

Contents

Preface	v
Chapter 0. PRELIMINARIES	1
0.1. Partial Differential Equations	1
0.1.1. What is a partial differential equation?	1
0.1.2. Superposition principle and subtraction principle	3
0.1.3. Sources of PDEs in classical physics	4
0.1.4. The one-dimensional heat equation	5
0.1.5. Classification of second-order PDEs	9
0.2. Separation of Variables	10
0.2.1. What is a separated solution?	10
0.2.2. Separated solutions of Laplace's equation	11
0.2.3. Real and complex separated solutions	13
0.2.4. Separated solutions with boundary conditions	18
0.3. Orthogonal Functions	21
0.3.1. Inner product space of functions	21
0.3.2. Projection of a function onto an orthogonal set	24
0.3.3. Orthonormal sets of functions	28
0.3.4. Parseval's equality, completeness, and mean square convergence	29
0.3.5. Weighted inner product	30
0.3.6. Gram-Schmidt orthogonalization	31
0.3.7. Complex inner product	32
Chapter 1. FOURIER SERIES	35
1.1. Definitions and Examples	35
1.1.1. Orthogonality relations	35
1.1.2. Definition of Fourier coefficients	36
1.1.3. Even functions and odd functions	37
1.1.4. Periodic functions	41
1.1.5. Implementation with Mathematica	41
1.1.6. Fourier sine and cosine series	42
1.2. Convergence of Fourier Series	46
1.2.1. Piecewise smooth functions	47
1.2.2. Dirichlet kernel	51
1.2.3. Proof of convergence	52
1.3. Uniform Convergence and the Gibbs Phenomenon	58
1.3.1. Example of Gibbs overshoot	58
1.3.2. Implementation with Mathematica	61
1.3.3. Uniform and nonuniform convergence	64
1.3.4. Two criteria for uniform convergence	64
1.3.5. Differentiation of Fourier series	65
1.3.6. Integration of Fourier series	66
1.3.7. A continuous function with a divergent Fourier series	67
1.4. Parseval's Theorem and Mean Square Error	71
1.4.1. Statement and proof of Parseval's theorem	71
1.4.2. Application to mean square error	72
1.4.3. Application to the isoperimetric theorem	74

1.5. Complex Form of Fourier Series	78
1.5.1. Fourier series and Fourier coefficients	78
1.5.2. Parseval's theorem in complex form	79
1.5.3. Applications and examples	79
1.5.4. Fourier series of mass distributions	81
1.6. Sturm-Liouville Eigenvalue Problems	84
1.6.1. Examples of Sturm-Liouville eigenvalue problems	85
1.6.2. Some general properties of S-L eigenvalue problems	86
1.6.3. Example of transcendental eigenvalues	87
1.6.4. Further properties: completeness and positivity	89
1.6.5. General Sturm-Liouville problems	92
1.6.6. Complex-valued eigenfunctions and eigenvalues	95
 Chapter 2. BOUNDARY-VALUE PROBLEMS IN RECTANGULAR COORDINATES	 99
2.1. The Heat Equation	99
2.1.1. Fourier's law of heat conduction	99
2.1.2. Derivation of the heat equation	100
2.1.3. Boundary conditions	101
2.1.4. Steady-state solutions in a slab	102
2.1.5. Time-periodic solutions	103
2.1.6. Applications to geophysics	105
2.1.7. Implementation with Mathematica	106
2.2. Homogeneous Boundary Conditions on a Slab	110
2.2.1. Separated solutions with boundary conditions	110
2.2.2. Solution of the initial-value problem in a slab	112
2.2.3. Asymptotic behavior and relaxation time	113
2.2.4. Uniqueness of solutions	114
2.2.5. Examples of transcendental eigenvalues	116
2.3. Nonhomogeneous Boundary Conditions	121
2.3.1. Statement of problem	122
2.3.2. Five-stage method of solution	122
2.3.3. Temporally nonhomogeneous problems	130
2.4. The Vibrating String	134
2.4.1. Derivation of the equation	134
2.4.2. Linearized model	137
2.4.3. Motion of the plucked string	138
2.4.4. Acoustic interpretation	140
2.4.5. Explicit (d'Alembert) representation	141
2.4.6. Motion of the struck string	145
2.4.7. d'Alembert's general solution	146
2.4.8. Vibrating string with external forcing	148
2.5. Applications of Multiple Fourier Series	152
2.5.1. The heat equation (homogeneous boundary conditions)	153
2.5.2. Laplace's equation	155
2.5.3. The heat equation (nonhomogeneous boundary conditions)	157
2.5.4. The wave equation (nodal lines)	159
2.5.5. Multiplicities of the eigenvalues	162
2.5.6. Implementation with Mathematica	164
2.5.7. Application to Poisson's equation	165

Chapter 3. BOUNDARY-VALUE PROBLEMS IN CYLINDRICAL COORDINATES	171
3.1. Laplace's Equation and Applications	171
3.1.1. Laplacian in cylindrical coordinates	171
3.1.2. Separated solutions of Laplace's equation in ρ, φ	173
3.1.3. Application to boundary-value problems	174
3.1.4. Regularity	177
3.1.5. Uniqueness of solutions	177
3.1.6. Exterior problems	178
3.1.7. Wedge domains	178
3.1.8. Neumann problems	179
3.1.9. Explicit representation by Poisson's formula	179
3.2. Bessel Functions	183
3.2.1. Bessel's equation	183
3.2.2. The power series solution of Bessel's equation	184
3.2.3. Integral representation of Bessel functions	188
3.2.4. The second solution of Bessel's equation	191
3.2.5. Zeros of the Bessel function J_0 .	192
3.2.6. Asymptotic behavior and zeros of Bessel functions	193
3.2.7. Fourier-Bessel series	197
3.2.8. Implementation with Mathematica	202
3.3. The Vibrating Drumhead	209
3.3.1. Wave equation in polar coordinates	209
3.3.2. Solution of initial-value problems	211
3.3.3. Implementation with Mathematica	214
3.4. Heat Flow in the Infinite Cylinder	216
3.4.1. Separated solutions	217
3.4.2. Initial-value problems in a cylinder	217
3.4.3. Initial-value problems between two cylinders	221
3.4.4. Implementation with Mathematica	224
3.4.5. Time-periodic heat flow in the cylinder	224
3.5. Heat Flow in the Finite Cylinder	227
3.5.1. Separated solutions	227
3.5.2. Solution of Laplace's equation	228
3.5.3. Solutions of the heat equation with zero boundary conditions	231
3.5.4. General initial-value problems for the heat equation	232
Chapter 4. BOUNDARY-VALUE PROBLEMS IN SPHERICAL COORDINATES	235
4.1. Spherically Symmetric Solutions	235
4.1.1. Laplacian in spherical coordinates	236
4.1.2. Time-periodic heat flow: Applications to geophysics	237
4.1.3. Initial-value problem for heat flow in a sphere	240
4.1.4. The three-dimensional wave equation	247
4.1.5. Convergence of series in three dimensions	249
4.2. Legendre Functions and Spherical Bessel Functions	251
4.2.1. Separated solutions in spherical coordinates	251
4.2.2. Legendre polynomials	253
4.2.3. Legendre polynomial expansions	258

4.2.4. Implementation with Mathematica	259
4.2.5. Associated Legendre functions	261
4.2.6. Spherical Bessel functions	263
4.3. Laplace's Equation in Spherical Coordinates	267
4.3.1. Boundary-value problems in a sphere	268
4.3.2. Boundary-value problems exterior to a sphere	269
4.3.3. Applications to potential theory	272
Chapter 5. FOURIER TRANSFORMS AND APPLICATIONS	277
5.1. Basic Properties of the Fourier Transform	277
5.1.1. Passage from Fourier series to Fourier integrals	277
5.1.2. Definition and properties of the Fourier transform	279
5.1.3. Fourier sine and cosine transforms	285
5.1.4. Generalized h -transform	287
5.1.5. Fourier transforms in several variables	288
5.1.6. The uncertainty principle	289
5.1.7. Proof of convergence	291
5.2. Solution of the Heat Equation for an Infinite Rod	294
5.2.1. First method: Fourier series and passage to the limit	294
5.2.2. Second method: Direct solution by Fourier transform	295
5.2.3. Verification of the solution	296
5.2.4. Explicit representation by the Gauss-Weierstrass kernel	297
5.2.5. Some explicit formulas	300
5.2.6. Solutions on a half-line: The method of images	303
5.2.7. The Black-Scholes model	310
5.2.8. Hermite polynomials	314
5.3. Solutions of the Wave Equation and Laplace's Equation	318
5.3.1. One-dimensional wave equation and d'Alembert's formula	318
5.3.2. General solution of the wave equation	321
5.3.3. Three-dimensional wave equation and Huygens' principle	323
5.3.4. Extended validity of the explicit representation	327
5.3.5. Application to one- and two-dimensional wave equations	329
5.3.6. Laplace's equation in a half-space: Poisson's formula	332
5.4. Solution of the Telegraph Equation	335
5.4.1. Fourier representation of the solution	336
5.4.2. Uniqueness of the solution	338
5.4.3. Time-periodic solutions of the telegraph equation	339
Chapter 6. ASYMPTOTIC ANALYSIS	345
6.1. Asymptotic Analysis of the Factorial Function	345
6.1.1. Geometric mean approximation: Analysis by logarithms	346
6.1.2. Refined method using functional equations	347
6.1.3. Stirling's formula via an integral representation	348
6.2. Integration by Parts	350
6.2.1. Two applications	351
6.3. Laplace's Method	354
6.3.1. Statement and proof of the result	354
6.3.2. Three applications to integrals	357
6.3.3. Applications to the heat equation	358
6.3.4. Improved error with gaussian approximation	359

6.4. The Method of Stationary Phase	362
6.4.1. Statement of the result	363
6.4.2. Application to Bessel functions	364
6.4.3. Proof of the method of stationary phase	364
6.5. Asymptotic Expansions	368
6.5.1. Extension of integration by parts	368
6.5.2. Extension of Laplace's method	369
6.6. Asymptotic Analysis of the Telegraph Equation	371
6.6.1. Asymptotic behavior in case $\alpha = 0$	372
6.6.2. Asymptotic behavior in case $\alpha > 0$	372
6.6.3. Asymptotic behavior in case $\alpha < 0$	374
 Chapter 7. NUMERICAL ANALYSIS	 379
7.1. Numerical Analysis of Ordinary Differential Equations	379
7.1.1. The Euler method	380
7.1.2. The Heun method	384
7.1.3. Error analysis	387
7.2. The One-Dimensional Heat Equation	393
7.2.1. Formulation of a difference equation	393
7.2.2. Computational molecule	394
7.2.3. Examples and comparison with the Fourier method	395
7.2.4. Stability analysis	398
7.2.5. Other boundary conditions	399
7.3. Equations in Several Dimensions	403
7.3.1. Heat equation in a triangular region	404
7.3.2. Laplace's equation in a triangular region	405
7.4. Variational Methods	409
7.4.1. Variational formulation of Poisson's equation	409
7.4.2. More general variational problems	411
7.4.3. Variational formulation of eigenvalue problems	411
7.4.4. Variational problems, minimization, and critical points	412
7.5. Approximate Methods of Ritz and Kantorovich	415
7.5.1. The Ritz method: Rectangular regions	416
7.5.2. The Kantorovich method: Rectangular regions	417
7.6. Orthogonality Methods	420
7.6.1. The Galerkin method: Rectangular regions	420
7.6.2. Nonrectangular regions	422
7.6.3. The finite element method	425
 Chapter 8. GREEN'S FUNCTIONS	 427
8.1. Green's Functions for Ordinary Differential Equations	427
8.1.1. An example	427
8.1.2. The generic case	429
8.1.3. The exceptional case: Modified Green's function	431
8.1.4. The Fredholm alternative	432
8.2. The Three-Dimensional Poisson Equation	433
8.2.1. Newtonian potential kernel	434
8.2.2. Single- and double-layer potentials	436
8.2.3. Green's function of a bounded region	437
8.2.4. Solution of the Dirichlet problem	440

8.3. Two-Dimensional Problems	443
8.3.1. The logarithmic potential	443
8.3.2. Green's function of a bounded plane region	444
8.3.3. Solution of the Dirichlet problem	445
8.3.4. Green's functions and separation of variables	446
8.4. Green's Function for the Heat Equation	450
8.4.1. Nonhomogeneous heat equation	450
8.4.2. The one-dimensional heat kernel and the method of images	452
8.5. Green's Function for the Wave Equation	454
8.5.1. Derivation of the retarded potential	454
8.5.2. Green's function for the Helmholtz equation	459
8.5.3. Application to the telegraph equation	461
APPENDIXES	465
A.1. Review of Ordinary Differential Equations	465
A.1.1. First-order linear equations	465
A.1.2. Second-order linear equations	466
A.1.3. Second-order linear equations with constant coefficients	468
A.1.4. Euler's equidimensional equation	470
A.1.5. Power series solutions	471
A.1.6. Steady state and relaxation time	474
A.2. Review of Infinite Series	476
A.2.1. Numerical series	476
A.2.2. Taylor's theorem	478
A.2.3. Series of functions: Uniform convergence	480
A.2.4. Abel's lemma	483
A.2.5. Double series	484
A.2.6. Big- O notation	485
A.3. Review of Vector Integral Calculus	489
A.3.1. Implementation with Mathematica	491
A.4. Using Mathematica	492
A.4.1. Introduction	492
A.4.2. The notebook front end	492
A.4.3. Textual interface: Direct access through a terminal window	495
A.4.4. Mathematica notation versus ordinary mathematical notation	495
A.4.5. Functional notation in Mathematica	500
ANSWERS TO SELECTED EXERCISES	503
INDEX	521
ABOUT THE AUTHOR	527