 + 1}, \tag{3.34}$$

which separates to

$$\Bigl(1 + \frac{2}{y^2 - 1}\Bigr)\, dy = \frac{dt}{t}. \tag{3.35}$$

To integrate the left side of (3.35), write

$$\frac{2}{y^2 - 1} = \frac{1}{y + 1} - \frac{1}{y - 1}, \tag{3.36}$$

to get

$$\begin{aligned} \int \frac{2}{y^2 - 1}\, dy &= \log|y + 1| - \log|y - 1| \\ &= \log\Bigl|\frac{y + 1}{y - 1}\Bigr|, \end{aligned} \tag{3.37}$$

the latter identity by (1.28). Thus the solution to (3.33) is given implicitly by

$$\frac{x}{t} + \log\Bigl|\frac{x + t}{x - t}\Bigr| = \log|t| + C. \tag{3.38}$$

Exercises

Solve the following initial value problems. Do the integrals, if you can.

(1)
$$\frac{dx}{dt} = \frac{x^2+1}{t^2+1}, \quad x(0) = 0.$$

(2)
$$\frac{dx}{dt} = (x^2-1)e^t, \quad x(0) = 1.$$

(3)
$$\frac{dx}{dt} = e^{x-t}, \quad x(0) = 0.$$

(4)
$$\frac{dx}{dt} = \sqrt{x^2+1}, \quad x(0) = 0.$$

(5)
$$\frac{dx}{dt} = \frac{xt}{x^2+t^2}, \quad x(1) = 1.$$

4. Second order equations – reducible cases

Second order differential equations have the form

$$x'' = f(t, x, x'), \quad x(t_0) = x_0, \quad x'(t_0) = v_0. \tag{4.1}$$

There are some important cases, with special structure, which reduce to first order equations for

$$v(t) = \frac{dx}{dt}. \tag{4.2}$$

One such case is

$$x'' = f(t, x'), \tag{4.3}$$

which for v given by (4.2) yields

$$\frac{dv}{dt} = f(t, v), \quad v(t_0) = v_0. \tag{4.4}$$

Depending on the nature of $f(t, v)$, methods discussed in §§2–3 might apply to (4.4). Once one has $v(t)$, then

$$x(t) = x_0 + \int_{t_0}^{t} v(s)\, ds. \tag{4.5}$$

The following is a more significant special case:

$$x'' = f(x, x'). \tag{4.6}$$

Direct substitution of v, given by (4.2), yields

$$\frac{dv}{dt} = f(x, v), \tag{4.7}$$

which is not satisfactory, since (4.7) contains too many variables. One route to success is to rewrite the equation as one for v as a function of x, using

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}. \tag{4.8}$$

Substitution into (4.7) gives the first order equation

$$\frac{dv}{dx} = \frac{f(x, v)}{v}, \quad v(x_0) = v_0. \tag{4.9}$$

Again, depending on the nature of $f(x, v)/v$, methods developed in §§2–3 might apply to (4.9).

An important special case of (4.6) is

$$x'' = f(x), \tag{4.10}$$

in which case (4.9) becomes

$$\frac{dv}{dx} = \frac{f(x)}{v}, \tag{4.11}$$

which is separable:

$$v\, dv = f(x)\, dx, \tag{4.12}$$

hence

$$\frac{1}{2}v^2 = g(x) + C, \quad \int f(x)\, dx = g(x) + C. \tag{4.13}$$

Thus

$$\frac{dx}{dt} = v = \pm\sqrt{2g(x) + 2C}, \tag{4.14}$$

which in turn is separable:

$$\pm \int \frac{dx}{\sqrt{2g(x) + 2C}} = t + C_2. \tag{4.15}$$

The constants C and C_2 are determined by the initial conditions.

Exercises

Use $v = dx/dt$ to transform each of the following equations to first order equations, either for $v = v(t)$ or for $v = v(x)$, as appropriate. Solve these first order equations, if you can.

(1)
$$\frac{d^2x}{dt^2} = t\frac{dx}{dt}.$$

(2)
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + t.$$

(3)
$$\frac{d^2x}{dt^2} = x\frac{dx}{dt}.$$

(4)
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + x.$$

(5)
$$\frac{d^2x}{dt^2} = x^2.$$

Reconsider (4) when you get to §9.

5. Newton's equations for motion in 1D

Newton's law for motion in 1D of a particle of mass m, subject to a force F, is

$$F = ma, \tag{5.1}$$

where a is acceleration:

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}, \tag{5.2}$$

the rate of change of the velocity $v(t) = dx/dt$. In general one might have $F = F(t, x, x')$. If F is t-independent, $F = F(x, x')$, which puts us in the setting of (4.6).

Frequently one has $F = F(x)$, which puts us in the setting of (4.10). We revisit this setting, bringing in some more concepts from physics. We set

$$F(x) = -V'(x). \tag{5.3}$$

$V(x)$, defined up to an additive constant, is called the potential energy. The total energy is the sum of the potential energy and the kinetic energy, $mv^2/2$:

$$E = \frac{1}{2}mv(t)^2 + V(x(t)). \tag{5.4}$$

Note that

$$\begin{aligned}\frac{dE}{dt} &= mv(t)v'(t) + V'(x(t))x'(t) \\ &= ma(t)v(t) - F(x(t))v(t) \\ &= 0,\end{aligned} \tag{5.5}$$

the last identity by (5.1). This identity celebrates energy conservation. Given that x solves

$$m\frac{d^2x}{dt^2} = -V'(x), \quad x(t_0) = x_0,\ x'(t_0) = v_0, \tag{5.6}$$

one has from (5.5) that for all t,

$$\frac{1}{2}mx'(t)^2 + V(x(t)) = E_0, \tag{5.7}$$

where

$$E_0 = \frac{1}{2}mv_0^2 + V(x_0). \tag{5.8}$$

The equation (5.7) is equivalent to

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\big(E_0 - V(x)\big)}, \tag{5.9}$$

which separates to

$$\int \frac{dx}{\sqrt{E_0 - V(x)}} = \pm\sqrt{\frac{2}{m}}t + C, \tag{5.10}$$

or, alternatively,

$$\int_{x_0}^{x} \frac{dy}{\sqrt{E_0 - V(y)}} = \pm\sqrt{\frac{2}{m}}(t - t_0). \tag{5.11}$$

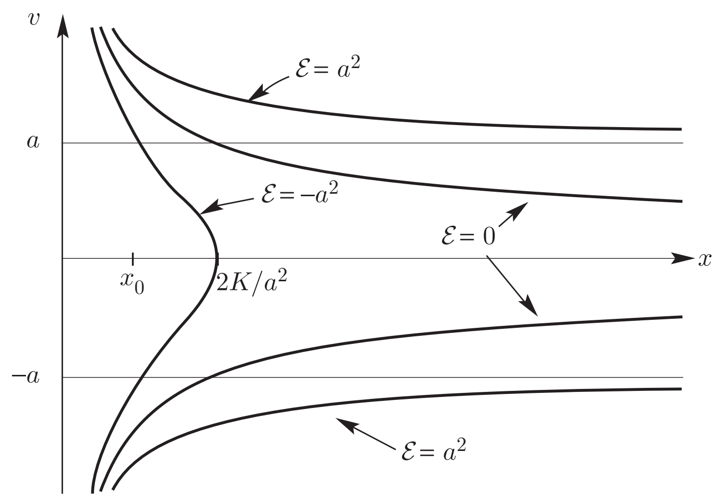

Figure 5.1

Note that (5.7) and (5.10) recover (4.13) and (4.15).

Projectile problem

Let us look in more detail at a special case, modeling the motion of a projectile of mass m traveling directly away from (or toward) the earth. In such a case, Newton's law of gravity gives

$$F(x) = -\frac{Km}{x^2}, \quad \text{hence} \quad V(x) = -\frac{Km}{x}, \quad x \in (0,\infty). \tag{5.12}$$

The conserved energy is

$$E_0 = \frac{m}{2}\Big(v^2 - \frac{2K}{x}\Big) = \frac{m}{2}\mathcal{E}(x,v). \tag{5.13}$$

See Figure 5.1 for a sketch of level curves of the function $\mathcal{E}(x,v)$. There are three cases to consider:

$$\begin{gathered} \mathcal{E} = -a^2 < 0, \quad \mathcal{E} = 0, \quad \mathcal{E} = a^2 > 0, \quad \text{i.e.,} \\ E_0 = -\frac{m}{2}a^2 < 0, \quad E_0 = 0, \quad E_0 = \frac{m}{2}a^2 > 0. \end{gathered} \tag{5.14}$$

In the first case, $x(t)$ has a maximum at $x_{\max} = 2K/a^2$. In the other two cases, $x(t) \to +\infty$ as $t \to +\infty$ (if $v_0 > 0$) or as $t \to -\infty$ (if $v_0 < 0$). Given $x_0 \in (0,\infty)$, the velocity $v_0 \in (0,\infty)$ for which $\mathcal{E}(x_0,v_0) = 0$ is called the "escape velocity."

We investigate the integral on the left side of (5.10), i.e.,

$$\int \frac{dx}{\sqrt{E_0 + Km/x}}, \tag{5.15}$$

which in the three cases in (5.14) is $\sqrt{2/m}$ times

$$\int \frac{x\,dx}{\sqrt{2Kx - a^2x^2}}, \quad \int \sqrt{\frac{x}{2K}}\,dx, \quad \int \frac{x\,dx}{\sqrt{2Kx + a^2x^2}}, \tag{5.16}$$

respectively. The second integral in (5.16) is easy; we investigate how to compute the other two, which we rewrite as

$$\frac{1}{a}\int \frac{x\,dx}{\sqrt{2kx - x^2}}, \quad \frac{1}{a}\int \frac{x\,dx}{\sqrt{2kx + x^2}}, \quad k = \frac{K}{a^2}. \tag{5.17}$$

We can compute these integrals by completing the square:

$$x^2 - 2kx = (x-k)^2 - k^2, \quad x^2 + 2kx = (x+k)^2 - k^2. \tag{5.18}$$

The respective change of variables $y = x - k$ and $y = x + k$ turn the integrals in (5.17) into the respective integrals

$$\int \frac{(y+k)\,dy}{\sqrt{k^2 - y^2}}, \quad \int \frac{(y-k)\,dy}{\sqrt{y^2 - k^2}}. \tag{5.19}$$

By inspection,

$$\int \frac{y\,dy}{\sqrt{k^2 - y^2}} = -\sqrt{k^2 - y^2} + C, \quad \int \frac{y\,dy}{\sqrt{y^2 - k^2}} = \sqrt{y^2 - k^2} + C. \tag{5.20}$$

The remaining parts of (5.19), after a change of variable $y = kz$, become

$$k\int \frac{dz}{\sqrt{1 - z^2}}, \quad k\int \frac{dz}{\sqrt{z^2 - 1}}. \tag{5.21}$$

To do these integrals, use

$$\begin{aligned} z = \sin s &\Longrightarrow \int \frac{dz}{\sqrt{1 - z^2}} = \int \frac{\cos s}{\cos s}\,ds = s + C, \\ z = \cosh s &\Longrightarrow \int \frac{dz}{\sqrt{z^2 - 1}} = \int \frac{\sinh s}{\sinh s}\,ds = s + C. \end{aligned} \tag{5.22}$$

Exercises

1. Make calculations analogous to (5.12)–(5.15) for each of the following forces. Examine whether you can do the resulting integrals.

(a)
$$F(x) = -Kx.$$

(b)
$$F(x) = -Kx^2.$$

(c)
$$F(x) = -\frac{K}{x}.$$

(d)
$$F(x) = x - x^3.$$

2. For such forces as given above, in each case find a potential energy $V(x)$ and sketch the level curves in the (x, v)-plane of the energy function

$$E(x, v) = \frac{m}{2}v^2 + V(x).$$

3. Use the substitution

$$x = k^2 \sin^2 \theta$$

to evaluate

$$\int \frac{dx}{\sqrt{\frac{k^2}{x} - 1}},$$

and use

$$x = k^2 \sinh^2 u$$

to evaluate

$$\int \frac{dx}{\sqrt{\frac{k^2}{x} + 1}}.$$

Use these calculations as alternatives for evaluating (5.15), for $E_0 < 0$ and $E_0 > 0$, respectively.

6. The pendulum

We produce a differential equation to describe the motion of a pendulum, which will be modeled by a rigid rod, of length ℓ, suspended at one end. We assume the rod has negligible mass, except for an object of mass m at the other end. See Figure 6.1. The rod is held at an angle $\theta = \theta_0$ from the downward pointing vertical, and released at time $t = 0$, after which it moves