Power Series Expansions

In this section we present several striking applications of Cauchy's theorems. The first of these is the existence of local power series expansions for analytic functions. This leads to a number of other results, including the Fundamental Theorem of Algebra and detailed information about the zeroes and singularities of analytic functions.

Before we show that analytic functions have power series expansions we need to develop a deeper understanding of convergence issues for power series. The first section of the chapter is devoted to this task.

3.1. Uniform Convergence

We would like to be able to integrate and differentiate power series term by term. This is shown to be legitimate in the case of real power series in the typical advanced calculus or foundations of analysis course. The key to doing this is to show that power series *converge uniformly* on certain sets. We will give a brief development of these ideas in the context of complex-valued functions of a complex variable.

Definition 3.1.1. Let $\{f_n\}$ be a sequence of functions defined on a set $S \subset \mathbb{C}$. Then

(a) the sequence $\{f_n\}$ converges *pointwise* to the function f on S if, for each $z \in S$, the sequence of numbers $\{f_n(z)\}$ converges to the number $f(z)$ – that is, for each $z \in S$ and each $\epsilon > 0$, there is an N such that

$$|f_n(z) - f(z)| < \epsilon \quad \text{for all} \quad n \geq N;$$

(b) the sequence $\{f_n\}$ converges *uniformly* on S if for each $\epsilon > 0$ there exists an N such that

$$|f_n(z) - f(z)| < \epsilon \quad \text{for all} \quad n \geq N \quad \text{and all} \quad z \in S.$$

There is a subtle but crucial difference between statements (a) and (b) in the above definition: In (b), given ϵ, there must be an N that works for all $z \in S$. In (a), for each z there must be an N, but N depends on z, in general, and there may not be an N that works simultaneously for all $z \in S$.

The importance of uniform convergence stems primarily from two facts that are proved below and are extensively used thereafter: (1) The limit of a uniformly convergent sequence of continuous functions is also continuous; and (2) the integral along a path of the limit of a uniformly convergent sequence of continuous functions is the limit of their integrals.

Theorem 3.1.2. *If E is a subset of $\mathbb{C}$ and $\{f_n\}$ is a sequence of continuous functions on E which converges uniformly on E to a function f, then f is also continuous on E.*

Proof. Let z_0 be a point of E. Given $\epsilon > 0$, we choose N such that

$$n \geq N \quad \text{implies} \quad |f(z) - f_n(z)| < \frac{\epsilon}{3} \quad \text{for all} \quad z \in E.$$

We can do this because $\{f_n\}$ converges uniformly to f on E.

We next choose a $\delta > 0$ such that

$$|f_N(z) - f_N(z_0)| < \frac{\epsilon}{3} \quad \text{whenever} \quad z \in E \quad \text{and} \quad |z - z_0| < \delta.$$

We can do this because f_N is continuous on E.

Then $|z - z_0| < \delta$ and $z \in E$ imply

$$\begin{aligned} |f(z) - f(z_0)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

We conclude that $\lim_{z \to z_0} f(z) = f(z_0)$, and so f is continuous at z_0. Since z_0 was a general point of E, f is continuous on E. □

Theorem 3.1.3. *If $\gamma : I \to \mathbb{C}$ is a path and $\{f_n\}$ is a sequence of continuous functions on $\gamma(I)$ which converges uniformly on $\gamma(I)$ to f, then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz. \tag{3.1.1}$$

Proof. Given $\epsilon > 0$, we choose N such that

$$|f(z) - f_n(z)| < \epsilon/\ell(\gamma) \quad \text{for all} \quad n \geq N,\ z \in \gamma(I).$$

Then Theorem 2.4.9 implies that, for $n \geq N$,

$$\left|\int_\gamma f(z)\,dz - \int_\gamma f_n(z)\,dz\right| = \left|\int_\gamma (f(z) - f_n(z))\,dz\right| \leq \frac{\epsilon}{\ell(\gamma)}\ell(\gamma) = \epsilon$$

and this proves (3.1.1). □

Example 3.1.4. If $f_n(z) = |z|^n$, then prove that the sequence $\{f_n\}$ converges pointwise on $\overline{D}_1(0)$ but not uniformly. Show that it does converge uniformly on any disc $\overline{D}_r(0)$ with $r < 1$.

Solution: If $|z| < 1$, then $|z|^n \to 0$. If $|z| = 1$, then $|z|^n \to 1$. Thus, the sequence converges pointwise and the limit function $f(z)$ is 0 if $|z| < 1$ and 1 if $|z| = 1$. The convergence is not uniform because the limit function is not continuous.

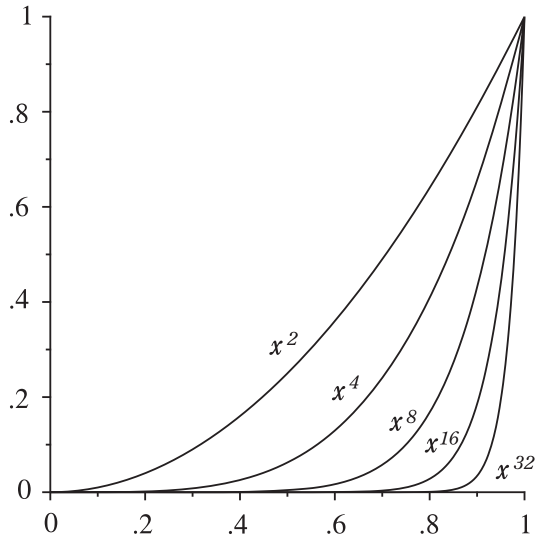

Figure 3.1.1. The Sequence $\{x^n\}$ does not Converge Uniformly on $[0,1]$.

The fact that the convergence is not uniform can also be seen directly: No matter how large n is chosen, we can always find a z with $|z| < 1$ such that $|f_n(z) - f(z)| = |z|^n \geq 1/2$. In fact, $(1/2)^{1/n}$ is such a z. Thus, the condition for uniform convergence fails to hold for $\epsilon = 1/2$.

On the other hand, if $z \in \overline{D}_r(0)$ with $r < 1$, then $|z| \leq r$ and $|z|^n \leq r^n$. Given $\epsilon > 0$, if N is chosen larger than $\log \epsilon / \log r$, then $n \geq N$ implies

$$|f_n(z) - f(z)| = |z|^n \leq r^n < \epsilon.$$

Since N was chosen independent of z, the convergence is uniform on $\overline{D}_r(0)$.

The real analogue of the above example is the sequence $\{x^n\}$ on $[0,1]$, which is illustrated in Figure 3.1.1.

Uniform Convergence of Series. We say that an infinite series $\sum_{k=0}^{\infty} f_k(z)$ of functions, defined on a set E, converges uniformly on E if the sequence of partial sums $\{s_n\}$ converges uniformly on E, where we recall that

$$s_n(z) = \sum_{k=0}^{n} f_k(z).$$

There is a very useful criterion which insures uniform convergence of such a sequence. This is the *Weierstrass M-test*:

Theorem 3.1.5 (Weierstrass M-Test)**.** *Let*

$$\sum_{k=0}^{\infty} f_k(z) \tag{3.1.2}$$

be an infinite series of functions defined on a set E. If there is a convergent series of non-negative numbers

$$\sum_{k=0}^{\infty} M_k \tag{3.1.3}$$

such that $|f_k(z)| \le M_k$ for all k and all $z \in E$, then (3.1.2) *converges uniformly on E.*

Proof. The comparison test, comparing (3.1.2) to (3.1.3), shows that, for each $z \in E$, the series (3.1.2) converges. Let $s(z)$ be the number it converges to, and let $s_n(z)$ denote the nth partial sum of (3.1.2). Then

$$|s(z) - s_n(z)| \le \sum_{k=n+1}^{\infty} |f_k(z)| \le \sum_{k=n+1}^{\infty} M_k. \tag{3.1.4}$$

Since the series (3.1.3) converges, given $\epsilon > 0$, we can choose N such that the right side of (3.1.4) is less than ϵ for all $n \ge N$. Then (3.1.4) implies that $|s(z)-s_n(z)| < \epsilon$ for all $n \ge N$ and all $z \in E$. Since N was chosen independently of z, this shows that the convergence is uniform. □

Example 3.1.6. Show that the series $\sum_{k=1}^{\infty} z^k/k^2$ converges uniformly on the closed unit disc $\overline{D}_1(0)$.

Solution: We have $|z^k/k^2| \le 1/k^2$ if $|z| \le 1$. Furthermore, $\sum_{k=1}^{\infty} 1/k^2$ converges, because it is a p-series with $p = 2$. Hence, by the Weierstrass M-test, the series $\sum_{k=1}^{\infty} z^k/k^2$ converges uniformly on $\overline{D}_1(0)$.

Uniform Convergence of Power Series. In Section 1.2 we stated without proof that a complex power series converges on a certain open disc and diverges at all points in the complement of the corresponding closed disc. The radius of this disc is called the *radius of convergence* of the power series. We are now prepared to prove this result and give a formula for the radius of convergence.

The theorem that does this uses the notion of lim sup of a sequence. This is defined as follows.

If S is a non-empty set of real numbers, then an *upper bound* for S is a number M such that $s \le M$ for every $s \in S$. If there is an upper bound for S, then S is said to be *bounded above*. The completeness axiom for the real number system states that each non-empty set of real numbers S that is bounded above, has a least upper bound L. This means L has two properties: (1) it is greater than or equal to each element of S, and (2) it is less than every other number with this property. We will denote the least upper bound of a non-empty set S which is bounded above by $\sup(S)$. If the set is non-empty, but not bounded above, we set $\sup(S) = \infty$. The notion of inf or *greatest lower bound* is defined analogously, but the inequalities are all reversed.

Definition 3.1.7. If $\{a_k\}$ is a sequence of real numbers, then $\limsup\{a_k\}$ is the limit of the non-increasing sequence $\{u_n\}$ defined by

$$u_n = \sup\{a_k : k \ge n\}.$$

The sequence $\{u_n\}$ of this definition is non-increasing because as n increases, the set of numbers $\{a_k : k \geq n\}$ gets smaller. The sequence $\{u_n\}$ may not, however, be bounded below and so $\limsup\{a_k\}$ may be $-\infty$. Also, the u_n could all be $+\infty$, in which case $\limsup\{a_k\} = +\infty$.

Of course, there is an analogous notion, lim inf, which uses inf instead of sup in the above definition.

The lim sup and lim inf of a sequence $\{a_k\}$ always exist (but may be infinite), even though $\lim a_k$ itself may not exist. In fact the sequence has a limit (which may be infinite) if and only if $\limsup a_k = \liminf a_k$. In this case $\lim a_k$ is this common value.

Theorem 3.1.8. *Given a power series $\sum_{k=0}^{\infty} c_k(z-z_0)^k$, let*

$$R = \left(\limsup |c_k|^{1/k}\right)^{-1}.$$

Then the series converges absolutely if $z \in D_R(z_0)$ and diverges if $z \notin \overline{D}_R(z_0)$. Furthermore, it converges uniformly on each closed disc $\overline{D}_r(z_0)$ with $r < R$. Thus, R is the radius of convergence of the given power series.

Proof. Since we can always make a change of variables which replaces $z - z_0$ by z, we may as well assume that $z_0 = 0$.

By definition, $\limsup |c_k|^{1/k}$ is the limit of the non-increasing sequence $\{u_n\}$ where

$$u_n = \sup\{|c_k|^{1/k} : k \geq n\}.$$

If $r < R$, we choose a number t with $r < t < R$. Then $t^{-1} > R^{-1} = \lim u_n$. This implies that for large enough n the numbers u_n are less than t^{-1}. If n is one such integer, then, for all $k \geq n$,

$$|c_k|^{1/k} < t^{-1} \quad \text{and, hence,} \quad |c_k| < t^{-k}.$$

Now if $|z| \leq r$, then this implies that

$$|c_k z^k| < \left(\frac{r}{t}\right)^k \quad \text{for all} \quad k \geq n. \tag{3.1.5}$$

Since $|r/t| < 1$, the geometric series $\sum_{k=n}^{\infty}(r/t)^k$ converges. Then the Weierstrass M-test implies that the series $\sum_{k=n}^{\infty} c_k z^k$ converges uniformly in the disc $\overline{D}_r(0)$. The same is true of the original series $\sum_{k=0}^{\infty} c_k z^k$, since the convergence or uniform convergence of a series is unaffected by the first n terms if n is fixed.

Since the series converges on each closed disk $\overline{D}_r(0)$, with radius less than R, it converges at each point z in the open disc $D_R(0)$.

It remains to prove that the series diverges at each z with $|z| > R$. Given such a z, we have

$$|z|^{-1} < \lim u_n.$$

This implies that, for each n, there is a $k > n$ with

$$|z|^{-1} < |c_k|^{1/k}, \quad \text{so that} \quad |c_k z^k| > 1.$$

But this means that there is a subsequence of the sequence of terms $\{c_k z^k\}$ consisting of numbers with modulus greater than 1. Since the sequence of terms does not converge to 0, the series diverges by the term test (Exercise 1.2.9). $\square$

The above theorem has the following corollary, the proof of which is left as an exercise.

Corollary 3.1.9. *If f has a power series expansion about z_0 which converges on the disc $D_R(z_0)$, then f is continuous on this disc.*

Example 3.1.10. Prove that if $\sum_{k=0}^{\infty} c_k z^k$ is a power series with radius of convergence R, then the power series $\sum_{k=1}^{\infty} kc_k z^{k-1}$ also has radius of convergence R.

Solution: If we multiply the second series by z, the set on which the series converges does not change, and so its radius of convergence does not change. The resulting series is $\sum_{k=0}^{\infty} kc_k z^k$. Let R_1 be its radius of convergence. By the previous theorem,

$$R = \left(\limsup |c_k|^{1/k}\right)^{-1},$$

and

$$R_1 = \left(\limsup |kc_k|^{1/k}\right)^{-1} = \left(\limsup k^{1/k}|c_k|^{1/k}\right)^{-1}.$$

The sequence $k^{1/k}$ has limit 1 (Exercise 3.1.11), and so the factor $k^{1/k}$ does not effect the lim sup (Exercise 3.1.12). Hence, $R_1 = R$.

Theorem 3.1.8 and Theorem 3.1.3 combine to prove that it is legitimate to integrate a power series term by term.

Theorem 3.1.11. *Let $f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$, where the radius of convergence of this power series is R. Then*

$$\int_{z_0}^{z} f(w)\,dw = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(z - z_0)^{k+1} \tag{3.1.6}$$

for all $z \in D_R(z_0)$.

Proof. If we set $s_n(z) = \sum_{k=0}^{n} c_k(z - z_0)^k$ for each positive integer n, then

$$\int_{z_0}^{z} s_n(w)\,dw = \sum_{k=0}^{n} \frac{c_k}{k+1}(z - z_0)^{k+1}$$

because the integral is linear and we know how to integrate $(z - z_0)^k$. To finish the proof of (3.1.6), we just need to take the limit of both sides and use Theorem 3.1.3 to bring the limit inside the integral on the left. Of course, we need to know that the convergence of $\{s_n\}$ to f is uniform on $[z_0, z]$. This, however, follows from Theorem 3.1.8, since the interval $[z_0, z]$ is inside the closed disc $\overline{D}_r(z_0)$, where $r = |z| < R$. □

Exercise Set 3.1

1. Show that the sequence $\{1/(nz)\}$ converges uniformly to 0 on every set of the form $\{z : |z| \geq r\}$ for fixed $r > 0$, but it does not converge uniformly on $\{z : z \neq 0\}$.

2. Show that the sequence $\{\sin(x/n)\}$ converges uniformly to 0 on any interval of the form $[0, k]$, but it does not converge uniformly on $[0, \infty)$.
3. Show that the sequence $\{\arctan(nx)\}$ converges pointwise but not uniformly on $\mathbb{R}$.
4. Use the Weierstrass M-test (but not Theorem 3.1.8) to show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k!}$ converges uniformly on $D_R(0)$ for each $R > 0$.
5. Use the Weierstrass M-test to show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{k+z}{k^3+1}$ converges uniformly on $\overline{D}_1(0)$.
6. Show that for each $r > 0$ the series
$$\sum_{k=0}^{\infty} \frac{1}{k^2 - z}$$
converges uniformly on the set
$$E_r = \{z : |z| \leq r,\ z \neq k^2 \text{ for } k = 0, 1, 2, \cdots\}.$$
7. Prove that the series $\sum_{k=1}^{\infty} k^{-z}$ converges uniformly on each set of the form $\{z \in \mathbb{C} : \mathrm{Re}(z) > s\}$, with $s > 1$. The function to which it converges is called the Riemann Zeta Function.
8. If the series of the previous exercise is differentiated term by term, does the resulting series still converge uniformly on $\{z \in \mathbb{C} : \mathrm{Re}(z) > s\}$ if $s > 1$?
9. For each n find $\sup\{1 + (-1)^k + 1/k : k \geq n\}$.
10. Find the radius of convergence of the power series $\sum_{k=0}^{\infty} (2 + (-1)^k)^k z^k$.
11. Prove that $\lim k^{1/k} = 1$.
12. Prove that if $\{a_k\}$ and $\{b_k\}$ are two sequences of non-negative numbers with $\lim a_k = a$ and $\limsup b_k = b$, then $\limsup a_k b_k = ab$.
13. Prove Corollary 3.1.9.
14. Can a power series of the form $\sum_{k=0}^{\infty} c_k (z-1)^k$ converge at $z = 3$ and diverge at $z = 0$? Why?
15. Using the power series expansion $\displaystyle\frac{1}{1+w} = \sum_{k=0}^{\infty} (-1)^k w^k$, find a power series expansion for $\displaystyle\int_0^z \frac{1}{1+w}\, dw$ about 0. What is the radius of convergence of this power series? What function does it converge to?
16. If a function $E(z)$ is defined on $\mathbb{C}$ by
$$E(z) = \int_0^z \mathrm{e}^{-w^2}\, dw,$$
find a power series expansion for $E(z)$ about 0. Where does this power series converge?

3.2. Power Series Expansions

A function of a real variable can be differentiable, even infinitely differentiable on an interval and still not have a convergent power series expansion in that interval. For functions of a complex variable, the situation is quite different. We will prove that every analytic function has convergent power series expansions about each point of its domain. In fact, we will prove that a function f is analytic if and only if it has such expansions. First, we show that a function which has a power series expansion on a disc is analytic on that disc. This involves showing that we can differentiate power series term by term.

Differentiating Power Series. The next theorem and its corollaries concern a function f defined by a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{3.2.1}$$

with radius of convergence R.

Theorem 3.2.1. *If f is defined as above, then f is analytic on $D_R(z_0)$ and f' has a convergent power series expansion*

$$f'(z) = \sum_{n=1}^{\infty} nc_n (z - z_0)^{n-1},$$

which converges to f' on $D_R(z_0)$.

Proof. If f has a power series expansion (3.2.1) with radius of convergence R, let g be the function which we hope turns out to be the derivative of f – that is, we set

$$g(z) = \sum_{n=1}^{\infty} nc_n (z - z_0)^{n-1}.$$

This series has the same radius of convergence as the series for f (see Example 3.1.10) and so it converges on $D_R(z_0)$ and converges uniformly on any smaller closed disc. By Corollary 3.1.9, g is continous on $D_R(z_0)$, and by Theorem 3.1.11,

$$\int_{z_0}^{z} g(w)\, dw = \sum_{n=1}^{\infty} c_n z^n = f(z) - f(z_0)$$

on $D_R(z_0)$. Theorem 2.6.1 tells us this function is an antiderivative for $g(z)$. Since $f(z_0)$ is a constant, this means $f' = g$ on $D_R(z_0)$, as required. □

Example 3.2.2. Find a power series expansion for the principal branch of the log function about the point $z_0 = 1$.

Solution: We know that the derivative of $\log(z)$ is $1/z$ (Exercise 2.2.14). We also know that, since

$$\frac{1}{z} = \frac{1}{1 - (1 - z)},$$

the power series expansion of $1/z$ about $z_0 = 1$ is

$$\frac{1}{z} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n. \tag{3.2.2}$$

Since we can differentiate term by term, a series which has this as its derivative is the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n+1}}{n+1} = -\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(z-1)^n}{n}. \tag{3.2.3}$$

The series (3.2.2) and (3.2.3) both have radius of convergence 1. If $f(z)$ denotes the sum of series (3.2.3), then $f'(z) = 1/z = \log'(z)$. It follows that f and log differ by a constant. Since they both have the value 0 at $z = 1$, they are the same. Therefore, (3.2.3) is the power series expansion of $\log(z)$ about 1.

Using Theorem 3.2.1, the following can be proved in the same way as its real variable counterpart. The details are left to the exercises.

Corollary 3.2.3. *If f has a power series expansion* (3.2.1) *about z_0, with radius of convergence R, then it has derivatives of all orders on $D_R(z_0)$. Its kth derivative is*

$$f^{(k)}(z) = \sum_{n=k}^{\infty}\frac{n!\,c_n}{(n-k)!}(z-z_0)^{n-k}.$$

In particular, its kth derivative at z_0 is given by

$$f^{(k)}(z_0) = k!\,c_k.$$

This immediately implies:

Corollary 3.2.4. *If f has a power series expansion* (3.2.1) *about z_0, with positive radius of convergence, then it has only one such expansion. In fact, the coefficients $\{c_n\}$ for such an expansion are uniquely determined by the equations*

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Power Series Expansions of Analytic Functions. We are now ready to present the most important application of Cauchy's theorems – the proof of the existence of local power series expansions of analytic functions.

Theorem 3.2.5. *Let f be analytic in an open set U and suppose $D_r(z_0)$, $r > 0$, is an open disc contained in U. Then there is a power series expansion for f*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n,$$

which converges to $f(z)$ on $D_r(z_0)$. Furthermore, the coefficients of this power series are the numbers

$$c_n = \frac{1}{2\pi i}\int_{|w-z_0|=s}\frac{f(w)}{(w-z_0)^{n+1}}\,dw, \tag{3.2.4}$$

where s is any number with $0 < s < r$.

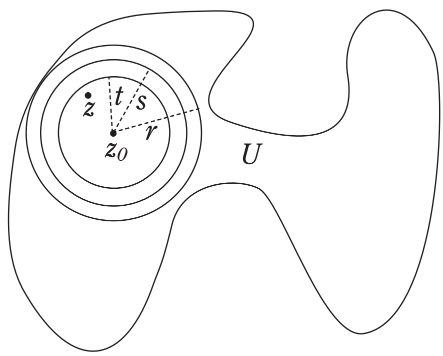

Figure 3.2.1. Setup for the Proof of the Existence of Power Series Expansions.

Proof. If $0 < t < s < r$, $|w - z_0| = s$, and $|z - z_0| \leq t$, then

$$\left| \frac{z - z_0}{w - z_0} \right| \leq \frac{t}{s} < 1,$$

and so we have

$$\frac{w - z_0}{w - z} = \left(1 - \frac{z - z_0}{w - z_0}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n, \tag{3.2.5}$$

where the geometric series on the right is dominated by the constant geometric series $\sum_{n=0}^{\infty}(t/s)^n$. By Theorem 3.1.5, this implies that (3.2.5) converges uniformly as a function of $z \in D_t(z_0)$ and also as a function of $w \in \partial D_s(z_0)$.

If we multiply (3.2.5) by $f(w)/(w - z_0)$ and integrate around the circle of radius s, then, since the series converges uniformly in w, we may integrate term by term. Using Cauchy's Integral Formula, this yields

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|z_0 - w| = s} \frac{f(w)}{w - z}\, dw \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\int_{|w - z_0| = s} \frac{f(w)}{(w - z_0)^{n+1}}\, dw \right) (z - z_0)^n. \end{aligned}$$

This gives us a power series expansion of f on $D_s(z_0)$ with coefficients given by (3.2.4).

Now given any $z \in D_r(z_0)$, we may choose s such that $|z - z_0| < s < r$. With this choice of s, the above series is defined and converges at z. However, it follows from Cauchy's Integral Theorem that the integrals defining the coefficients of this series do not depend on the choice of s (see Exercise 2.6.9). Thus, we have a power series expansion for f which converges on $D_r(z_0)$, with coefficients given by (3.2.4). This completes the proof. □

Corollary 3.2.6. *If f is analytic on an open set U, then f has derivatives of all orders on U and they are all analytic.*

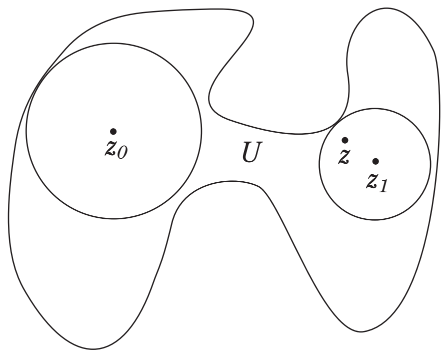

Figure 3.2.2. Shift to an Expansion About a Different Point of U.

Proof. Given any point $z_0 \in U$, there is a disc $D_r(z_0)$ centered at z_0 which is contained in U. On this disc, f has a convergent power series expansion. By Corollary 3.2.3 f has derivatives of all orders on this disc. These derivatives are analytic because each of them has a complex derivative. Therefore, f has analytic derivatives of all orders on all of U. □

It is important to emphasize that the power series expansion of an analytic function f about a point z_0 converges in the largest open disc, centered at z_0, that is contained in the domain U of f. In general, it will not converge at other points of U. To obtain a power series expansion of f that converges in a neighborhood of a point z outside this disc, we have to shift to a power series expansion about a different point of U – a point z_1 with the property that the largest disc centered at z_1 and contained in U contains the point z (see Figure 3.2.2). For example, the power series expansion of $\log z$ about $z = 1$ given in Example 3.2.2 has radius of convergence 1, and so it does not converge at $z = i$. There is a power series expansion of $\log z$ about $z = i$ (Exercise 3.2.7), but it is a different power series than the one about $z = 1$.

Example 3.2.7. Show that if f is a function which is analytic in an open set U, $z_0 \in U$, and $f(z_0) = 0$, then $g(z) = f(z)/(z - z_0)$ can be given a value at z_0 which makes it analytic on U.

Solution: By the above theorem f has a power series expansion about z_0 which converges in the largest open disc D which is centered at z_0 and contained in U. Since $f(z_0) = 0$, the constant term of this series is 0. Hence, this expansion has the form

$$f(z) = c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n + \cdots .$$

If we give the function $g(z)$ the value c_1 at $z = z_0$, then it agrees on D with the sum of the power series

$$c_1 + c_2(z - z_0) + c_3(z - z_0)^2 + \cdots + c_n(z - z_0)^{n-1} + \cdots .$$

Since it has a power series expansion on D, g is analytic on D. Also, g is analytic in $U \setminus \{z_0\}$ since f is analytic, and $z - z_0$ is analytic and non-vanishing on this set.

A function, defined on the union of two open sets and analytic on each of them, is clearly also analytic on their union; and so g is analytic on U.

Cauchy's Estimates. We now know that a function which is analytic in an open disc $D_r(z_0)$ has a power series expansion about z_0 which converges in that disc. We also have integral formulas (3.2.4) for the coefficients of this power series. However, we also have the formulas

$$c_n = \frac{f^{(n)}(z_0)}{n!} \tag{3.2.6}$$

for these coefficients from Corollary 3.2.4. Combining these yields integral formulas for the derivatives of an analytic function.

Theorem 3.2.8. *Let f be analytic in an open set containing the closed disc $\overline{D}_R(z_0)$. Then,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{|w-z_0|=R} \frac{f(w)}{(w-z_0)^{n+1}}\,dw \tag{3.2.7}$$

for $n = 0, 1, 2, \cdots$.

Proof. We just need to observe that if $\overline{D}_R(z_0) \subset U$, with U open, then there is an open disc $D_r(z_0)$ with $\overline{D}_R(z_0) \subset D_r(z_0) \subset U$ (Exercise 3.2.4). We can then apply Theorem 3.2.5 with $s = R$ to obtain a power series expansion of f with coefficients given by (3.2.4). Then Corollary 3.2.3 relates these coefficients to the derivatives of f at z_0. □

This leads to a very powerful tool. By estimating the size of the integrands in this formula, we can get estimates on the size of the derivatives of f. These estimates are called *Cauchy's estimates.*

Theorem 3.2.9 (Cauchy's Estimates)**.** *If f is analytic on an open set containing the closed disc $\overline{D}_R(z_0)$, and if $|f(z)| \le M$ on the boundary of this disc, then*

$$|f^{(n)}(z_0)| \le \frac{n!M}{R^n} \tag{3.2.8}$$

for $n = 0, 1, 2, \cdots$.

Proof. We use the previous theorem. Since $|w - z_0| = R$ and $|f(w)| \le M$ for w on the path $|w - z_0| = R$, the integrand of (3.2.7) is bounded by M/R^{n+1}. The length of the path is the circumference of a circle of radius R and so it is $2\pi R$. Thus, Theorem 2.4.9 implies that

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\frac{M}{R^{n+1}}2\pi R = \frac{n!M}{R^n}$$

for each non-negative integer n. □

This theorem will provide the crucial step in the proof of Liouville's Theorem in the next section.

Example 3.2.10. Find upper bounds on the derivatives at 0 of a function f which is analytic on the unit disc $D_1(0)$ and has modulus bounded by one on this disc. Also find bounds on the moduli of coefficients in the power series expansion of this function about 0.

Solution: We apply Cauchy's estimates. If $r < 1$, then f is analytic in the open set $D_1(0)$, which contains $\overline{D}_r(0)$. Since $|f(z)| \leq 1$ on $D_1(0)$, we may choose $M = 1$ and $R = r$ in Cauchy's estimates. We conclude

$$|f^{(n)}(0)| \leq \frac{n!}{r^n}.$$

However, since r was any positive number less than 1, we may pass to the limit as $r \to 1$ and conclude that

$$|f^{(n)}(0)| \leq n!.$$

By (3.2.6), the corresponding estimate on the power series coefficients is

$$|c_n| \leq 1.$$

Morera's Theorem. This is a very handy tool for showing that a function is analytic.

Theorem 3.2.11 (Morera's Theorem)**.** *Let f be a continuous function defined on an open set U. If the integral of f is 0 around the boundary of every triangle that is contained in U, then f is analytic in U.*

Proof. Theorem 2.6.1 says that a function f that is continuous on a convex open set U, and has the property that its integral around any triangle in U is 0, has a complex antiderivative g in U. However, the fact that $g' = f$ on U means, in particular, that g is analytic on U. But then f is analytic on U by Corollary 3.2.6.

The hypothesis that U is convex in the above argument is not necessary, since every open set is a union of convex open sets (open discs, in fact). Thus, we can apply the argument of the previous paragraph to each open disc contained in U and conclude that f is analytic on each of them. In particular, it has a derivative at each point of U and, hence, is analytic on U. □

An example of how Morera's Theorem is used is provided by the following theorem.

Theorem 3.2.12. *Let $\{f_n\}$ be a sequence of analytic functions on an open set U and suppose this sequence converges uniformly to f on each compact subset of U. Then f is analytic on U.*

Proof. Since $f_n \to f$ uniformly on each compact subset of U, f is continuous on U. The convergence is uniform, in particular, on $\partial\Delta$ for every triangle Δ contained in U. Given such a triangle Δ, Theorem 3.1.3 implies that

$$\int_{\partial\Delta} f(z)\,dz = \lim_{n\to\infty} \int_{\partial\Delta} f_n(z)\,dz.$$

Since each f_n is analytic, Theorem 2.5.8 implies

$$\int_{\partial\Delta} f_n(z)\,dz = 0$$

for each n. We conclude that

$$\int_{\partial\Delta} f(z)\,dz = 0,$$

for every triangle $\Delta \subset U$. By Morera's Theorem, f is analytic on U. □

Exercise Set 3.2

1. Use the power series expansion for $\dfrac{1}{1-z}$ about 0 to find the power series expansion of $\dfrac{1}{(1-z)^2}$ about 0.
2. Find a power series expansion of $\sqrt{1+z}$ about 0, where the square root function is defined in terms of the principal branch of the log function. What is the radius of convergence of this series?
3. Prove Corollary 3.2.3.
4. Prove that if $\overline{D}_R(z_0)$ is a closed disc contained in an open set U, then there is an open disc $D_r(z_0)$ such that $\overline{D}_R(z_0) \subset D_r(z_0) \subset U$.
5. Show that if f is analytic on an open set U, then, as a function on $\mathbb{R}^2$, it is $\mathcal{C}^\infty$ – that is, its partial derivatives of all orders exist and are continuous.
6. The function $\dfrac{1}{\cos z}$ has a power series expansion about $z_0 = 0$. Without finding the series, show that its radius of convergence is $\pi/2$.
7. Find the power series expansion of the principal branch of the log function about the point $z = i$. There are several ways to do this, one of which is really easy (see Example 3.2.2).
8. Use power series methods to show that the function which is $\dfrac{\sin z}{z}$ when $z \neq 0$ and 1 when $z = 0$ is analytic on the whole complex plane.
9. If f is analytic and not identically 0 on a disc $D_r(z_0)$, show that there is a non-negative integer k and a function g, which is also analytic in $D_r(z_0)$, such that $f(z) = (z-z_0)^k g(z)$ and $g(z_0) \neq 0$.
10. Prove that if f is analytic on the disc $D_R(z_0)$ and $|f(z)| \le M$ on $D_R(z_0)$, then $|f'(z_0)| \le M/R$.
11. Suppose $p(z) = a_3z^3 + a_2z^2 + a_1z + a_0$ is a polynomial of degree 3. If $|p(z)| \le 1$ on the unit circle $\{z : |z| = 1\}$, then show that $|a_3| \le 1$.
12. Suppose f is analytic in an open set U. Also, suppose $z \in U$ and the distance from z to the complement of U is d. If $|f(w)| \le M$ for all $w \in U$, find estimates, similar to Cauchy's estimates, on the size of $|f^{(n)}(z)|$ in terms of M and d.
13. Suppose f is analytic on a disc $D_r(z_0)$ and unbounded (there is no M such that $|f(z)| \le M$ on $D_r(z_0)$). Then prove that the radius of convergence of the power series expansion of f about z_0 is r.

14. Use Morera's Theorem to show that if f is continuous on an open set U and analytic on $U \setminus E$, where E is either a point or a line segment, then f is actually analytic on all of U.
15. Use Cauchy's estimates to prove that if $\{f_n\}$ is a sequence of analytic functions on an open set U, converging uniformly to f on each compact subset of U, then $\{f_n^{(k)}\}$ converges uniformly to $f^{(k)}$ on each compact subset of U.
16. Use Morera's Theorem to prove that if U is an open subset of $\mathbb{C}$, $I = [a, b]$ is an interval on the real line, and $g(z, t)$ is a continuous function on $U \times I$ which is analytic in z for each $t \in I$, then the function

$$f(z) = \int_a^b g(z, t)\, dt$$

is analytic in U.

3.3. Liouville's Theorem

Liouville's Theorem is simple to state, very easy to prove (given what we know at this point), and extremely powerful. It concerns entire functions, where an *entire function* is a function which is analytic in the entire complex plane. It also concerns bounded functions, where a function is bounded on a set E if there is a positive constant M such that $|f(z)| \leq M$ for every $z \in E$. If f is bounded on its domain, we simply say it is *bounded*. Thus, a bounded entire function is a function which is analytic and bounded on $\mathbb{C}$.

Theorem 3.3.1 (Liouville's Theorem)**.** *The only bounded entire functions are the constant functions.*

Proof. The reader who did Exercise 3.2.10 of the preceding section has nearly completed the proof of Liouville's Theorem. The exercise states a simple consequence of the Cauchy estimates: If a function f is analytic on a disc of radius R, centered at z_0, and if $|f(z)| \leq M$ for all z in this disc, then

$$|f'(z_0)| \leq \frac{M}{R}. \tag{3.3.1}$$

If f is bounded and entire, then $f(z)$ is analytic on the entire plane and $|f(z)|$ is bounded by some number M on the entire plane. This implies that (3.3.1) holds for all positive numbers R and all $z_0 \in \mathbb{C}$. If we take the limit as $R \to \infty$ in (3.3.1), we conclude that $|f'(z_0)| = 0$ for all $z_0 \in \mathbb{C}$. In other words, the derivative of f is identically zero. This implies f is a constant (see Exercise 2.6.1). □

One has to see the consequences of this theorem to appreciate its power. In the remainder of this section and the exercises we will introduce a number of these. Others will occur later.

The Fundamental Theorem of Algebra. At the very beginning of the text, we promised that we would prove that every non-constant polynomial with complex coefficients has a complex root. This is the *Fundamental Theorem of Algebra.* Before proving this theorem, it will be convenient to introduce limits at infinity.

Definition 3.3.2. If f is a function defined on an unbounded set E (so E contains points of arbitrarily large modulus) and $L \in \mathbb{C}$, then we say

$$\lim_{z \to \infty} f(z) = L$$

if, for every $\epsilon > 0$, there is an $R > 0$ such that $|f(z) - L| < \epsilon$ whenever $|z| > R$ and $z \in E$.

This concept satisfies the same basic rules as other kinds of limits: limit of the sum is sum of the limits, limit of the product is product of the limits, limit of the quotient is quotient of the limits if the denominator does not have limit zero, etc. These facts, as well as the fact that

$$\lim_{z \to \infty} \frac{1}{z} = 0$$

(Exercise 3.3.1), are used in the proof of the Fundamental Theorem of Algebra. Before proving that theorem, we prove the following simple result.

Theorem 3.3.3. *If a function f is defined and continuous on the entire plane and if $\lim_{z \to \infty} f(z)$ exists, then f is bounded on $\mathbb{C}$.*

Proof. If $\lim_{z \to \infty} f(z) = L$, there exists an $R > 0$ such that $|f(z) - L| < 1$ whenever $|z| > R$. By the triangle inequality, this implies

$$|f(z)| < |L| + 1 \quad \text{if} \quad |z| > R.$$

Since f is continuous on $\mathbb{C}$ and $\overline{D}_R(0)$ is closed and bounded, hence compact, f is bounded on $\overline{D}_R(0)$. Since f is bounded on $\overline{D}_R(0)$ and on its exterior, it is bounded on all of $\mathbb{C}$. □

Theorem 3.3.4 (Fundamental Theorem of Algebra). *Every non-constant complex polynomial has a complex root.*

Proof. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a non-constant complex polynomial of degree n. Then $n \geq 1$ and $a_n \neq 0$. We will show the assumption that p has no root leads to a contradiction.

If p has no root, then $p(z) \neq 0$ for every $z \in \mathbb{C}$. This implies that $1/p$ is an entire function. We will show that it is also bounded.

If we define a function $h(z)$ by

$$h(z) = \frac{z^n}{p(z)} = \frac{1}{a_n + a_{n-1} z^{-1} + \cdots + a_1 z^{n-1} + a_0 z^{-n}},$$

then

$$\frac{1}{p(z)} = \frac{h(z)}{z^n} \tag{3.3.2}$$

for $z \neq 0$. Furthermore

$$\lim_{z \to \infty} h(z) = 1/a_n$$

and so

$$\lim_{z\to\infty} \frac{1}{p(z)} = \lim_{z\to\infty} \frac{h(z)}{z^n} = 0.$$

Since the limit of $1/p$ exists at infinity, the previous theorem implies that $1/p$ is bounded on all of $\mathbb{C}$. So Liouville's Theorem implies that it is a constant. In fact, the constant must be zero, since $1/p(z)$ has limit 0 at ∞. This is clearly a contradiction, since $1/p(z)$ cannot take on the value 0 on $\mathbb{C}$. We conclude that $p(z)$ must have a root. □

The Fundamental Theorem of Algebra has a number of important consequences. We will discuss only a few of them.

Factoring Polynomials. A polynomial is said to factor completely if it can be written as a product

$$p(z) = b(z - z_1)(z - z_2)\cdots(z - z_n)$$

of linear factors. Here, n is the degree of p and the numbers $z_1, z_2, \cdots, z_n$ are the roots of p. The roots need not all be distict. If a root occurs k times in this factorization, it is said to be a root of *mutiplicity k*. Necessarily, the number of roots is the degree of the polynomial if each root is counted as many times as its multiplicity.

Corollary 3.3.5. *Each complex polynomial factors completely.*

Proof. The proof is by induction on the degree n of the polynomial. Obviously a polynomial of degree 0 or 1 factors completely. If every polynomial of degee n factors completely and p is a polynomial of degree $n+1$, then we use the previous theorem to assert that p has a root – call it z_{n+1}. It follows that p factors as

$$P(z) = (z - z_{n+1})q(z),$$

where q is a polynomial of degree n. By assumption, q factors as

$$q(z) = b(z - z_1)(z - z_2)\cdots(z - z_n).$$

Then p factors as

$$p(z) = b(z - z_1)(z - z_2)\cdots(z - z_n)(z - z_{n+1}).$$

This completes the induction step and finishes the proof of the corollary. □

Eigenvalues of Matrices. An eigenvalue for an $n \times n$ matrix A is a complex number λ such that the matrix $\lambda I - A$ is singular (has no inverse). Here, I is the $n \times n$ identity matrix.

Corollary 3.3.6. *An $n \times n$ matrix A has at least one complex eigenvalue.*

Proof. The determinant $\det(\lambda I - A)$ of $\lambda I - A$ is a polynomial in λ of degree n. This is the *characteristic polynomial* of A. By Kramer's Rule, $\lambda I - A$ is singular for a given λ if and only if $\det(\lambda I - A) = 0$ – that is, if and only if λ is a root of the charcteristic polynomial. By the Fundamental Theorem of Algebra, the characteristic polynomial has a root. Thus, A has an eigenvalue. □

The above result is the essential ingredient in the proof that every complex matrix can be put in upper triangular form (Exercise 3.3.14).

Example 3.3.7. Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$.

Solution: The characteristic polynomial of this matrix is

$$\det \begin{pmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 3 \end{pmatrix} = \lambda^2 - 4\lambda + 5.$$

By the Quadratic Formula, the roots of this polynomial are $2 \pm 2i$. Thus, the eigenvalues of the above matrix are $2 + 2i$ and $2 - 2i$.

Differential Equations. The Fundamental Theorem of Algebra also has applications to Differential Equations. A homogeneous constant-coefficient differential equation is an equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{3.3.3}$$

Example 3.3.8. Prove that each homogeneous constant-coefficient differential equation has a solution of the form $y = \mathrm{e}^{\lambda x}$, where λ is a complex number. Which numbers λ yield solutions?

Solution: We simply plug $y = \mathrm{e}^{\lambda x}$ into the equation (3.3.3) and see if there are values of λ that yield solutions. Since $(\mathrm{e}^{\lambda x})' = \lambda\, \mathrm{e}^{\lambda x}$, the result is

$$(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0)\, \mathrm{e}^{\lambda x} = 0.$$

The polynomial in parentheses is called the *auxiliary polynomial* for (3.3.3). Clearly $\mathrm{e}^{\lambda x}$ is a solution of (3.3.3) if and only if λ is a root of this polynomial. Since every non-constant polynomial has a root, equation (3.3.3) has a solution of the form $y = \mathrm{e}^{\lambda x}$.

If the auxiliary polynomial has distinct roots $\lambda_1, \cdots, \lambda_n$, then the general solution to equation (3.3.3) is a linear combination of the solutions $\mathrm{e}^{\lambda_j x}$.

If the coefficients of (3.3.3) are real numbers and we are looking only for real solutions to this differential equation, then we exploit the fact that the non-real roots of a polynomial with real coefficients occur in conjugate pairs (Exercise 3.3.10). If $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$ are roots of the auxilliary equation for (3.3.3), then

$$\mathrm{e}^{\alpha x} \cos \beta x = \frac{\mathrm{e}^{\lambda x} + \mathrm{e}^{\overline{\lambda} x}}{2} \quad \text{and}$$
$$\mathrm{e}^{\alpha x} \sin \beta x = \frac{\mathrm{e}^{\lambda x} - \mathrm{e}^{\overline{\lambda} x}}{2i}$$

both give real solutions to (3.3.3), as does any linear combination

$$C\, \mathrm{e}^{\alpha x} \cos \beta x + D\, \mathrm{e}^{\alpha x} \sin \beta x = \mathrm{e}^{\alpha x} (C \cos \beta x + D \sin \beta x),$$

where C and D are arbitrary real constants.

Example 3.3.9. Find all solutions to the differential equation $y'' - 2y' + 5y = 0$. Then find all real solutions.

Solution: The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0$, which has solutions $\lambda = 1 \pm 2i$. Thus, the general solution to the differential equation is

$$y = A\,\mathrm{e}^{(1+2i)x} + B\,\mathrm{e}^{(1-2i)x},$$

where A and B are complex constants. The general real solution is

$$y = \mathrm{e}^{x}(C\cos 2x + D\sin 2x),$$

where C and D are real constants.

Characterization of Polynomials. In the proof of Liouville's Theorem, we only use Cauchy's estimate on the first derivative of an analytic function on a disc. Here we present a generalization of Liouville's Theorem that uses Cauchy's estimates on higher derivatives.

Theorem 3.3.10. *An entire function f is a polynomial of degree at most n if and only if there are positive constants A and B such that*

$$|f(z)| \le A + B|z|^n \tag{3.3.4}$$

for all $z \in \mathbb{C}$.

Proof. We will prove that every polynomial satisfies an inequality of the form (3.3.4). We leave the converse as an exercise in the application of Cauchy's estimates.

Suppose $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree at most n. Then

$$\lim_{z\to\infty} \frac{p(z)}{z^n} = a_n.$$

If we apply the definition of limit, using $\epsilon = 1$, we conclude that there is an $R > 0$ such that $|z| > R$ implies

$$\left|\frac{p(z)}{z^n} - a_n\right| < 1.$$

It follows from the triangle inequality that

$$\frac{|p(z)|}{|z|^n} < |a_n| + 1 \quad \text{if} \quad |z| > R.$$

If we set $B = |a_n| + 1$, then

$$|p(z)| < B|z|^n \quad \text{if} \quad |z| > R.$$

This gives a bound on $|p|$ on the complement of the closed disc $\overline{D}_R(0)$. A closed disc of finite radius is a compact set and so p is bounded on $\overline{D}_R(0)$ – say, $|p(z)| \le A$ on this disc. If we combine this with our bound on the complement of $\overline{D}_R(0)$, we conclude that

$$|p(z)| \le A + B|z|^n$$

on all of $\mathbb{C}$, as required. □

Exercise Set 3.3

1. Prove that $\lim_{z\to\infty} \dfrac{1}{z} = 0$.
2. We say $\lim_{z\to\infty} f(z) = \infty$ if for each $K > 0$, there is an $M > 0$ such that $|f(z)| > K$ whenever $|z| > M$. Prove that $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to\infty} 1/f(z) = 0$.
3. Show that if f is an entire function and $\lim_{z\to\infty} f(z) = \infty$, then f must have a zero somewhere in $\mathbb{C}$. Hint: See the previous exercise.
4. Prove that if f is an entire function which satisfies $|f(z)| \geq 1$ on the entire plane, then f is constant.
5. Prove that if an entire function has real part which is bounded above, then the function is constant.
6. Prove that if an entire function f is not constant, then its range $f(\mathbb{C})$ is dense in $\mathbb{C}$ – meaning that the closure of $f(\mathbb{C})$ is $\mathbb{C}$. Hint: If $f(\mathbb{C})$ is not dense, then there is a point z_0 and a disc $D_r(z_0)$, centered at z_0, such that $f(\mathbb{C}) \cap D_r(z_0) = \emptyset$.
7. Show that if f is an entire function and $|f(z)| \leq K|z|$ for all $z \in \mathbb{C}$, where K is a positive constant, then $f(z) = Cz$ for some constant C.
8. Show that if f is an entire function and $|f(z)| \leq K|\,e^z\,|$ for all $z \in \mathbb{C}$, where K is a positive constant, then $f(z) = C\,e^z$ for some constant C.
9. Finish the proof of Theorem 3.3.10 by using Cauchy's estimates to prove that the only entire functions f that satisfy an inequality of the form $|f(z)| \leq A + B|z|^n$ for all $z \in \mathbb{C}$ are polynomials of degree at most n.
10. Show that if p is a polynomial with real coefficients, then the non-real roots of p occur in conjugate pairs. That is, show that if w is a root, then so is $\overline{w}$.
11. Find the roots of the polynomial $p(z) = z^2 - 2z + 2$.
12. Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.
13. Find all solutions of the differential equation $y'' - 4y' + 13y = 0$. Then find all real solutions.
14. Use Corollary 3.3.6 to prove that if A is an $n \times n$ matrix, then A is conjugate to an upper triangular matrix.
15. Use the result of Exercise 10 to prove that every polynomial with real coefficients factors as a product of polynomials of degree at most 2, also with real coefficients.
16. Are there any non-constant entire functions f that satisfy an inequality of the form

$$|f(z)| \leq A + B\log|z| \quad \text{for all } z \text{ with } |z| \geq 1,$$

where A and B are positive constants?

3.4. Zeroes and Singularities

The existence of power series expansions leads to a great deal of information about the local structure of analytic functions. For example, Exercise 3.2.9 makes an assertion that is important enough to be expanded and restated as a theorem.

Theorem 3.4.1. *If f is a function which is analytic in a open set U, then for each $z_0 \in U$, exactly one of the following statements is true:*

(1) *there is an open disc, centered at z_0, on which f is identically 0;*

(2) *there is a non-negative integer k, an open disc $D_r(z_0)$, and a function g, analytic on U, such that*

$$f(z) = (z - z_0)^k g(z) \quad \textit{for all} \quad z \in D_r(z_0) \tag{3.4.1}$$

and $g(z)$ is non-zero at each point of $D_r(z_0)$.

Proof. The function f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{3.4.2}$$

which converges in a disc $D_R(z_0)$ of positive radius R. Some of the coefficients c_n may be zero. If they are all zero, then f is identically zero on $D_R(z_0)$ and so (1) holds in this case.

If the coefficients c_n are not all zero, there is a smallest n for which $c_n \neq 0$ – call it k. Then (3.4.2) becomes

$$f(z) = \sum_{n=k}^{\infty} c_n (z - z_0)^n = (z - z_0)^k \sum_{n=0}^{\infty} c_{n+k} (z - z_0)^n, \tag{3.4.3}$$

where $c_k \neq 0$. We may then define g by

$$g(z) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} c_{n+k}(z - z_0)^n, & \text{if } z \in D_R(z_0); \\ \dfrac{f(z)}{(z - z_0)^k}, & \text{if } z \in U \setminus \{z_0\}. \end{cases}$$

This is a consistent definition, since on $D_R(z_0) \setminus \{z_0\}$ the functions $f(z)/(z - z_0)^k$ and $\sum_{n=0}^{\infty} c_{n+k}(z - z_0)^n$ are equal.

Since g is continuous and $g(z_0) = c_k \neq 0$, there is a positive $r < R$ such that $|g(z) - g(z_0)| < |c_k|$ whenever $|z - z_0| < r$. This implies $g(z) \neq 0$ if $|z - z_0| < r$. In other words, $g(z) \neq 0$ for every $z \in D_r(z_0)$. In this case, (2) holds. □

Zeroes of Analytic Functions. By a *zero* of a function, we mean a point in its domain at which it vanishes (has value 0). The previous theorem leads to an important and somewhat surprising result about the zeroes of an analytic function f on a connected open set: Unless f is identically zero, each zero of f has no neighboring zeroes. This is made precise in Part (b) of the following theorem.

Theorem 3.4.2. *Let f be a function which is analytic on a connected open set U and is not identically zero. Then*

(a) *for each* $z_0 \in U$*, there is a non-negative integer* k*, an* $r > 0$*, and a function* g*, analytic in* U*, such that*

$$f(z) = (z - z_0)^k g(z) \quad \text{for all} \quad z \in U,$$

and $g(z)$ *has no zeroes on* $D_r(z_0)$*;*

(b) *each* $z_0 \in U$ *is the center of an open disc* $D_r(z_0) \subset U$ *in which there are no zeroes of* f *except possibly* z_0 *itself;*

(c) *the set of zeroes of* f *is at most countable.*

Proof. Theorem 3.4.1 shows that, at each point z_0 of U, there are two possibilities: (1) there is a disc centered at z_0 in which f is identically 0, and (2) there is a disc centered at z_0 in which f has no zeroes except possibly at z_0 itself. Let V_j, $j = 1, 2$ be the set of points z_0 for which the jth possibility is the one which occurs. Obviously, V_1 and V_2 are both open sets, $V_1 \cap V_2 = \emptyset$, and $V_1 \cup V_2 = U$. Thus, either V_1 or V_2 is empty, since, otherwise, they would separate the connected set U. If $V_2 = \emptyset$, then f is identically 0 on U. Since f is not identically 0, we conclude that $V_1 = \emptyset$. Since the second of the possibilities in Theorem 3.4.1 is the only one that occurs in this situation, we conclude that (a) and (b) above both hold at every $z_0 \in U$.

We can modify the disc $D_r(z_0)$ in statement (a) of the theorem so that it has a center (which may no longer be z_0) with rational coordinates and a rational radius and still has the property that it contains z_0 and has no zeroes other than possibly z_0. We simply choose a point z_0' with rational coordinates, and a positive rational number ρ such that $|z_0 - z_0'| < \rho < r/2$. Then $z_0 \in D_\rho(z_0') \subset D_r(z_0)$ and so $D_\rho(z_0')$ contains z_0 but no other zeroes of f. Since we may do this for each zero of f in U and since there are only countably many discs with rational centers and rational radii, we conclude that f can have only countably many zeroes in U. □

Recall that an *isolated point* of a set E is a point which is contained in an open disc which contains no other points of E. Thus, if $Z(f)$ denotes the set of zeroes of an analytic function f on its domain U, then Part (b) of the above theorem implies that each point of $Z(f)$ is an isolated point of $Z(f)$. Actually it says something much stronger. It says that every point z_0 of U (whether in $Z(f)$ or not) is the center of a disc containing no points of $Z(f)$ other than z_0. There is a term to describe subsets with this property:

Definition 3.4.3. Let U be an open subset of $\mathbb{C}$ and E a subset of U. We say that E is a *discrete subset* of U if every point z_0 of U has a neighborhood which contains no points of E other than possibly z_0 itself.

Thus, Part (b) of the above theorem says that the zero set $Z(f)$ of a non-constant analytic function on a connected open set U is a discrete subset of U.

It turns out that a subset of an open set U is discrete if and only if no sequence of distinct points of E converges to a point of U (see Exercise 3.4.1). Of course, there may be sequences in E which converge to points not in U.

Theorem 3.4.2 has the following easy but important consequence. We leave the proof as an exercise (Exercise 3.4.2).

Theorem 3.4.4 (Identity Theorem)**.** *Suppose f and g are two analytic functions with domain a connected open set U. If $f(w) = g(w)$ at each point w of a non-discrete subset E of U, then $f(z) = g(z)$ at every point of U.*

If the f of Theorem 3.4.2 actually has a zero at z_0, then the integer k is positive. We call it the *order* of the zero of f at z_0.

Example 3.4.5. What is the order of the zero of the function $f(z) = \cos z - 1$ at 0? What is the function g of Part (a) of Theorem 3.4.2 in this case if $z_0 = 0$?

Solution: If we subtract 1 from the power series expansion of $\cos z$ about 0, we obtain

$$\begin{aligned}\cos z - 1 &= -\frac{z^2}{2!} + \frac{z^4}{4!} + \cdots (-1)^n \frac{z^{2n}}{2n!} + \cdots \\ &= z^2 \left(-\frac{1}{2!} + \frac{z^2}{4!} + \cdots (-1)^n \frac{z^{2n-2}}{2n!} + \cdots \right).\end{aligned}$$

We conclude that the order of the zero of $\cos z - 1$ at 0 is 2 and the function g of Part (a) of Theorem 3.4.2 is the function given by the power series in parentheses above. This power series has infinite radius of convergence by the ratio test.

Theorem 3.4.6. *If an analytic function g is not zero at a point z_0 in its domain, then in some neighborhood V of z_0 there is an analytic function h such that $g(z) = \mathrm{e}^{h(z)}$ in V.*

Proof. To find such an h, we simply choose a branch of the log function that does not have $g(z_0)$ on its cut line. Then the set on which this branch of the log function is analytic is an open set W which contains $g(z_0)$. If we set $V = g^{-1}(W)$ and define h on V by

$$h(z) = \log(g(z)),$$

then V is a neighborhood of z_0 and $g(z) = \mathrm{e}^{h(z)}$ on V. □

If we combine this result with Theorem 3.4.2, we have:

Theorem 3.4.7. *Let f be an analytic function on an open set U and let z_0 be a point of U such that f is not identically zero in a neighborhood of z_0. Then there exist a non-negative integer k, a neighborhood $V \subset U$ of z_0, and an analytic function h on V, such that*

$$f(z) = (z - z_0)^k \, \mathrm{e}^{h(z)}$$

for all $z \in V$.

Isolated Singularities. If U is an open set, $z_0 \in U$, and f is a function which is analytic on $U \setminus \{z_0\}$, then f is said to have an *isolated singularity* at z_0. If f can be given a value at z_0 which makes it analytic everywhere on U, then the singularity is said to be *removable*.

Theorem 3.4.8. *If f has an isolated singularity at z_0 and is bounded in some deleted neighborhood of z_0, then z_0 is a removable singularity of f.*

Proof. Suppose f is analytic and bounded on $U \setminus \{z_0\}$. If we define a function g by $g(z) = (z - z_0)^2 f(z)$ for $z \neq z_0$ and $g(z_0) = 0$, then g is differentiable at z_0. In fact,

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0.$$

That this limit is 0 is proved as follows. Let M be a bound on $|f(z)|$ on $U \setminus \{z_0\}$. Then $|(z - z_0) f(z)| \leq M|z - z_0|$ on U. Since $\lim_{z \to z_0} M|z - z_0| = 0$, it follows that $\lim_{z \to z_0} (z - z_0) f(z) = 0$ as well.

The function g is differentiable at every point of $U \setminus \{z_0\}$ and, by the above, is also differentiable at z_0. Thus, it is analytic on all of U. Since $g(z_0) = g'(z_0) = 0$, the first two terms of its power series expansion about z_0 are 0. It follows that we may factor $(z - z_0)^2$ out of every term of its power series expansion and write $g(z) = (z - z_0)^2 h(z)$ for an analytic function h defined by a power series in a disc centered at z_0. Clearly h and f are the same in this disc, except at z_0, where f is not defined. Thus, setting $f(z_0) = h(z_0)$ serves to define f at z_0 in such a way that it becomes analytic on all of U. □

Example 3.4.9. Show that $f(z) = \dfrac{\cos z - 1}{z^2}$ has a removable singularity at 0.

Solution: This follows immediately from the factorization of $\cos z - 1$ obtained in Example 3.4.5. If f is given the value $-1/2$ at $z = 0$, it becomes analytic on the entire plane.

There are two types of isolated singularities that are not removable. A function f defined on $U \setminus \{z_0\}$ of the form

$$f(z) = \frac{g(z)}{(z - z_0)^k},$$

where g is analytic on U, $g(z_0) \neq 0$, and k is a positive integer, is said to have a *pole of order k*. If the order of the pole is 1, then it is called a *simple pole*. An isolated singularity which is not a pole and is not a removable singularity is called an *essential singularity*.

Example 3.4.10. If U is an open set and $z_0 \in U$, then analyze the singularities of a function of the form f/g, where f and g are analytic on U.

Solution: The singularities of f/g in U are all isolated because the zeroes of g are isolated. If we factor f and g as in Theorem 3.4.2, then

$$f(z) = (z - z_0)^j p(z) \quad \text{and} \quad g(z) = (z - z_0)^k h(z),$$

where j and k are the orders of the zeroes of f and g at z_0 and p and h are analytic in U and non-vanishing in some neighborhood of z_0. Then $f(z)/g(z) = (z - z_0)^{j-k} p(z)/h(z)$ with $p(z)/h(z)$ analytic and non-vanishing in a neighborhood of z_0. The point z_0 is a removable singularity for f/g if $j \geq k$ and, otherwise, is a pole of order $k - j$.

Example 3.4.11. Analyze the singularities of the function $f(z) = \dfrac{1}{1 - e^z}$.

Solution: The denominator of this fraction has zeroes at the points $\{2\pi k i\}$ for k an integer. Each of these is a zero of order 1 because the derivative of $1 - e^z$

is $-\mathrm{e}^z$ and this is non-zero for every z and, in particular, is non-zero at the points $\{2\pi ki\}$. It follows from the preceding example that each of these points is a simple pole for f.

Essential singularities are quite wild. In fact, the Big Picard Theorem states that a function with an essential singularity at z_0 takes on every complex number but one as a value in every open disc centered at z_0. We will prove the Big Picard Theorem and its little brother – the Little Picard Theorem – in Chapter 7. The following is a very much weaker statement than the Big Picard Theorem, but it is still enough to show that an analytic function behaves very wildly near an essential singularity.

Theorem 3.4.12. *If f is analytic in $U\setminus\{z_0\}$ and has an essential singularity at z_0, then for every open disc D, centered at z_0 and contained in U, the set $f(D\setminus\{z_0\})$ has closure equal to the entire complex plane.*

Proof. Suppose there is a disc D, centered at z_0 and contained in U and there is some complex number w which is not in the closure of $f(D\setminus\{0\})$. Then there is an $r>0$ such that $f(D\setminus\{z_0\})\cap D_r(w)=\emptyset$. This means $|f(z)-w|\geq r$ for every $z\in D\setminus\{z_0\}$. Then

$$\left|\frac{1}{f(z)-w}\right|\leq\frac{1}{r}\quad\text{for all}\quad z\in D\setminus\{z_0\}.$$

Thus, $1/(f(z)-w)$ is analytic and bounded on $D\setminus\{z_0\}$ and, hence, has a removable singularity at z_0. In other words, there is an analytic function h on D such that

$$\frac{1}{f(z)-w}=h(z)$$

on $D\setminus\{z_0\}$. If h has no zero at z_0, set $k=0$; otherwise, let k be the order of the zero of h at 0. Then

$$h(z)=(z-z_0)^k g(z)$$

for some analytic function g on D which does not have a zero at z_0. Solving for f, we find

$$f(z)=w+\frac{1}{h(z)}=\frac{w(z-z_0)^k+1/g(z)}{(z-z_0)^k}$$

in some disc containing z_0 where g is non-vanishing. Since the numerator is analytic in this disc, f has a pole of order k at z_0 (or a removable singularity if $k=0$). Since neither of these things is true, we conclude that there is no w which fails to be in the closure of $f(D)$. This completes the proof. □

The following theorem gives an easy way to identify whether a given isolated singularity is removable, a pole, or essential.

Theorem 3.4.13. *Let f be an analytic function with an isolated singularity at z_0. Then*

(a) *f has a removable singularity at z_0 if and only if $\lim_{z\to z_0} f(z)$ exists and is finite;*
(b) *f has a pole at z_0 if and only if $\lim_{z\to z_0} f(z)=\infty$;*
(c) *f has an essential singularity at z_0 if and only if $\lim_{z\to z_0} f(z)$ does not exist, even as an infinite limit.*

Proof. The function f has a removable singularity at z_0 if and only if it can be given a value w at z_0 which makes it analytic in a neighborhood of z_0. In this case, $\lim_{z\to z_0} f(z) = w$, since f with its new value at z_0 must be continuous at z_0.

On the other hand, if f has a finite limit as $z \to z_0$, then f is bounded in a neighborhood of z_0 and, by Theorem 3.4.8, the singularity at z_0 is removable. This proves (a).

If f has a pole at z_0, then $f(z) = \dfrac{g(z)}{(x - z_0)^k}$ for some $k > 0$ and some g analytic in a neighborhood of z_0 with $g(z_0) \neq 0$. Then $\lim_{z\to z_0} g(z) = g(z_0) \neq 0$ and $\lim_{z\to z_0}(z - z_0)^k = 0$. It follows that $\lim_{z\to z_0} f(z) = \infty$.

On the other hand, if $\lim_{z\to z_0} f(z) = \infty$ the singularity is not removable since f does not have a finite limit as $z \to z_0$. It cannot be essential either, since the previous theorem says that, if it were essential, f would take on values arbitrarily close to any given complex number in every open disc centered at z_0. Thus, f could not have limit ∞ at z_0 if the singularity were essential. This proves (b).

If $\lim_{z\to z_0} f(z)$ does not exist, even as a finite limit, then by (a) and (b) f does not have a removable singularity or a pole. Thus, it has an essential singularity. Conversely, if f has an essential singularity at z_0, then it does not have a removable singularity or a pole. Then, also by (a) and (b), f does not have a finite limit at z_0 nor does it have limit ∞. This proves (c). □

Example 3.4.14. Analyze the singularity of the function $f(z) = e^{1/z}$ at $z = 0$.

Solution: This is an essential singularity. The function $f(z) = e^{1/z}$ takes on the value 1 at all points of the form $z = (2\pi n)^{-1}$ and the value e^n at all points of the form $z = n^{-1}$. Thus, $f(z)$ approaches 1 as z approaches 0 along one sequence of points and it approaches ∞ as z approaches 0 along another sequence of points. Thus, f certainly does not have a limit, finite or infinite, as $z \to 0$.

Meromorphic Functions. For many reasons, it is important to study functions which are analytic on an open set U except on a subset of U consisting of points where poles occur. Necessarily, such a subset is a discrete subset of U.

Definition 3.4.15. Let U be an open set and E a discrete subset of U. If f is a function which is analytic on $U \setminus E$ and has a removable singularity or a pole at each point of E, then f is called a meromorphic function on U.

One reason the set of meromorphic functions on U is interesting is that it is a field, if U is a connected open set. Obviously, we can add and multiply meromorphic functions and the results are still meromorphic. Just as obviously, the appropriate commutative, associative and distributive laws hold and there are zero and identity elements and additive inverses. All of these things are also true of the class of analytic functions on U. However, in the class of meromorphic functions we also have the last field axiom satisfied: every non-zero element has a multiplicative inverse.

Theorem 3.4.16. *If U is a connected open set and f is a meromorphic function on U which is not identically zero, then $1/f$ is also meromorphic.*

Proof. By Theorem 3.4.2 the set $Z(f)$ of zeroes of f is a discrete subset of U. By definition, the set $P(f)$ of poles of f is a discrete subset of U. It follows that the set $E = Z(f) \cup P(f)$ is also a discrete subset of U. The function $1/f$ is analytic on $U \setminus E$ and so the only thing we need to establish is that at points of E it has only removable singularities or poles.

Let z_0 be a point of E. Since, E is discrete, there is a disc D, centered at z_0 in which z_0 is the only point of E. If $z_0 \in E$, then f is analytic everwhere in D except at z_0 where is has either a zero or a pole.

If f has a zero of order k at z_0, then it factors as $(z - z_0)^k g(z)$ where g is analytic and non-vanishing in D. Obviously then $1/f(z) = (z - z_0)^{-k}/g(z)$ has a pole of order k at z_0.

If f has a pole of order k at z_0, then f factors as $f(z) = (z - z_0)^{-k}h(z)$ where h is analytic and non-vanishing in D. Then $1/f(z) = (z - z_0)^k/h(z)$ has a zero of order k at z_0 and, hence, a removable singularity at z_0. □

Exercise Set 3.4

1. Prove that a set E is a discrete subset of an open set U if and only if no sequence of distinct points of E converges to a point in U.
2. Prove Theorem 3.4.4.
3. Is there a function, analytic on $\mathbb{C}$, which is 0 on the set of points $\{1/n\}$ for n a positive integer and not identically 0? Justify your answer. What if the function is only required to be analytic on $\mathbb{C} \setminus \{0\}$?
4. If $f(z) = \sin z - z$, find the order of the zero of f at 0. Then give the factorization of f at 0, as in Theorem 3.4.1.
5. If $f(z) = \cos z - 1 + z^2/2$, find the order of the zero and the factorization at 0 as in the previous exercise.
6. Give an example to show that Theorem 3.4.2 no longer holds if we drop the hypothesis that U is connected.
7. Prove that if f is an analytic function on a connected open set U and if K is a compact subset of U, then there can be at most finitely many zeroes of f in K.
8. Show that if f is an analytic function with a zero of order k at z_0, then there is a neighborhood V of z_0 and an analytic function g on V such that $f = g^k$ on V and $g'(z_0) \neq 0$.
9. Suppose f and g are analytic functions on an open set U, $z_0 \in U$, and $f(z_0) = g(z_0) = 0$. Show that
$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$
and that this limit exists if ∞ is allowed as a possible value.
10. Suppose U is a connected open set and $z_0 \in U$. Prove that if f is a non-constant analytic function on $U \setminus \{z_0\}$ and f takes on a certain value c at least once in every deleted neighborhood of z_0, then f has an essential singularity at z_0.

11. Prove that if f is a function which is analytic in the exterior of the closed disc $\overline{D}_r(0)$ and if $\lim_{z\to\infty} f(z) = 0$, then f has a power series expansion of the form
$$f(z) = \sum_{n=1}^{\infty} a_n z^{-n},$$
which converges on the set $\{z \in \mathbb{C} : |z| > r\}$. Hint: Consider the function $g(z) = f(1/z)$.

In the next five problems, analyze each singularity z_0 of f. Is it removable, a pole, or essential? If it is a pole, what is its order? If it is removable, what value should you give the function at z_0 to make it analytic?

12. $f(z) = \dfrac{1}{z - z^3}$.
13. $f(z) = \sin 1/z$.
14. $f(z) = \dfrac{e^z - 1 - z}{z^2}$.
15. $f(z) = \dfrac{1}{e^z - 1} - \dfrac{1}{z}$.
16. $f(z) = \dfrac{\log z}{(1-z)^2}$, where log is the principal branch of the log function.

3.5. The Maximum Modulus Principle

This is another important application of Cauchy's theorems. Before stating it, we state and prove a technical lemma that will be used in its proof. The lemma states that if the average value of a continuous function on an interval is as large, in modulus, as all values of the function on the interval, then the function must be constant.

Lemma 3.5.1. *Let f be a continuous complex-valued function defined on an interval $I = [a, b]$ on the real line. If $|f(t)| \le M$ on I, where*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt \right| = M, \tag{3.5.1}$$

then f is a constant of modulus M on I.

Proof. If we choose a complex number u of modulus 1 such that
$$u \int_a^b f(t)\, dt = \left| \int_a^b f(t)\, dt \right|,$$
then (3.5.1) may be written as

$$\int_a^b (M - uf(t))\, dt = 0. \tag{3.5.2}$$

Let $uf = g + ih$, where g and h are real-valued continuous functions on I. Note that $|f(t)| \le M$ implies that $g(t) \le M$ and, hence, that $M - g(t) \ge 0$. Equation (3.5.2) implies

$$\int_a^b (M - g(t))\, dt = 0.$$

However, $\int_a^x (M - g(t))\, dt$ is a differentiable function of x with non-negative derivative $M - g(x)$ on I and, hence, is a non-decreasing function. Since it is 0 at a and b, it must be identically zero. This implies that its derivative $M - g(x)$ is identically 0.

We now have that the real part of $uf = g + ih$ is the constant M. However, we also have that

$$M^2 \ge |f(t)|^2 = |uf(t)|^2 = g^2(t) + h^2(t) = M^2 + h^2(t),$$

and this implies that $h(t) \equiv 0$ on I. Thus, f is the constant $u^{-1}M$, which has modulus M. □

The Maximum Modulus Theorem.

Theorem 3.5.2. *If f is analytic on a connected open set U and $|f|$ has a local maximum at $z_0 \in U$, then f is constant on U.*

Proof. If f has a local maximum at $z_0 \in U$, then we may choose an $r > 0$ such that $\overline{D}_r(z_0) \subset U$ and such that $|f(z_0)|$ is a maximum for $|f(z)|$ on $\overline{D}_r(z_0)$. Then Cauchy's Integral Theorem tells us that

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\,\mathrm{e}^{it})\, dt \end{aligned} \tag{3.5.3}$$

Since $|f(z_0 + r\,\mathrm{e}^{it})| \le |f(z_0)|$ for each t, by our choice of r, we have the hypotheses of the previous lemma satisfied with $M = |f(z_0)|$, $[a, b] = [0, 2\pi]$, and $f(z_0 + r\,\mathrm{e}^{it})$ playing the role of the function f of the lemma. Based on the lemma, we conclude that f is a constant c on the circle $z_0 + r\,\mathrm{e}^{it}$. Since this circle is a non-discrete subset of U, it follows from the Identity Theorem (Theorem 3.4.4) that f is the constant c on all of U. □

Recall that a subset of the plane is compact if it is both closed and bounded. Suppose U is an open set which is bounded. Then its closure $\overline{U}$ is both closed and bounded and, hence, is compact. Suppose U is connected. If f is a continuous complex-valued function on $\overline{U}$, then the continuous real valued function $|f(z)|$ has a maximum value on $\overline{U}$. If f is also analytic on U and non-constant, then the previous theorem implies that this maximum cannot occur at a point of U. This means it must occur only on the boundary ∂U. This proves the following corollary of Theorem 3.5.2.

Corollary 3.5.3. *Suppose U is a connected, bounded, open subset of $\mathbb{C}$. If f is a function which is continuous on $\overline{U}$, analytic on U, and non-constant, then the maximum value of $|f(z)|$ on $\overline{U}$ is attained on ∂U and nowhere else.*

The typical example of a set U of the type described in the above corollary is an open disc $\overline{D}_r(z_0)$ of finite radius.

Example 3.5.4. Find where the function $f(z) = z^2 - z$ attains its maximum modulus on the closed unit disc.

Solution: By Corollary 3.5.3, the maximum occurs only on the unit circle $\{z = \mathrm{e}^{it} : t \in [0, 2\pi]\}$. Thus, we need to find where the maximum modulus of the function $|f(\mathrm{e}^{it})|$ occurs for $t \in [0, 2\pi]$. This is equivalent to finding where the square of the function has a maximum. Thus, we wish to maximize the function

$$h(t) = |\,\mathrm{e}^{2it} - \mathrm{e}^{it}\,|^2 = |\,\mathrm{e}^{it} - 1|^2 = 2 - 2\cos t.$$

Clearly, the maximum occurs at $t = \pi$ and only there. Thus, the maximum of $|z^2 - z|$ on the closed unit disc is 2 and it occurs only at $z = -1$.

Schwarz's Lemma. Schwarz's Lemma is a nice application of the Maximum Modulus Theorem. It will be quite useful later in the theory of conformal mappings.

Lemma 3.5.5 (Schwarz's Lemma)**.** *Let f be analytic on $D_1(0)$, with $f(0) = 0$ and $|f(z)| \le 1$ for every $z \in D_1(0)$. Then,*

$$|f(z)| \le |z| \quad \textit{for all} \quad z \in D_1(0) \tag{3.5.4}$$

and $|f'(0)| \le 1$. If $|f'(0)| = 1$, then there is a constant c of modulus one such that $f(z) = cz$.

Proof. Since $f(0) = 0$, Theorem 3.4.2 implies that $f(z) = zg(z)$, where g is also analytic in $D_1(0)$. Since $|f(z)| \le 1$ on $D_1(0)$, it follows that

$$|g(z)| \le \frac{1}{r} \quad \text{on the circle} \quad |z| = r$$

for each $r < 1$. The Maximum Modulus Theorem implies that this inequality also holds inside the disc of radius r. Since this is true of each $r < 1$, we conclude that $|g(z)| \le 1$ on all of $D_1(0)$ and this implies (3.5.4).

The inequality $|f'(0)| \le 1$ follows from (3.5.4), since

$$f'(0) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} g(z) = g(0).$$

If $|f'(0)| = 1$, then $|g(0)| = 1$ and this is the maximum value of g on $D_1(0)$. The Maximum Modulus Theorem says this cannot happen unless g is a constant c. Then $f(z) = cz$. □

We now give a simple application of Schwarz's Lemma, which is a precursor of its later use in conformal mapping theory. Let U and V be open sets in $\mathbb{C}$. A *bi-analytic map* from U to V is an analytic function $f : U \to V$ with an analytic inverse $f^{-1} : V \to U$. That is, $f^{-1} \circ f(z) = z$ for every $z \in U$ and $f \circ f^{-1}(w) = w$ for every $w \in V$.

Theorem 3.5.6. *The only bi-analytic maps from the unit disc to itself that take 0 to 0 are of the form $f(z) = cz$ for a constant c of modulus 1. That is, the only bi-analytic maps of the unit disc onto itself which fix 0 are the rotations.*

Proof. If $f : D_1(0) \to D_1(0)$ is bi-analytic with inverse f^{-1}, then both f and f^{-1} satisfy the hypotheses of Schwarz's Lemma. Thus,

$$|f'(0)| \leq 1 \quad \text{and} \quad |(f^{-1})'(0)| \leq 1. \tag{3.5.5}$$

However, it follows from the chain rule, applied to the composition $f^{-1} \circ f(z) = z$, that

$$(f^{-1})'(0) = \frac{1}{f'(0)}.$$

Combining this with (3.5.5), we conclude that $|f'(0)| = 1$. By Schwarz'e Lemma, $f(z) = cz$ for some constant c of modulus 1. □

Harmonic Functions. Theorems about analytic functions, like Cauchy's Formula and the Maximum Modulus Theorem, imply things about harmonic functions. This is due to the fact that, locally at least, each harmonic function is the real part of an analytic function:

Theorem 3.5.7. *Let u be a function which is of class $\mathcal{C}^2$ and harmonic on a convex open set U. Then u has a harmonic conjugate v on U. That is, there is a harmonic function v on U such that the function $f = u + iv$ is analytic on U.*

Proof. Consider the function $g = u_x - iu_y$. It is $\mathcal{C}^1$, therefore differentiable, and satisfies the Cauchy-Riemann equations, since

$$u_{xx} = -u_{yy} \quad \text{and} \quad u_{xy} = u_{yx}.$$

Thus, g is analytic in U. Since U is convex, Theorem 2.5.6 implies that g has an antiderivative h in U. This means h is analytic in U and $h' = g$. If $h = w + iv$, with w and v real, then

$$u_x - iu_y = g = h' = w_x + iv_x = w_x - iw_y.$$

On equating real and imaginary parts in this equation, we conclude that

$$u_x = w_x,$$
$$u_y = w_y.$$

It follows that $w = u + c$ for some real constant c. Thus, $f = h - c = u + iv$ is an analytic function in U with real part u. □

This theorem leads to two important results about harmonic functions: a maximum principle for harmonic functions and an integral formula.

Theorem 3.5.8. *If u is a harmonic function on a connected open set U and u has a local maximum at some point $z_0 \in U$, then u is constant on U.*

Proof. Let V be a convex neighborhood of z_0 with $V \subset U$. The previous theorem implies that u has a harmonic conjugate v on V. Then, $f = u + iv$ is analytic on V, as is the function

$$g(z) = e^{f(z)}.$$

Since $|g(z)| = e^{u(z)}$, if u has a local maximum at z_0, then so does $|g(z)|$. By the Maximum Modulus Theorem, this implies that g is constant on V. But if g is constant on the connected open set V, then f is also constant on V (Exercise 3.5.10). Hence u is constant on V.

The completion of the proof involves showing that if a harmonic function on a connected open set U is constant on a non-empty open subset of U, then it is constant on all of U. We leave this as an exercise (Exercise 3.5.12). □

The next theorem shows that a harmonic function u has the mean value property: the value of u at a point is equal to its mean value over any circle centered at the point.

Theorem 3.5.9. *If u is harmonic on an open set U and $\overline{D}_r(z_0) \subset U$, then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\,\mathrm{e}^{it})\,dt. \tag{3.5.6}$$

Proof. Choose $R > r$ such that $D_R(z_0) \subset U$. Then Theorem 3.5.7 implies that u is the real part of an analytic function f on $D_R(z_0)$. The Cauchy Integral Formula applied to f and the path $\gamma(t) = z_0 + r\,\mathrm{e}^{it}$ yields

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\,\mathrm{e}^{it})\,dt.$$

Equation (3.5.6) follows from this by equating real parts. □

Example 3.5.10. Find a harmonic congugate for the harmonic function $x^2 - y^2$.

Solution: We recognize $x^2 - y^2$ as the real part of the analytic function $z^2 = (x + iy)^2$. The imaginary part of this function is $2xy$ and so $2xy$ is a harmonic conjugate of $x^2 - y^2$.

Example 3.5.11. Prove that

$$u(x, y) = \frac{x}{x^2 + y^2}$$

is harmonic on $\mathbb{C} \setminus \{0\}$ and find a harmonic conjugate for it.

Solution: We simply observe that $u(x, y)$ is the real part of

$$\frac{x - iy}{x^2 + y^2} = \frac{1}{z}.$$

Therefore, u is harmonic and has

$$\frac{-y}{x^2 + y^2}$$

as a harmonic conjugate.

Exercise Set 3.5

1. Find where the function $z^2 - 1$ attains its maximum modulus on the closed unit disc.
2. Find where the function $|\,\mathrm{e}^z\,|$ attains its maximum value on the closed unit disc.
3. Find where the function $(z-1)^2$ attains its maximum modulus on the triangle with vertices at $0, 1+i, 1-i$.

4. Show that if f is a non-constant analytic function on a connected open set U and if f has no zeroes on U, then there are no points of U where $|f(z)|$ has a local minimum.
5. Show that if f is a non-constant, continuous function on $\overline{D}_1(0)$, which is analytic on $D_1(0)$ and $|f(z)| = 1$ for all z on the unit circle, then f has a zero somewhere in $D_1(0)$.
6. Prove that if f is a non-constant entire function, and $c > 0$ is a constant such that the closed set $K = \{z \in \mathbb{C} : |f(z)| \leq c\}$ is bounded, then the open set $U = \{z \in \mathbb{C} : |f(z)| < c\}$ contains at least one zero of f.
7. Supppose f is an analytic function on the unit disc $D_1(0)$. Prove that if $|f(z)| \leq 1$ on $D_1(0)$ and f has a zero of order 2 at 0, then $|f(z)| \leq |z|^2$ for all $z \in D_1(0)$.
8. If f is a harmonic function u on a connected open set U, prove that any two harmonic conjugates for u must differ by a constant.
9. Prove that if U is a connected open set with compact closure $\overline{U}$, and u is a continuous function on $\overline{U}$ which is harmonic on U, then u attains its maximum and minimum values on ∂U. Also, prove that if u attains either its maximum or its minimum at a point of U, then it is constant.
10. Prove that if f is a continuous function on a connected open set V and if $\mathrm{e}^{f(z)}$ is constant on V, then f is also constant on V.
11. Show that $f(z) = \dfrac{2z-1}{z-2}$ is a bi-analytic map from the unit disc onto itself which takes 0 to 1/2. Hint: Show that $|f(z)| = 1$ on the unit circle and conclude from this that f maps the unit disc into itself. Then show it has an inverse function which also maps the unit disc into itself by directly solving the equation $w = f(z)$ for z as a function of w.
12. Prove that if a harmonic function on a connected open set U is constant on a non-empty open subset of U, then it is constant on all of U.
13. Find a harmonic conjugate for $u(x, y) = \mathrm{e}^x \cos y$ on $\mathbb{C}$.
14. Find a harmonic conjugate for $u(x, y) = 1/2 \log(x^2 + y^2)$ on the complement of the non-positive real axis in $\mathbb{C}$.
15. Give an example of an open subset U of $\mathbb{C}$ and a harmonic function on U which has no harmonic conjugate on U.

Chapter 9

The Gamma and Zeta Functions

This chapter is devoted to developing some of the properties of two special functions of a complex variable – the gamma function and the zeta function. These functions are of great importance in modern mathematics. The developement of their properties provides a very instructive practical application of many of the techniques developed in the preceding chapters.

The zeta function is the subject of one of the most famous unsolved problems in mathematics – the Riemann Hypothesis. This conjecture arose from Riemann's attempt to settle an old conjecture concerning the rate of growth of the number $\pi(x)$ of primes less than or equal to x as the positive number x increases. In the process, Riemann developed (but did not completely prove) a formula for $\pi(x)$. This formula involves the zeroes of the zeta function in the strip $0 < \mathrm{Re}(z) < 1$, and its study led Riemann to conjecture that all these zeroes lie on the line $\mathrm{Re}(z) = 1/2$. If true, this would have been helpful in both the proof of Riemann's Formula and its use in analyzing the growth of $\pi(x)$.

The methods introduced by Riemann eventually led to proofs by others of the result on the growth of $\pi(x)$ that he was seeking. This result is now known as the Prime Number Theorem. These proofs use information about the location of the zeroes of the zeta function, but not, of course, the information proposed in the Riemann Hypothesis, since it has never been proved.

One of the reasons this chapter is included in the text is so that we may describe the Riemann Hypothesis and its connection to the Prime Number Theorem. For completeness, we conclude the chapter with a proof of the Prime Number Theorem. This proof makes strong use of the results on infinite products presented in the previous chapter.

We begin the chapter with a discussion of the gamma function.

9.1. Euler's Gamma Function

We define Euler's gamma function for $\mathrm{Re}(z) > 0$ by the integral formula

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt. \tag{9.1.1}$$

Of course, Γ is defined by an improper integral and so we must show that this integral actually converges if $\mathrm{Re}(z) > 0$. In fact, it not only converges, but the resulting function of z is analytic.

Theorem 9.1.1. *The integral* (9.1.1) *converges and defines an analytic function* $\Gamma(z)$ *for* $\mathrm{Re}(z) > 0$.

Proof. For $0 < r < s$ we define a function $\Gamma_{r,s}$ on the right half-plane by

$$\Gamma_{r,s}(z) = \int_r^s e^{-t}\, t^{z-1}\, dt.$$

The function $e^{-t}\, t^{z-1} = e^{-t+(z-1)\log t}$ is continuous as a function of (t, z) in $[r, s]\times\mathbb{C}$ and is analytic in z for each fixed value of t. By Exercise 3.2.16, $\Gamma_{r,s}$ is analytic on the entire plane. We will show that as $s \to \infty$ and $r \to 0$ the functions $\Gamma_{r,s}$ converge uniformly on each strip of the form

$$S = \{z : a \le \mathrm{Re}(z) \le b\} \quad \text{with} \quad 0 < a < b.$$

The limit function is then necessarily analytic on the right half-plane and is, by definition, Euler's function Γ.

If $x = \mathrm{Re}(z)$, then

$$|\, e^{-t}\, t^{z-1}| = e^{-t}\, t^{x-1}.$$

Thus, if z is in the strip S, then

$$|\, e^{-t}\, t^{z-1}| \le t^{a-1} \quad \text{for} \quad t \le 1. \tag{9.1.2}$$

If $t \ge 1$, then

$$|\, e^{-t}\, t^{z-1}| \le e^{-t}\, t^{b-1} \quad \text{on} \quad S.$$

The function $e^{-t}\, t^{b+1}$ is continuous and has limit 0 at infinity. It is, therefore, bounded on $[1, \infty)$, by a positive number K. Thus,

$$|\, e^{-t}\, t^{z-1}| \le Kt^{-2} \quad \text{for} \quad t \ge 1. \tag{9.1.3}$$

Since t^{a-1} is integrable on $(0, 1]$ and Kt^{-2} is integrable on $[1, \infty)$, inequalities (9.1.2) and (9.1.3) imply that the improper integrals of $e^{-t}\, t^{z-1}$ on $(0, 1]$ and on $[1, \infty)$ both exist (see Theorem 5.2.2). Hence, the improper integral defining Γ exists for each $z \in S$.

To show that $\Gamma(z)$ is analytic, we will show that $\Gamma_{r,s}$ converges uniformly to Γ on each strip of the form S as $r \to 0$ and $s \to \infty$. In fact, from (9.1.2) and (9.1.3) we conclude

$$|\Gamma(z) - \Gamma_{r,s}(z)| \le \int_0^r t^{a-1}\, dt + \int_s^\infty Kt^{-2}\, dt \le r^a/a + K/s.$$

Given $\epsilon > 0$ the right side of this inequality is less than ϵ whenever $r < (a\epsilon/2)^{1/a}$ and $s > 2K/\epsilon$. It follows from this that $\Gamma_{r,s}(z)$ converges uniformly to $\Gamma(z)$ for $z \in S$ as $r \to 0$ and $s \to \infty$. This completes the proof. □

Analytic Continuation of Gamma. We will continue Γ to a meromorphic function defined on the entire plane. The key to doing this is the fact that Γ satisfies a functional equation, as specified in the following theorem.

Theorem 9.1.2. *The gamma function satisfies the functional equation*

$$\Gamma(z+1) = z\Gamma(z)$$

for all z in the right half-plane.

Proof. We have

$$\Gamma(z+1) = \int_0^\infty \mathrm{e}^{-t}\, t^z\, dt.$$

Integrating by parts with $u = t^z$ and $dv = \mathrm{e}^{-t}\, dt$ yields

$$\Gamma(z+1) = -t^z\, \mathrm{e}^{-t}\Big|_0^\infty + \int_0^\infty \mathrm{e}^{-t}\, zt^{z-1}\, dt = z\Gamma(z). \qquad \square$$

Corollary 9.1.3. *If n is a positive integer, then $\Gamma(n) = (n-1)!$.*

We leave the proof of this corollary as an exercise (Exercise 9.1.2).

Theorem 9.1.4. *The gamma function has a meromorphic continuation to the complex plane which has simple poles at the points $\{0, -1, -2, \cdots\}$.*

Proof. We prove by induction that Γ has a meromorphic continuation with the indicated poles and satisfying the functional equation $\Gamma(z+1) = z\Gamma(z)$ on the set $\{z : \mathrm{Re}(z) > -n\}$ for $n = 0, 1, 2, \cdots$. This is trivially true for $n = 0$. If it is true for n, then

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

defines Γ on $\{z : \mathrm{Re}(z) > -n-1\}$ in a fashion which is consistent with its definition on the smaller set $\{z : \mathrm{Re}(z) > -n\}$, because of the functional equation. Clearly the poles of this continuation are as required and the functional equation continues to hold. $\square$

Zeroes of Gamma. It turns out that Γ has no zeroes. To prove this requires deriving another functional equation. The derivation involves a pair of computational lemmas.

We define Euler's beta function, $B(z, w)$, by

$$B(z, w) = \int_0^1 (1-s)^{z-1} s^{w-1}\, ds,$$

for z and w with positive real part.

Lemma 9.1.5. *If z and w have positive real parts, then*

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(z, w).$$

Proof. For z with $\mathrm{Re}(z) > 0$, the substitution $t = u^2$ leads to

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}dt = 2\int_0^\infty u^{2z-1}\mathrm{e}^{-u^2}du.$$

Then for two points z, w with positive real parts we have

$$\Gamma(z)\Gamma(w) = 4\int_0^\infty \int_0^\infty \mathrm{e}^{-(u^2+v^2)}u^{2z-1}v^{2w-1}\,du\,dv.$$

We pass to polar coordinates, setting $u = r\cos(\theta)$ and $v = r\sin(\theta)$. Then

$$\begin{aligned}\Gamma(z)\Gamma(w) &= 4\int_0^\infty \int_0^{\pi/2} \mathrm{e}^{-r^2}r^{2(z+w)-2}\cos^{2z-1}(\theta)\sin^{2w-1}(\theta)\ r\,dr\,d\theta \\ &= 2\int_0^\infty \mathrm{e}^{-r^2}r^{2(z+w)-1}\,dr\cdot 2\int_0^{\pi/2}\cos^{2z-1}(\theta)\sin^{2w-1}(\theta)\,d\theta \\ &= \Gamma(z+w)\cdot 2\int_0^{\pi/2}\cos^{2z-1}(\theta)\sin^{2w-1}(\theta)\,d\theta.\end{aligned}$$

The substitution $s = \sin^2(\theta)$ leads to

$$2\int_0^{\pi/2}\cos^{2z-1}(\theta)\sin^{2w-1}(\theta)\,d\theta = \int_0^1 (1-s)^{z-1}s^{w-1}\,ds.$$

The latter expression is Euler's beta function $B(z,w)$. Thus,

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(z,w).$$ □

We will use the above identity in the case $w = 1 - z$ to derive a functional equation for Γ. To do this, we will need to evaluate $B(z, 1-z)$. This involves the following integral evaluation.

Lemma 9.1.6. *If* $x \in (0,1)$*, then* $\displaystyle\int_0^\infty \frac{t^{-x}}{1+t}\,dt = \frac{\pi}{\sin \pi x}$.

Proof. We essentially proved this back in Chapter 5 where we discussed the Mellin transform. In fact, the integral in the theorem is just the Mellin transform of $\dfrac{1}{1+t}$ evaluated at $1-x$. In Example 5.4.3, as a special case of Theorem 5.4.2, we proved that the Mellin transform of $\dfrac{1}{1+x}$ is

$$\int_0^\infty \frac{x^{t-1}}{1+x}\,dx = \frac{\pi}{\sin \pi t}.$$

If the roles of t and x are reversed, this becomes

$$\int_0^\infty \frac{t^{x-1}}{1+t}\,dt = \frac{\pi}{\sin \pi x}.$$

The identity of the theorem is then obtained by replacing x by $1-x$ and using the identity $\sin(\pi - \pi x) = \sin \pi x$. □

Theorem 9.1.7. *The gamma function satisfies the functional equation*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Proof. If we set $z = x$ and $w = 1 - x$ for $x \in (0,1)$, then since $\Gamma(1) = 1$, Lemma 9.1.5 implies

$$\Gamma(x)\Gamma(1-x) = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)} = B(x, 1-x) = \int_0^1 (1-s)^{x-1} s^{-x}\, ds.$$

Then the substitution $s = \dfrac{t}{t+1}$ leads to

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \left(1 - \frac{t}{t+1}\right)^{x-1} \frac{t^{-x}}{(t+1)^{-x}} \frac{dt}{(t+1)^2} = \int_0^\infty \frac{t^{-x}}{1+t}\, dt.$$

We use the previous lemma to evaluate the last integral and conclude that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Since this identity holds for $x \in (0,1)$, the Identity Theorem implies that it continues to hold when x is replaced by any complex number z for which the functions involved are defined. □

This theorem has the following corollary, the proof of which is left as an exercise (Exercise 9.1.4).

Corollary 9.1.8. *The gamma function has no zeroes.*

Product Formula for Γ. The fact that $\mathrm{e}^{-t} = \lim_{n\to\infty}(1-t/n)^n$ can be exploited to express Γ as an infinite product of the sort studied in the previous chapter. The first step in deriving this formula is to show the following:

Theorem 9.1.9. *The identity*

$$\Gamma(x) = \lim_{n\to\infty} \int_0^n (1 - t/n)^n t^{x-1}\, dt$$

holds for all $x > 0$.

Proof. The function $\mathrm{e}^{-s} - 1 + s$ is 0 at $s = 0$ and has a positive derivative $1 - \mathrm{e}^{-s}$ for $s > 0$. It is, therefore positive for $s > 0$. Thus, $1 - s \le \mathrm{e}^{-s}$ for $s > 0$. With $s = t/n$ this implies $1 - t/n < \mathrm{e}^{-t/n}$ for $t > 0$ and, on taking nth powers,

$$(1 - t/n)^n \le \mathrm{e}^{-t}.$$

Furthermore, an elementary calculus argument (Exercise 9.1.8) shows that

$$\mathrm{e}^{-t} - (1 - t/n)^n \le \frac{1}{n\,\mathrm{e}}, \tag{9.1.4}$$

for $t \ge 0$.

If we fix $a > 0$, then

$$\Gamma(x) - \int_0^n (1-t/n)^n t^{x-1}\, dt \le \int_0^a (\mathrm{e}^{-t} - (1-t/n)^n) t^{x-1}\, dt + \int_a^\infty \mathrm{e}^{-t}\, t^{x-1}\, dt$$

for $n > a$. The first term on the right converges to 0 as $n \to \infty$ by (9.1.4) and the second term can be made less than any given ϵ by choosing a large enough, because the improper integral defining Γ converges. □

Theorem 9.1.10. *The entire function* $1/\Gamma$ *can be represented as the infinite product*

$$\frac{1}{\Gamma(z)} = z \prod_{k=1}^{\infty} \frac{1+z/k}{(1+1/k)^z},$$

where this product converges uniformly on each disc of finite radius.

Proof. The integral in the previous theorem may be evaluated using a repeated application of integration by parts (Exercise 9.1.9). The result is

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}.$$

for $x > 0$.

If we invert this, divide both numerater and denomenator by $n!$, and note that $n^x = \prod_{k=1}^{n-1}(1+1/k)^x$, we obtain

$$\frac{1}{\Gamma(x)} = x \prod_{k=1}^{\infty} \frac{1+x/k}{(1+1/k)^x}$$

for $x > 0$.

If we can show that this infinite product converges, not just for $x > 0$, but uniformly on each disc of finite radius in the complex plane, then the result will be an entire function which agrees with the entire function $1/\Gamma(z)$ on the positive real axis. This implies the two entire functions agree on all of $\mathbb{C}$. Thus, the proof will be complete if we can show that

$$z \prod_{k=1}^{\infty} \frac{1+z/k}{(1+1/k)^z} \tag{9.1.5}$$

converges uniformly on each compact disc. This product is very nearly a Weierstrass product, as studied in the previous chapter. This fact can be used to prove the uniform convergence on compact discs. The details are left to Exercise 9.1.10.

□

Exercise Set 9.1

1. Show that, for z real and positive, $\Gamma(z)$ is the Mellin transform of a certain function. What function? (see Section 5.4).
2. Prove that $\Gamma(n) = (n-1)!$ if n is a positive integer.
3. Prove that $z(z+1)(z+2)\cdots(z+n) = \dfrac{\Gamma(z+n+1)}{\Gamma(z)}$.
4. Prove Corollary 9.1.8.
5. Prove that the residue of $\Gamma(z)$ at $-n$ is $\dfrac{(-1)^n}{n!}$ for $n = 0, 1, 2, \cdots$.
6. Prove that, for $r > 0$ and $\mathrm{Re}(z) > 0$, $\int_0^\infty e^{-rt}\, t^{z-1}\, dt = r^{-z}\Gamma(z)$.
7. Prove that $\Gamma(z)\Gamma(-z) = \dfrac{-\pi}{z \sin \pi z}$.

8. Prove that $\mathrm{e}^{-t} - (1 - t/n)^n \le \dfrac{1}{n\,\mathrm{e}}$ for all $t \in [0, n]$. Hint: Show that the maximum of the function $h(t) = \mathrm{e}^{-t} - (1 - t/n)^n$ on $[0, n]$ occurs at a point t_0 where $h(t_0) = \mathrm{e}^{-t_0}\, t_0/n$. Then show that this number is less than or equal to $\dfrac{1}{ne}$.
9. Using integration by parts, prove that if $x > 0$, then
$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt = \frac{n^x n!}{x(x+1)\cdots(x+n)}.$$
10. Prove that the infiinite product
$$\frac{1}{\Gamma(z)} = z \prod_{k=1}^{\infty} \frac{1 + z/k}{(1 + 1/k)^z}$$
converges uniformly on each compact disc. Hint: Show that
$$\frac{1 + z/k}{(1 + 1/k)^z} = (1 + z/k)\,\mathrm{e}^{-z/k}\,\mathrm{e}^{a_k z},$$
where $a_k = 1/k - \log(1 + 1/k)$. Then show that $\prod_k (1 + z/k)\,\mathrm{e}^{-z/k}$ is a convergent Weierstrass product and $\sum_k |a_k|$ converges.

9.2. The Riemann Zeta Function

The Riemann zeta function is defined on the set $\{z : \mathrm{Re}(z) > 1\}$ by the infinite series

(9.2.1) $$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.$$

If $z = x + iy$, then $|n^{-z}| = n^{-x}$, and so this series converges uniformly absolutely on each set of the form $\{z : \mathrm{Re}(z) \ge r\}$ for $r > 1$. It follows that $\zeta(z)$ is defined and analytic on the set $\{z : \mathrm{Re}(z) > 1\}$.

A Product Formula for the Zeta Function. Let $\{p_1, p_2, p_3, \cdots\}$ be the set of prime numbers written in increasing order. Then we have

Theorem 9.2.1. *For* $\mathrm{Re}(z) > 1$, $\zeta(z) = \prod_{n=1}^{\infty}(1 - p_n^{-z})^{-1}$.

Proof. The fact that the infinite product converges for $\mathrm{Re}(z) > 1$ follows from Theorem 8.1.4 and the fact that $\sum_{n=1}^{\infty} p_n^{\mathrm{Re}(z)}$ converges if $\mathrm{Re}(z) > 1$. Then
$$\zeta(z)(1 - 2^{-z}) = \sum_1^{\infty} n^{-z} - \sum_1^{\infty} (2n)^{-z} = \sum_{n \in S_1} n^{-z},$$
where S_1 is the set of odd natural numbers. An induction argument using the same technique then shows that for each natural number k
$$\zeta(z) \prod_{n=1}^{k} (1 - p_n^{-z}) = \sum_{n \in S_k} n^{-z},$$

where S_k is the set of natural numbers not divisible by any of the first k primes. Since the right side of this equation has limit 1 as $k \to \infty$, the theorem follows. $\square$

This theorem has the following two corollaries. We leave the proofs to the exercises (Exercises 9.2.2 and 9.2.3).

Corollary 9.2.2. *There are infinitely many primes.*

Corollary 9.2.3. *The zeta function has no zeroes in the region* $\mathrm{Re}(z) > 1$.

The Function ξ. Our next goal is to extend the zeta function to be a meromorphic function on the entire plane. We will do this by expressing the zeta function in terms of the gamma function and a certain entire function ξ.

We begin the development of ξ by making the substitution $t = n^2 s^2 \pi$ in the formula (9.1.1) defining Γ. The result is

$$\Gamma(z) = 2n^{2z}\pi^z \int_0^\infty \mathrm{e}^{-n^2s^2\pi}\, s^{2z}\frac{ds}{s}.$$

If we divide by $n^{2z}\pi^z$ and sum over $n = 1, 2, 3, \cdots$, we obtain

$$\zeta(2z)\Gamma(z)\pi^{-z} = 2\sum_{n=1}^\infty \int_0^\infty \mathrm{e}^{-n^2s^2\pi}\, s^{2z}\,\frac{ds}{s} \quad \text{if} \quad \mathrm{Re}(z) > 1. \tag{9.2.2}$$

We will use the result of Exercise 9.2.7 to prove that it is legitimate to move the summation inside the integral in the expression on the right. We estimate the size of each integrand in this series as follows:

$$\left|\mathrm{e}^{-n^2s^2\pi}\, s^{2z-1}\right| \le \mathrm{e}^{-ns^2}\, s^{2\,\mathrm{Re}(z)-1}.$$

The functions on the right are positive and their sum is

$$\sum_{n=1}^\infty \mathrm{e}^{-ns^2}\, s^{2\,\mathrm{Re}(z)-1} = \frac{s^{2\,\mathrm{Re}(z)-1}}{\mathrm{e}^{s^2}-1}. \tag{9.2.3}$$

For each z, this series converges uniformly on each closed subinterval of $(0, \infty)$. Furthermore, since $\mathrm{e}^{s^2}-1 \ge s^2$, if $\mathrm{Re}(z) > 2$, the function on the right in (9.2.3) is less than or equal to $s^{2\,\mathrm{Re}(z)-3}$ and, hence, has finite integral over $[0, 1]$ if $\mathrm{Re}(z) > 2$. Since $\mathrm{e}^{s^2}-1 \ge \mathrm{e}^{s^2}/2$ if $s \ge 1$, the function on the right in (9.2.3) is less than or equal to $2\,\mathrm{e}^{-s^2}\, s^{\mathrm{Re}(z)-1}$ on $[1, \infty)$ and, hence, has finite integral on $[1, \infty)$. It follows that this function has finite integral on $[0, \infty)$. Thus, by the result of Exercise 9.2.7, the sum can be taken inside the integral in (9.2.2). Doing so yields

$$\zeta(2z)\Gamma(z)\pi^{-z} = 2\int_0^\infty \sum_{n=1}^\infty \mathrm{e}^{-n^2s^2\pi}\, s^{2z}\,\frac{ds}{s} \quad \text{for} \quad \mathrm{Re}(z) > 2. \tag{9.2.4}$$

If we replace z by $z/2$ and set

$$H(s) = \sum_{n=1}^\infty \mathrm{e}^{-n^2s^2\pi}, \tag{9.2.5}$$

then (9.2.4) may be rewritten as

$$\zeta(z)\Gamma(z/2)\pi^{-z/2} = 2\int_0^\infty H(s)s^z\,\frac{ds}{s} \quad \text{for} \quad \mathrm{Re}(z) > 2. \tag{9.2.6}$$

The function ξ is obtained by multiplying this expression by $z(z-1)/2$. Thus, for $\mathrm{Re}(z) > 2$,

$$\xi(z) = \frac{z(z-1)}{2}\zeta(z)\Gamma(z/2)\pi^{-z/2} = z(z-1)\int_0^\infty H(s)s^z\frac{ds}{s}. \tag{9.2.7}$$

The Poisson Summation Formula. We pause to prove a technical result about Fourier transforms (see Section 5.3). It will be used in the upcoming proof that ξ extends to an entire function.

Theorem 9.2.4. *If f is a continuous function on $\mathbb{R}$ with the property that the series $\sum_{n=-\infty}^{\infty} f(x+2\pi n)$ converges absolutely and uniformly for $x \in [-\pi, \pi]$ and the series $\sum_{n=-\infty}^{\infty} f\hat{\ }(n)$ converges absolutely, then*

$$\sum_{n=1}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty} f\hat{\ }(n),$$

where $f\hat{\ }$ is the Fourier transform of f.

Proof. We set $g(\theta) = \sum_{n=1}^{\infty} f(\theta + 2\pi n)$ whenever $\theta \in [-\pi, \pi]$, and then integrate this function against the Poisson kernel (see Section 6.5)

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = \sum_{-\infty}^{\infty} r^{|n|}\,\mathrm{e}^{in\theta}$$

over the interval $[-\pi, \pi]$. The result is

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)P_r(\theta)\,d\theta &= \int_{-\pi}^{\pi}\sum_{-\infty}^{\infty} f(\theta+2\pi n)P_r(\theta)\,d\theta \\
&= \sum_{-\infty}^{\infty}\int_{-\pi}^{\pi} f(\theta+2\pi n)P_r(\theta)\,d\theta \\
&= \sum_{-\infty}^{\infty}\int_{(2n-1)\pi}^{(2n+1)\pi} f(\theta)P_r(\theta)\,d\theta \\
&= \int_{-\infty}^{\infty} f(\theta)P_r(\theta)\,d\theta \\
&= \sqrt{2\pi}\sum_{-\infty}^{\infty} r^{|n|} f\hat{\ }(n).
\end{aligned} \tag{9.2.8}$$

Note that the third step in this calculation uses the hypothesis that the series defining g converges uniformly absolutely .

As $r \to 1$ the integral on the left in (9.2.8) converges to

$$2\pi g(0) = 2\pi\sum_{n=1}^{\infty} f(2\pi n),$$

by Lemma 6.5.5, while the sum on the right converges to

$$\sqrt{2\pi}\sum_{-\infty}^{\infty} f\hat{\ }(n)$$

since, by hypothesis, the series $\sum_{-\infty}^{\infty} f\hat{}(n)$ converges absolutely. We conclude that

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f\hat{}(n),$$

as required. □

The Symmetry of ξ. We can now prove that ξ extends to be an entire function and that it satisfies the symmetry relation $\xi(z) = \xi(1-z)$. We first derive a symmetry relation for H.

Lemma 9.2.5. *The function H satisfies the relation*

$$H(s^{-1}) = sH(s) + (s-1)/2. \tag{9.2.9}$$

Proof. Note that $H(s) = (G(s)-1)/2$, where

$$G(s) = \sum_{-\infty}^{\infty} \mathrm{e}^{-n^2 s^2 \pi} = 1 + 2\sum_{n=1}^{\infty} \mathrm{e}^{-n^2 s^2 \pi}.$$

If $g(x) = \dfrac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$ is the normal distribution function from Example 5.3.6, then

$$G(s) = \sqrt{2\pi} \sum_{-\infty}^{\infty} g(ns\sqrt{2\pi}).$$

Note that the function g is its own Fourier transform, by Example 5.3.6. This implies that the function f, defined by $f(x) = g(xs/\sqrt{2\pi})$, satisfies the hypotheses of the previous theorem. If we apply that theorem to f, we conclude that

$$G(s) = \sum_{-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} f\hat{}(n).$$

A change of variables in the integral defining the Fourier transform shows that $f\hat{}(n) = s^{-1}\sqrt{2\pi}\, g\hat{}(ns^{-1}\sqrt{2\pi})$. Thus,

$$G(s) = \sum_{-\infty}^{\infty} s^{-1}\sqrt{2\pi}\, g(ns^{-1}\sqrt{2\pi}) = s^{-1}G(s^{-1}).$$

The identity (9.2.9) follows from this. □

The consequence for the function ξ is the following:

Theorem 9.2.6. *The function ξ extends to an entire function which is symmetric about the point $z = 1/2$; that is, $\xi(1-z) = \xi(z)$.*

Proof. If we break the integral on the right side of (9.2.6) into an integral over $[1,\infty)$ and an integral over $[0,1]$ and make the substitution $s \to s^{-1}$ in the latter integral, the result is

$$\int_1^{\infty} H(s)s^z \frac{ds}{s} + \int_1^{\infty} H(s^{-1})s^{-z}\frac{ds}{s}.$$

Using (9.2.9), this becomes

$$\int_1^\infty H(s)s^z \frac{ds}{s} + \int_1^\infty (sH(s) + (s-1)/2)s^{-z}\frac{ds}{s}$$
$$= \int_1^\infty H(s)s^z \frac{ds}{s} + \int_1^\infty H(s)s^{(1-z)}\frac{ds}{s} + \frac{1}{2}\int_1^\infty (s^{(1-z)} - s^{-z})\frac{ds}{s}.$$

Since

$$\frac{1}{2}\int_1^\infty (s^{(1-z)} - s^{-z})\frac{ds}{s} = \frac{1/2}{z(z-1)} \quad \text{if} \quad \mathrm{Re}(z) > 1,$$

we conclude that

$$\xi(z) = 1/2 - z(1-z)\int_1^\infty H(s)(s^z + s^{1-z})\,\frac{ds}{s} \quad \text{for} \quad \mathrm{Re}(z) > 2. \tag{9.2.10}$$

However, the right side of this equation is defined and analytic on the entire plane, since $H(s)$ times any power of s is absolutely integrable on $[1,\infty)$. It is also obviously symmetric about $z = 1/2$. Thus, ξ has an extension to the whole plane with the required properties. □

Meromorphic Extension of ζ. The next theorem is an immediate consequence of the preceding theorem and (9.2.7).

Theorem 9.2.7. *The function ζ has a meromorphic extension to the plane given by the formula*

$$\zeta(z) = \frac{2\pi^{z/2}\xi(z)}{z(z-1)\Gamma(z/2)}. \tag{9.2.11}$$

It is useful to note that the above formula can be put in a slightly different form by using Theorem 9.1.2. This theorem, with z replaced by $z/2$, implies that

$$(z/2)\Gamma(z/2) = \Gamma(z/2+1).$$

Then (9.2.11) becomes

$$\zeta(z) = \frac{\pi^{z/2}\xi(z)}{(z-1)\Gamma(z/2+1)}. \tag{9.2.12}$$

Exercise Set 9.2

1. Show that $\lim_{x\to\infty}\zeta(x+iy) = 1$ and the convergence is uniform in y.
2. Use Theorem 9.2.1 to prove Corollary 9.2.2.
3. Use Theorem 9.2.1 to prove Corollary 9.2.3.
4. If $z = s + it$ with $s > 1$, prove that

$$\left|\frac{1}{\zeta(z)}\right| < \zeta(s).$$

   Hint: Use Theorem 9.2.1.

5. Use the result of the previous exercise to prove that

$$\left|\frac{1}{\zeta(z)}\right| < \zeta(2)$$

if $\mathrm{Re}(z) \geq 2$.

6. Let $u(t) = \sum_{n=1}^{\infty} u_n(t)$ be the sum of a series of positive continuous functions on $(0, \infty)$ and suppose this series converges uniformly on closed bounded intervals of $(0, \infty)$. Prove that

$$\sum_{n=1}^{\infty} \int_0^{\infty} u_n(t)\, dt = \int_0^{\infty} u(t)\, dt.$$

Hint: Either both sides are infinite or one of them is finite.

7. Let $h(t) = \sum_{n=1}^{\infty} h_n(t)$ be the sum of an infinite series of continuous functions on $(0, \infty)$ and suppose

$$|h_n(t)| \leq u_n(t) \quad \text{for all} \quad n, t,$$

where $\sum_{n=1}^{\infty} u_n$ is a positive termed series which satisfies the conditions of the previous exercise. Prove that the improper integral of h on $\mathbb{R}$ converges and

$$\int_0^{\infty} h(t)\, dt = \sum_{n=1}^{\infty} \int_0^{\infty} h_n(t)\, dt.$$

8. Show that the Poisson Summation Formula can, under appropriate hypotheses on f and $f\hat{}$, be reformulated as

$$\sum_{n=-\infty}^{\infty} f(n) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} f\hat{}(2\pi n).$$

9. Use the form of the Poisson Summation Formula derived in the previous exercise to show that

$$\sum_{-\infty}^{\infty} \frac{1}{1+n^2} = \pi \frac{\mathrm{e}^{2\pi} + 1}{\mathrm{e}^{2\pi} - 1}.$$

Hint: The Fourier transform of $\dfrac{1}{1+x^2}$ is calculated in Example 5.3.3.

The next five exercises outline an alternative to the approach used in Theorem 9.2.7 to prove that ζ extends to be meromorphic in the plane.

10. Formula (9.2.4) was developed using the substitution $t = n^2 s^{\pi}$ in the integral defining Γ. Use the substitution $t = ns$ in a similar way to derive the formula

$$\zeta(z)\Gamma(z) = \int_0^{\infty} \frac{1}{\mathrm{e}^s - 1} s^{z-1}\, ds \quad \text{for} \quad \mathrm{Re}(z) > 1.$$

11. For complex numbers w and z, define $(-w)^{z-1}$ to be $\mathrm{e}^{(z-1)\log(-w)}$, where log is the principal branch of the log function. Show that this function is analytic except for a cut on the positive real line, that its limit as w approaches the positive real number s from above is $\mathrm{e}^{(z-1)(\log s - \pi i)}$, and that its limit as w appproaches s from below is $\mathrm{e}^{(z-1)(\log s + \pi i)}$.

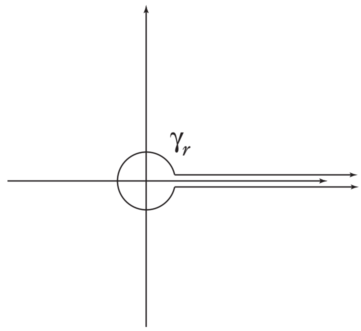

Figure 9.2.1. Contour for Exercise 9.2.12.

12. Using this definition for $(-w)^{z-1}$, consider the contour integral

$$\eta(z) = \int_{\gamma_r} \frac{1}{e^w - 1}(-w)^{z-1}\, dw, \tag{9.2.13}$$

where γ_r is the contour indicated in Figure 9.2.1, $r < 2\pi$ is the radius of the indicated circle, and the two horizontal lines are a distance $\epsilon \leq r$ above and below the positive real axis. Prove that this integral exists for all z, is independent of r and ϵ, and defines an entire function $\eta(z)$.

13. By passing to the limit as $\epsilon \to 0$ and $r \to 0$ in the integral 9.2.13, prove that $\eta(z) = -2\pi i \sin(\pi z)\Gamma(z)\zeta(z)$ if $\mathrm{Re}(z) > 1$.

14. Use the previous exercise and an identify involving Γ to prove that

$$\zeta(z) = -\frac{1}{2\pi i}\Gamma(1-z)\eta(z) \quad \text{for} \quad \mathrm{Re}(z) > 1.$$

Conclude that ζ has a meromorphic extension to the plane with a single simple pole at $z = 1$.

9.3. Properties of ζ

The expressions (9.2.11) and (9.2.12) for ζ and the properties of Γ and ξ lead to a wealth of information about ζ. Ultimately, this information will lead to a proof of the Prime Number Theorem in the last section of this chapter.

Zeroes and Poles.

Theorem 9.3.1. *The zeta function has a simple pole at $z = 1$, with residue 1, and no other poles.*

Proof. The function $\Gamma(z/2 + 1)$ has no zeroes, and the function ξ is entire. Thus, (9.2.12) implies that the only pole of ζ is at $z = 1$ and it is a simple pole. The fact that the residue is 1 follows from the fact, proved in the exercises, that $\Gamma(1/2) = \sqrt{\pi}$, and from (9.2.10), which implies $\xi(1) = 1/2$. $\square$

Theorem 9.3.2. *The zeta function has a zero of order* 1 *at each negative even integer.*

This is Exercise 9.3.1.

Corollary 9.3.3. *The zeta function has no zeroes outside the strip*

$$0 \le \mathrm{Re}(z) \le 1$$

except the ones that occur at negative even integers.

Proof. Except for the zeroes of ζ that occur at negative even integers (due to the poles of Γ), the functions ζ and ξ have the same zeroes. Since ζ has no zeroes in the region $\mathrm{Re}(z) > 1$ by Corollary 9.2.3, and since ξ is symmetric about $1/2$, it follows that ζ has no zeroes outside the strip $0 \le \mathrm{Re}(z) \le 1$ except the negative even integers. □

This result can be strengthened to exclude the existence of zeroes of ζ on the lines $\mathrm{Re}(z) = 0, 1$. The proof uses the following lemma:

Lemma 9.3.4. *For* $\mathrm{Re}(z) > 1$ *there is an analytic logarithm for* $\zeta(z)$*, defined by*

$$\log \zeta(z) = \sum_p \sum_{m=1}^{\infty} \frac{p^{-mz}}{m}. \tag{9.3.1}$$

The derivative of this function is

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_p \sum_{m=1}^{\infty} p^{-mz} \log p. \tag{9.3.2}$$

Here, in both equations, the summation on the left is over all primes p*.*

Proof. If $\mathrm{Re}(z) > 1$, then $|p^{-z}| < 1/2$ for all primes p and so $\log(1 - p^{-z})$ is defined and analytic if log is the principal branch of the log function. Furthermore, by Theorem 9.2.1 we have for $\mathrm{Re}(z) > 1$

$$\exp\left(-\sum_p \log(1 - p^{-z})\right) = \prod (1 - p^{-z})^{-1} = \zeta(z). \tag{9.3.3}$$

Hence, $-\sum_p \log(1 - p^{-z})$ is an analytic logarithm for $\zeta(z)$ on $\mathrm{Re}(z) > 1$. If we expand this function in a power series in p^{-z}, the result is (9.3.1). On differentiating (9.3.1), we obtain (9.3.2). □

Theorem 9.3.5. *The zeta function has no zeroes outside the strip*

$$0 < \mathrm{Re}(z) < 1$$

except those which occur at negative even integers.

Proof. We first note that the inequality

$$0 \le 3 + 4\cos\theta + \cos 2\theta \tag{9.3.4}$$

holds for all real numbers θ due to the identity

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2,$$

which follows from $\cos 2\theta = 2\cos^2\theta - 1$.

If $z = x + iy$, then $\mathrm{Re}(p^{-mz}) = p^{-mx}\cos(-my\log p)$. It follows from (9.3.4) that, for $x > 1$,

$$0 \le 3p^{-mx} + 4\,\mathrm{Re}(p^{-mx-myi}) + \mathrm{Re}(p^{-mx-2myi}),$$

which, when combined with (9.3.1), implies that, for $x > 1$,

$$3\,\mathrm{Re}(\log\zeta(x)) + 4\,\mathrm{Re}(\log\zeta(x+iy)) + \mathrm{Re}(\log\zeta(x+i2y)) \ge 0,$$

or, on exponentiating,

$$|\zeta(x)|^3|\zeta(x+iy)|^4|\zeta(x+i2y)| \ge 1. \tag{9.3.5}$$

We divide both sides of this inequality by $x - 1$ and write the result in the form

$$|(x-1)\zeta(x)|^3\left|\frac{\zeta(x+iy)}{x-1}\right|^4|\zeta(x+i2y)| \ge \frac{1}{x-1}. \tag{9.3.6}$$

Since $\zeta(z)$ has a simple pole at $z = 1$, $(1-z)\zeta(z)$ has a removable singularity at $z = 1$. This implies that the first factor on the left in (9.3.6) is bounded as $x \to 1$. Since there are no other poles of ζ, the factor on the right is also bounded as $x \to 1$, provided $y \ne 0$. If ζ has a zero at $1 + iy$, then the middle factor is also bounded as $x \to 1$. Since the right side of (9.3.6) is not bounded as $x \to 1$, we conclude that there can be no zero of ζ at $z = 1 + iy$.

Now that we know that there are no zeroes of ζ on the line $\mathrm{Re}(z) = 1$, we use the fact that the set of zeroes of ζ in the strip $0 \le \mathrm{Re}(z) \le 1$ is symmetric about the line $\mathrm{Re}(z) = 1/2$ (since these are also the zeroes of ξ) to conclude that there are no zeroes on the line $\mathrm{Re}(z) = 0$. In view of Corollary 9.3.3, this completes the proof. □

Estimate on the growth of ξ. Integration by parts in the integral appearing in (9.2.10) leads to

$$\begin{aligned}\xi(z) &= 1/2 + H(1) + \int_1^\infty H'(s)((1-z)s^z + zs^{1-z})\,ds\\ &= 1/2 + H(1) + \int_1^\infty s^2H'(s)((1-z)s^{(z-1)} + zs^{-z})\,\frac{ds}{s}.\end{aligned}$$

Another integration by parts leads to

$$\xi(z) = 1/2 + H(1) + 2H'(1) + \int_1^\infty (s^2H'(s))'(s^{z-1} + s^{-z})\,ds.$$

If we differentiate (9.2.9) and set $s = 1$, the result is

$$\frac{1}{2} + H(1) + 2H'(1) = 0.$$

Thus,

$$\xi(z) = \int_1^\infty (s^2H'(s))'(s^{z-1} + s^{-z})\,ds. \tag{9.3.7}$$

Since

$$\begin{aligned}(s^{z-1}+s^{-z}) &= s^{-1/2}(s^{z-1/2}+(s^{-z+1/2})\\ &= 2s^{-1/2}\cosh\left((z-1/2)\log(s)\right),\end{aligned}$$

(9.3.7) can be rewrittten as

$$\xi(z)=\int_1^\infty (s^2H'(s))'s^{-1/2}\cosh\left((z-1/2)\log(s)\right)ds. \tag{9.3.8}$$

Theorem 9.3.6. *If ξ is expanded as a power series in $z-1/2$, the coefficients of the power series are all real and non-negative.*

Proof. A direct calculation using (9.2.5) shows that

$$(s^2H'(s))'s^{-1/2}=\sum_{n=1}^\infty (4n^4\pi^2s^4-6n^2\pi s^2)s^{-1/2}\,\mathrm{e}^{-n^2s^2\pi}\,.$$

The terms of this series are clearly positive for $s\geq 1$ and so the function itself is positive. Also, the power series coefficients in the expansion of $\cosh w$ about 0 are real and non-negative. Since $\log(s)\geq 0$ for $s\geq 1$, the theorem follows (see Exercise 9.37). □

Theorem 9.3.7. *There is a constant R such that $|\xi(1/2+z)|\leq r^r$ for all $z\in\mathbb{C}$ with $|z|=r>R$.*

Proof. Since $\xi(z+1/2)$ has a power series expansion in z with real non-negative coefficients, its maximum absolute value on any disc $D_r(0)$ is achieved at $z=r$. However, if n is an integer such that $1/2+r\leq 2n\leq 5/2+r$, then by (9.2.7) and the fact that ξ is increasing on the positive real line (Exercise 9.3.2),

$$\xi(1/2+r)\leq \xi(2n)=n(2n-1)\zeta(2n)\Gamma(n)\pi^{-n}.$$

Now ζ is decreasing on $(1,\infty)$. Thus,

$$\zeta(2n)\leq\zeta(2)\quad\text{if}\quad n\geq 1,$$

and

$$\Gamma(n)=(n-1)!$$

if n is a positive integer. Thus,

$$\xi(1/2+r)\leq 2n\,n!\,\zeta(2)\leq 2\zeta(2)\,n^{n+1}\leq r^r$$

if r is sufficiently large (since $n\leq 5/4+r/2$) (Exercise 9.3.9). □

The following corollary is a direct consequence of the above theorem and the results of sections 8.3 and 8.4. We leave the details to the exercises.

Corollary 9.3.8. *The function ξ is an entire function of finite order at most* 1. *Consequently, the zeroes of ξ (and, hence, the zeroes of ζ that lie in the strip* $0<\mathrm{Im}(z)<1$*) form a sequence with exponent of convergence at most* 1.

A Product Expansion for ξ. In what follows, it will be convenient to make the change of variables $w = z - 1/2$. Then

$$\xi(z) = \xi(1/2 + w).$$

Since ξ is symmetric about $1/2$, the function $\xi(1/2 + w)$ is symmetric about 0.

By Corollary 9.3.8, $\xi(1/2 + w)$ is an entire function of finite order at most 1. Hence, the Hadamard Factorization Theorem applies with $\lambda = 1$. It tells us that ξ has a factorization of the form

$$\xi(1/2 + w) = \mathrm{e}^{q(w)} \prod_{\sigma} (1 - w/\sigma)\, \mathrm{e}^{w/\sigma},$$

where q is a polynomial of degree at most 1, and the product is over all zeroes σ of $\xi(1/2+w)$. However, the zeroes of this function are symmetric about 0 and so, if a zero σ appears in this product, then so does its negative. The exponential factors $\mathrm{e}^{w/\sigma}$ and $\mathrm{e}^{-w/\sigma}$ cancel and so

$$\xi(1/2 + w) = \mathrm{e}^{q(w)} \prod_{\sigma} (1 - w/\sigma)$$

as long as it is understood that the factors involving a given σ and its negative $-\sigma$ are to be grouped together. If this is done, then the product expansion becomes

$$\xi(1/2 + w) = \mathrm{e}^{q(w)} \prod_{\mathrm{Im}(\sigma)>0} (1 - w^2/\sigma^2).$$

It is this product that actually converges (see the discussion of the product expansion of $\sin(\pi z)$ in Example 8.2.6).

Now $\xi(1/2 + w)$ is an even function of w (symmetric about 0) and so is the above product. It follows that the polynomial q must also be an even function. Since the only even polynomials of degree at most 1 are constants, we conclude that

$$\xi(1/2 + w) = c \prod_{\sigma} (1 - w/\sigma) \tag{9.3.9}$$

for some constant c.

If we recall that $w = z - 1/2$ and $\sigma = \rho - 1/2$ and we use the identity

$$1 - \frac{(z - 1/2)}{(\rho - 1/2)} = \left(1 - \frac{z}{\rho}\right)\left(1 + \frac{1/2}{\rho - 1/2}\right) \tag{9.3.10}$$

(Exercise 9.3.10), then (9.3.9) becomes

$$\xi(z) = \prod_{\rho} \left(1 - \frac{z}{\rho}\right)\left(1 + \frac{1/2}{\rho - 1/2}\right).$$

Since the product

$$c_1 = c \prod_{\rho} \left(1 + \frac{1/2}{\rho - 1/2}\right) \tag{9.3.11}$$

converges as long as the factors involving $\rho - 1/2$ and its negative are grouped together (Exercise 9.3.11), we obtain an infinite product expansion

$$\xi(z) = c_1 \prod_\rho \left(1 - \frac{z}{\rho}\right).$$

By evaluating at $z = 0$, we see that $c_1 = \xi(0) = 1/2$. This proves the following theorem.

Theorem 9.3.9. *The function ξ has the following infinite product expansion which converges uniformly on each disc of finite radius:*

$$\xi(z) = \frac{1}{2} \prod_\rho \left(1 - \frac{z}{\rho}\right),$$

where the numbers ρ are the zeroes of $\xi(z)$ and the factors are arranged so that ρ and $1 - \rho$ are grouped together. The product converges uniformly on each disc of finite radius.

This, in turn, implies that the zeta function has the following infinite product expansion:

Theorem 9.3.10. *The zeta function satisfies*

$$\zeta(z) = \frac{1}{z(z-1)} \frac{\pi^{z/2}}{\Gamma(z/2)} \prod_\rho \left(1 - \frac{z}{\rho}\right),$$

where the product is over the zeroes ρ of ζ in the strip $0 < \mathrm{Re}(z) < 1$, and the factors involving ρ and $1 - \rho$ are grouped together in the infinite product.

Exercise Set 9.3

1. Prove that ζ has a zero of order 1 at each negative even integer.
2. Prove that ξ is increasing on the positive real line.
3. Show that $\xi(0) = 1/2$ even though (9.2.7) suggests it should be 0. Why is there no contradiction here?
4. Calculate $\zeta(0)$ using (9.2.12).
5. Calculate $\zeta(2)$ directly from the definition of ζ, using results from Section 5.5.
6. Prove Corollary 9.3.8 (see Section 8.3 and Exercise 8.3.8).
7. Suppose f is an entire function and the coefficients of the power series expansion of f about 0 are all non-negative. Show that if $g(t)$ and $h(t)$ are positive continuous functions on $[1, \infty)$, then

$$\int_1^\infty g(t) f(zh(t))\, dt$$

is also an entire function of z with non-negative coefficients in its power series expansion about 0, provided this integral exists for all positive real values of z.

8. Show that
$$\int_1^\infty e^{-t}\, t^z \frac{dt}{t}$$
is an entire function of z with all non-negative coefficients in its power series expansion about 0.
9. Verify the claim made in the last sentence of the proof of Theorem 9.3.7 – that is, show that $2\zeta(2)n^{n+1} \le r^r$ if r is sufficiently large and $n \le 5/4 + r/2$.
10. Prove the identity (9.3.10).
11. Show why the product in (9.3.11) converges if the terms are grouped as indicated.

9.4. The Riemann Hypothesis and Prime Numbers

The Riemann Hypothesis is the conjecture that all zeroes of the zeta function in the strip $0 < \text{Re}(z) < 1$ lie on the line $\text{Re}(z) = 1/2$. The significance of this conjecture lies in its connection with the problem of estimating the density of the prime numbers in the set of natural numbers. In this section we will discuss some of the history of the two problems and attempt to illustrate the connection between them without going into too much computational detail. For a more comprehensive and detailed account of the subject see the book by H. M. Edwards [**4**].

We let $\pi(x)$ denote the number of primes less than or equal to the positive real number x. It was recognized by Riemann that there is a connection between the rate of growth of $\pi(x)$ as x increases and the zeroes of the zeta function. That there is some connection between the growth of $\pi(x)$ and the zeta function is seen in the proof that there are infinitely many primes (Exercise 9.2.2).

Based on experimental evidence, Gauss and Legendre conjectured that

$$\lim_{x\to\infty} \frac{\pi(x)}{x} \log x = 1. \tag{9.4.1}$$

This means that the fraction of the natural numbers up to x that are prime, $\dfrac{\pi(x)}{x}$, is asymptotic to $\dfrac{1}{\log x}$ in the sense that their ratio has limit 1.

In his famous 1859 paper Riemann introduced a formula for $\pi(x)$. For a certain constant c, the function Li is defined to be

$$\text{Li}(x) = \begin{cases} 0, & x \le 2; \\ \displaystyle\int_2^x \frac{dt}{\log t} + c, & x > 2. \end{cases}$$

The formula of Riemann for $\pi(x)$ is then

$$\pi(x) = \sum_{n=1}^\infty \text{Li}(x^{1/n}) + \sum_\rho \sum_{n=1}^\infty \text{Li}(x^{\rho/n}) + \text{other}, \tag{9.4.2}$$

where the "other" terms are of lower order in x, and ρ ranges over all zeroes of ζ in the strip $0 < \text{Re}(z) < 1$. Note that the sums over n are actually finite sums for each fixed x and ρ, since $x^{1/n}$ and $x^{\rho/n}$ are less than 2 if n is large enough.

An integration by parts argument shows that the first (and presumably dominant) term in the expansion (9.4.2) may be rewritten as

$$\mathrm{Li}(x) = \frac{x}{\log x} - \int_2^x \frac{dt}{(\log t)^2} + c.$$

The second term in this expression, when divided by $\frac{x}{\log x}$, has limit 0 at infinity (Exercise 9.4.1). Thus, if the remaining terms in Riemann's Formula for $\pi(x)$, when divided by $\frac{x}{\log x}$, also have limit 0 at infinity, and if it is legitimate to take the limit inside the sum in the first term, then (9.4.1) follows.

In dealing with the terms involving the zeroes ρ of ζ in Riemann's Formula, it would be useful if the zeroes of ζ in the strip $0 < \mathrm{Re}(z) < 1$ all satisfied $\mathrm{Re}(z) < r$ for some $r < 1$. Riemann suspected that this was true and, in fact, he conjectured that all such zeroes actually lie on the line $\mathrm{Re}(z) = 1/2$. This is the Riemann Hypothesis.

Actually, Riemann did not give a complete proof of his formula (9.4.2) for $\pi(x)$ and, in fact, he did not even prove that the infinite series in this formula converges. Both facts were eventually proved, but the difficulties involved in these proofs and in determining the contribution of the terms involving the zeroes of ζ to the asymptotic behavior of $\pi(x)$ led to the introduction of another function $\psi(x)$ which also measures the density of primes and which satisfies a simpler and more natural formula analogous to (9.4.2).

Eventually, Hadamard and de la Vallée-Poussin in 1896 proved (9.4.1). It is now known as the Prime Number Theorem. The proofs of Hadamard and de la Vallée-Poussin as well as other classical proofs of this result are based on the fact that there are no zeroes of the zeta function on the line $\mathrm{Re}(z) = 1$. We will present one such proof in the next section.

The Function ψ. As mentioned above, the function $\pi(x)$ is closely related to another function $\psi(x)$ which also measures the density of primes and which has a more straightforward connection to the zeta function. The following equation gives two ways of expressing this function:

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{p \le x} m_p \log p, \tag{9.4.3}$$

where the first sum is over all prime powers $p^m \le x$. For a given prime p, the term $\log p$ appears in this sum as many times as there are positive powers of p less than or equal to x. This number is the m_p which appears in the second sum. It can also be described as the largest number m such that $p^m \le x$.

We will show that if the function ψ satisfies $\lim_{x\to\infty} \psi(x)/x = 1$, then the Prime Number Theorem follows.

Lemma 9.4.1. *Let x and y be real numbers greater than* 1*. Then*

$$\frac{\psi(x)}{\log x} \le \pi(x) \le y + \frac{\psi(x)}{\log y}.$$

Proof. We have

$$\text{(9.4.4)}\qquad \begin{aligned} \pi(x) = \pi(y) + \sum_{y<p\le x} 1 \le y + \sum_{y<p\le x} \frac{\log p}{\log y} \quad \text{if} \quad y < x,\\ \pi(x) \le y \quad \text{if} \quad y \ge x, \end{aligned}$$

where the sums are over primes p in the indicated range.

By definition,

$$\text{(9.4.5)}\qquad \psi(x) = \sum_{p\le x} m_p \log p = \sum_{p\le x} \log p^{m_p}.$$

The sum on the right satisfies the inequalities

$$\sum_{y<p\le x} \log p \le \sum_{p\le x} \log p^{m_p} \le \pi(x)\log x,$$

and so, by (9.4.5),

$$\sum_{y<p\le x} \log p \le \psi(x) \le \pi(x)\log x.$$

The left side of this inequality, combined with (9.4.4), yields

$$\pi(x) \le y + \frac{\psi(x)}{\log y}.$$

The right side, when divided by $\log x$, yields

$$\frac{\psi(x)}{\log x} \le \pi(x).$$

This completes the proof of the lemma. □

Theorem 9.4.2. *If* $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$, *then* $\lim_{x\to\infty} \frac{\pi(x)}{x}\log x = 1$.

Proof. Given $x > 1$, we use the previous lemma with

$$y = \frac{x}{(\log x)^2}.$$

According to that lemma,

$$\frac{\psi(x)}{\log x} \le \pi(x) \le \frac{x}{(\log x)^2} + \frac{\psi(x)}{\log x - 2\log\log x},$$

and so

$$1 \le \pi(x)\frac{\log x}{\psi(x)} \le \frac{1}{\log x}\frac{x}{\psi(x)} + \frac{\log x}{\log x - 2\log\log x}.$$

The first term on the right has limit 0 as $x \to \infty$, while the second term has limit 1. This is because

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1,$$

by hypothesis, while

$$\lim_{x\to\infty} \frac{\log x}{\log x - 2\log\log x} = 1.$$

The latter statement is left as an exercise (Exercise 9.4.4). It follows that

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{\psi(x)} = 1,$$

and, from this, that

$$\lim_{x\to\infty} \frac{\pi(x)}{x} \log x = 1.$$

This completes the proof. □

The above result says that the Prime Number Theorem will follow if we can prove that $\lim_{x\to\infty} \psi(x)/x = 1$. There are direct proofs of this; however, they involve serious difficulties with improper integrals and conditionally convergent series. It turns out that these difficulties can be made to disappear if we follow a similar approach, but use the integral of ψ rather than ψ itself as the main focus of attention. Thus, we set

$$\phi(x) = \int_1^x \psi(u)\, du. \tag{9.4.6}$$

This is another function which measures the density of primes. Furthermore, the Prime Number Theorem follows from an appropriate estimate on its asymptotic behavior.

Properties of ϕ. It turns out that, $\lim_{x\to\infty} \dfrac{\psi(x)}{x} = 1$, from which the Prime Number Theorem follows, provided

$$\lim_{x\to\infty} \frac{\phi(x)}{x^2} = \frac{1}{2}. \tag{9.4.7}$$

This is proved using a kind of reverse L'Hôpital's Rule. Specifically:

Lemma 9.4.3. *Let f be a positive, increasing function on $[1,\infty)$ and suppose $r > 0$. Then*

$$\lim_{x\to\infty} \frac{f(x)}{x^r} = \lim_{x\to\infty} \frac{r+1}{x^{r+1}} \int_1^x f(u)\, du,$$

provided the limit on the right exists and is finite.

Proof. If we were to assume that the limit on the left exists, then the equality would follow from applying L'Hôpital's rule to the limit on the right. However, we are assuming only that the limit on the right exists, and so we must proceed differently (see Exercise 9.4.6).

Using the fact that f is increasing and positive, we conclude that, for $\alpha < 1$ and $\beta > 1$,

$$\frac{1}{(1-\alpha)x} \int_{\alpha x}^x f(u)\, du \le f(x) \le \frac{1}{(\beta-1)x} \int_x^{\beta x} f(u)\, du.$$

On dividing by x^r, this becomes

$$\frac{1}{(1-\alpha)x^{r+1}} \int_{\alpha x}^x f(u)\, du \le \frac{f(x)}{x^r} \le \frac{1}{(\beta-1)x^{r+1}} \int_x^{\beta x} f(u)\, du.$$

If we set $F(x) = \int_1^x f(u)\, du$, then this can be rewritten as

$$\frac{F(x) - F(\alpha x)}{(1-\alpha)x^{r+1}} \le \frac{f(x)}{x^r} \le \frac{F(\beta x) - F(x)}{(\beta-1)x^{r+1}},$$

or as

$$\frac{1}{1-\alpha}\left(\frac{F(x)}{x^{r+1}} - \alpha^{r+1}\frac{F(\alpha x)}{(\alpha x)^{r+1}}\right) \le \frac{f(x)}{x^r} \le \frac{1}{\beta-1}\left(\beta^{r+1}\frac{F(\beta x)}{(\beta x)^{r+1}} - \frac{F(x)}{x^{r+1}}\right).$$

If we set

$$L = \lim_{x\to\infty}\frac{F(x)}{x^{r+1}} = \lim_{x\to\infty}\frac{F(\alpha x)}{(\alpha x)^{r+1}} = \lim_{x\to\infty}\frac{F(\beta x)}{(\beta x)^{r+1}},$$

then the above inequality implies that

$$\frac{1-\alpha^{r+1}}{1-\alpha}L \le \liminf_{x\to\infty}\frac{f(x)}{x^r} \le \limsup_{x\to\infty}\frac{f(x)}{x^r} \le \frac{\beta^{r+1}-1}{\beta-1}L.$$

The lemma then follows from this on taking the limit as α and β approach 1, since both $\dfrac{1-\alpha^{r+1}}{1-\alpha}$ and $\dfrac{\beta^{r+1}-1}{\beta-1}$ have limit $r+1$. □

This leads directly to the following theorem. The details are left to the exercises.

Theorem 9.4.4. *If* $\displaystyle\lim_{x\to\infty}\frac{\phi(x)}{x^2} = 1/2$, *then* $\displaystyle\lim_{x\to\infty}\frac{\pi(x)}{x}\log x = 1$.

Exercise Set 9.4

1. Prove that $\displaystyle\lim_{x\to\infty}\frac{\log x}{x}\int_2^x \frac{dt}{(\log t)^2} = 0$.
2. Prove Part (a) of Lemma 9.5.1.
3. Prove Part (b) of Lemma 9.5.1.
4. Prove that $\displaystyle\lim_{x\to\infty}\frac{\log x}{\log x - 2\log\log x} = 1$.
5. Use Lemma 9.4.3 to prove Theorem 9.4.4.
6. Find functions f and g on $\mathbb{R}$ such that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} \quad \text{exists, but} \quad \lim_{x\to\infty}\frac{f'(x)}{g'(x)} \quad \text{does not.}$$

9.5. A Proof of the Prime Number Theorem

In view of Theorems 9.4.2 and 9.4.4, to prove the Prime Number Theorem, it suffices to show that $\lim_{x\to\infty}\phi(x)/x^2 = 1/2$. The strategy for doing this involves expressing $\phi(x)$ as an integral involving ζ'/ζ. This is where the zeroes of the zeta function come in.

The integral formula relating ζ'/ζ and ϕ is derived from the series expansion (9.3.2) and the following integral formula.

Lemma 9.5.1. *Suppose* $p(z)$ *is a non-constant polynomial and* b *a real number such that no zero of* p *lies on the line* $\mathrm{Re}(z) = b$. *If* $y > 1$, *let* $A = \{z_1, z_2, \cdots, z_n\}$ *be the set of zeroes of* $p(z)$ *that lie to the left of the line* $\mathrm{Re}(z) = b$. *Then*

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{y^z}{p(z)}\,dz = \sum_{k=1}^{n}\mathrm{Res}(y^z/p(z), z_k). \tag{9.5.1}$$

If the set A is empty, then the integral is zero.

If $y < 1$, a similar formula holds, the only differences being: A is replaced by the set of zeroes to the right of $\mathrm{Re}(z) = b$ and the expression on the right is multiplied by -1.

Proof. In Chapter 5 we showed how to use residue theory to calculate the Fourier transforms of certain functions (Theorem 5.3.2). The integral that appears in (9.5.1) is actually the Fourier transform of a function to which Theorem 5.3.2 applies. To see this, we write

$$\frac{y^z}{p(z)} = \frac{\mathrm{e}^{z \log y}}{p(z)} = \frac{y^b}{p(b+it)}\, \mathrm{e}^{it \log y}$$

for $z = b + it$. Then

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{y^z}{p(z)}\, dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{it \log y}\, dt = \frac{1}{\sqrt{2\pi}} f\hat{\ }(-\log y),$$

where f is the restriction to the real line of the meromorphic function

$$f(z) = \frac{y^b}{p(b+iz)}.$$

The function f has limit 0 at infinity since p is a non-constant polynomial. Thus, by Theorem 5.3.2, if $y > 1$, then

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{y^z}{p(z)}\, dz = i \sum_{w \in B} \mathrm{Res}(f(z)\, \mathrm{e}^{iz \log y}, w),$$

where B is the set of poles of f in the upper half-plane. Since

$$f(z)\, \mathrm{e}^{z \log y} = \frac{y^{b+iz}}{p(b+iz)},$$

a calculation of the effect on a residue of the change of variables $z \to b+iz$ (Exercise 9.5.8) shows that

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{y^z}{p(z)}\, dz = \sum_{\lambda \in A} \mathrm{Res}(f(z)\, \mathrm{e}^{iz \log y}, \lambda),$$

where $A = \{\lambda = b + iw : w \in B\}$ – that is, A is the set of zeroes of p that lie to the left of the line $\mathrm{Re}(z) = b$. This completes the proof in the case $y > 1$. The proof in the case $y < 1$ proceeds in the same way. $\square$

Example 9.5.2. Prove that if $b > 0$ and $y > 0$, then

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{y^z}{z(z+1)}\, dz = \begin{cases} 1 - 1/y, & \text{if } y > 1; \\ 0, & \text{if } y < 1. \end{cases}$$

Solution: Since $b > 0$, by the previous lemma, if $y > 1$, the integral is the sum of the residues of $\dfrac{y^z}{z(z+1)}$ at 0 and -1, which is $1 - 1/y$. If $y < 1$, the lemma implies that the integral is 0, since $z(z+1)$ has no zeroes to the right of $\mathrm{Re}(z) = 0$.

Theorem 9.5.3. *If $x > 0$ is not a power of a prime and $b > 1$, then*

$$\phi(x) = -\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz.$$

Proof. We multiply equation (9.3.2) by $\dfrac{x^{z+1}}{z(z+1)}$ and integrate. The result is

$$\begin{aligned} &-\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz \\ &\qquad = \sum_{p^m}\left(\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{x}{p^m}\right)^z \frac{x\log p}{z(z+1)}\,dz\right), \end{aligned} \tag{9.5.2}$$

provided the integrals exist and the integral can be moved inside the summation on the right. Assuming these things for the moment, we have, by the previous example,

$$-\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz = \sum_{p^m\le x}(x-p^m)\log p.$$

The expression on the right is $\phi(x) = \int_1^x \psi(u)\,du$ (Exercise 9.5.1), and so the proof will be complete if we can verify that the integrals in (9.5.2) exist and the integral can be brought inside the summation on the right.

The integrand corresponding to $n = p^m$ on the right in (9.5.2) is less than or equal in modulus to

$$\frac{\log n}{n^b}\frac{x^{b+1}}{b^2+t^2}$$

on the vertical line $z = b + it$. This has the form $c_n f(t)$, where f is a positive integrable function of t on $(-\infty, \infty)$ and $\sum_1^n c_n$ is a convergent series of positive numbers (see Exercise 9.5.2). By Exercise 9.5.4 this implies that the series of integrals on the right in (9.5.2) converges and it converges to the integral on the left. □

A Series Expansion of ϕ. The Prime Number Theorem will follow directly from the following infinite series expansion of ϕ.

Theorem 9.5.4. *There are constants A and B such that*

$$\phi(x) = \frac{x^2}{2} - \sum_{k=1}^{\infty}\frac{x^{1-2k}}{2k(2k-1)} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - Ax + B,$$

where ρ ranges over the zeroes of ζ in the strip $0 < \mathrm{Re}(z) < 1$.

Proof. The integral that appears in Theorem 9.5.3 can also be evaluated by using the infinite product expansion of Theorem 9.3.10. This theorem implies that the logarithmic derivative of ζ can be written as

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{1-z} - \frac{1}{z} + \frac{\log\pi}{2} - \frac{\Gamma'(z/2)}{\Gamma(z/2)} + \sum_{\rho}\frac{1}{z-\rho},$$

where, in the last sum, terms involving ρ and $1-\rho$ must be grouped together for the series to converge. The product formula for $1/\Gamma$ given in Theorem 9.1.10 leads to

$$-\frac{\Gamma'(z/2)}{\Gamma(z/2)} = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{2k+z} - \frac{1}{2}\log(1+1/k)\right).$$

Thus,

$$\frac{\zeta'(z)}{\zeta(z)} = -\frac{1}{z-1} + \frac{\log\pi}{2} + \sum_{k=1}^{\infty}\left(\frac{1}{z+2k} + \frac{1}{2}\log(1+1/k)\right) + \sum_{\rho}\frac{1}{z-\rho}.$$

This simplifies significantly if we subtract $\zeta'(0)/\zeta(0)$:

$$\frac{\zeta'(z)}{\zeta(z)} - \frac{\zeta'(0)}{\zeta(0)} = -1 - \frac{1}{z-1} + \sum_{k=1}^{\infty}\left(\frac{1}{z+2k} - \frac{1}{2k}\right) + \sum_{\rho}\left(\frac{1}{z-\rho} + \frac{1}{\rho}\right),$$

or

$$\frac{\zeta'(z)}{\zeta(z)} = -\frac{z}{z-1} - \sum_{k=1}^{\infty}\frac{z}{2k(z+2k)} + \sum_{\rho}\frac{z}{\rho(z-\rho)} + \frac{\zeta'(0)}{\zeta(0)}. \tag{9.5.3}$$

We next multiply equation (9.5.3) by $\dfrac{1}{2\pi i}\dfrac{x^{z+1}}{z(z+1)}$ and integrate along the line $\mathrm{Re}(z) = b > 1$, obtaining

(9.5.4)

$$\begin{aligned}
\phi(x) &= -\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz\\
&= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{x^{z+1}}{(z-1)(z+1)}\,dz + \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\sum_{k=1}^{\infty}\frac{x^{z+1}}{2k(z+2k)(z+1)}\,dz\\
&\quad - \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\sum_{\rho}\frac{x^{z+1}}{\rho(z-\rho)(z+1)}\,dz - \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\zeta'(0)}{\zeta(0)}\frac{x^{z+1}}{z(z+1)}\,dz.
\end{aligned}$$

Assuming for the moment that the integral can be taken inside each of the infinite sums, the result is

$$\begin{aligned}
\phi(x) &= -\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{x^{z+1}}{z(z+1)}\frac{\zeta'(z)}{\zeta(z)}\,dz\\
&= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{x^{z+1}}{(z-1)(z+1)}\,dz + \sum_{k=1}^{\infty}\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{x^{z+1}}{2k(z+2k)(z+1)}\,dz\\
&\quad - \sum_{\rho}\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{x^{z+1}}{\rho(z-\rho)(z+1)}\,dz - \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{\zeta'(0)}{\zeta(0)}\frac{x^{z+1}}{z(z+1)}\,dz.
\end{aligned}$$

Each of these integrals can be evaluated using Lemma 9.5.1. This leads to

$$\phi(x) = \frac{x^2}{2} - \sum_{k=1}^{\infty}\frac{x^{1-2k}-1}{2k(2k-1)} - \sum_{\rho}\frac{x^{\rho+1}-1}{\rho(\rho+1)} - \frac{\zeta'(0)}{\zeta(0)}x,$$

or

$$\phi(x) = \frac{x^2}{2} - \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - Ax + B,$$

where $A = \dfrac{\zeta'(0)}{\zeta(0)}$ and $B = \displaystyle\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} + \sum_{\rho} \frac{1}{\rho(\rho+1)}$.

It remains to prove that the integral can be taken inside the infinite sums in (9.5.4). The kth term of the first sum is

$$\frac{x^{z+1}}{2k(z+2k)(z+1)}. \tag{9.5.5}$$

The numerator of this fraction is bounded on the vertical line $\mathrm{Re}(z) = b$. With $z = b + it$, we estimate the middle factor of the denominator as follows:

$$\begin{aligned} |z+2k|^2 &= t^2 + (b+2k)^2 = t^2 + b^2 + 4bk + 4k^2 \\ &\geq t^2 + b^2 + 4k^2 = |z|^2 + (2k)^2 \geq 4|z|k. \end{aligned}$$

Thus,

$$|z + 2k| \geq 4|z|^{1/2} k^{1/2}.$$

The right factor of the denominator satisfies $|z+1| \geq |z|$ since $z = b+it$ has positive real part. Hence, the fraction(9.5.5) has modulus less than or equal to a constant times $|z|^{-3/2}k^{-3/2}$. It follows that, in the first infinite sum the integral of each term over $\mathrm{Re}(z) = b$ exists and the integral of the sum is the sum of the integrals and the latter sum is absolutely convergent (see Exercise 9.5.4).

The term involving ρ of the second infinite sum in (9.5.4) is

$$\frac{x^{z+1}}{\rho(z-\rho)(z+1)}. \tag{9.5.6}$$

The numerator is bounded by x^{b+1} on $\mathrm{Re}(z) = b$. With $z = b + iy$, $\rho = \beta + i\gamma$, and $c = b - 1$, we estimate the denominator as follows: Since $|z - \rho| \geq (|y - \gamma| + c)/2$ and $|z + 1| \geq (|y| + b + 1)/2 \geq (|y| + c)/2$, we have

$$|\rho(z-\rho)(z+1)| \geq |\gamma|(|y-\gamma| + c)(|y| + c)/4.$$

Thus, the modulus of (9.5.6) is less than or equal to

$$g(y) = 4x^{b+1}|\gamma|^{-1}h(y) \quad \text{where} \quad h(y) = \frac{1}{(|y-\gamma|+c)(|y|+c)}.$$

If we divide the real line into three subintervals by cutting at $y = 0$ and $y = \gamma$, then, on each of these subintervals, the absolute values in h can be eliminated, and the integral of h with respect to y can be evaluated using the method of partial fractions. The result (Exercise 9.5.5) is that the integral of h over each of the unbounded subintervals is

$$|\gamma|^{-1}\log(|\gamma|/c + 1),$$

while the integral of h over the bounded subinterval is

$$2(|\gamma| + 2c)^{-1}\log(|\gamma|/c + 1) \leq 2|\gamma|^{-1}\log(|\gamma|/c + 1).$$

Thus, the integral of g over $(-\infty, \infty)$ is less than or equal to

$$16x^{b+1}|\gamma|^{-2}\log(|\gamma|/c + 1)$$

and, since $\log(|\gamma|/c) \le (|\gamma|/c)^{1/2}$, this integral is less than or equal to

$$16c^{1/2}x^{b+1}|\gamma|^{-3/2}. \tag{9.5.7}$$

Since, by Corollary 9.3.8, the sequence of roots $\rho = \beta + i\gamma$ has exponent of convergence at most 1, the same is true of the sequence of imaginary parts γ. It follows that the terms (9.5.7) are the terms of a convergent series. Thus, Exercise 9.2.7, modified to cover integrals over $(-\infty, \infty)$, implies that the integral of the sum is the sum of the integrals for the series of terms given by (9.5.6). □

The Prime Number Theorem.

Theorem 9.5.5. *If $\pi(x)$ is the number of primes less than or equal to x, then*

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1.$$

Proof. By Theorems 9.4.4 and 9.4.2, it suffices to prove that $\lim_{x\to\infty} \phi(x)/x^2 = 1/2$.

By Theorem 9.5.4,

$$\frac{\phi(x)}{x^2} = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{x^{-1-2k}}{2k(2k-1)} - \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho+1)} - \frac{A}{x} + \frac{B}{x^2}. \tag{9.5.8}$$

Each of the infinite sums in this expression involves only negative powers of x and each of them is absolutely convergent at $x = 1$. It follows that both infinite series converge uniformly in x on $[1, \infty)$. Thus, in taking the limit of $\phi(x)/x^2$ as $x \to \infty$, we may take the limit inside the infinite sums. Since each term on the right side of (9.5.8) has limit 0 except the term 1/2, the theorem is proved. □

Exercise Set 9.5

1. Verify that $\int_1^x \psi(u)\,du = \sum_{p^m \le x}(x - p^m)\log p$, where p is prime and m is a positive integer.
2. Prove that if p and r are arbitrary positive numbers, there is a constant C such that $\log^p(t) \le Ct^r$ for all $t > 1$.
3. Give a direct proof of Lemma 9.5.1, using residue theory, but without interpreting the integal as a Fourier transform and applying Theorem 5.3.2.
4. For each n, let g_n be be a continuous function on $(-\infty, \infty)$ satisfying $|g_n(t)| \le c_n f(t)$, where $f \ge 0$ is integrable and $\sum_1^\infty c_n$ is a convergent series of positive numbers. Use Exercise 9.2.7 to prove that
$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} g_n(t)\,dt = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} g_n(t)\,dt.$$
5. Verify the claims made near the end of the proof of Theorem 9.5.4 regarding the integral of
$$h(y) = \frac{1}{(|y-\gamma| + c)(|y| + c)}$$
over each of the subintervals of $(-\infty, \infty)$ created by cutting at $y = 0$ and $y = \gamma$.

6. Prove that $\log(s+1) \leq s^{1/2}$ for $s \in (0, \infty)$. This was used near the end of the proof of Theorem 9.5.4.
7. Verify the statement about taking the limit inside the integral in the proof of Theorem 9.5.5. That is, prove that if a series $\sum_{n=1}^{\infty} u_n(x)$ of functions on $[1, \infty)$ converges uniformly absolutely on $[1, \infty)$, then
$$\lim_{x \to \infty} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to \infty} u_n(x),$$
provided each limit on the right converges.
8. Prove that if f is a function analytic in an open set containing $b + iw$ and $g(z) = f(b + iz)$, then g is analytic in an open set containing w and
$$\operatorname{Res}(g, w) = -i \operatorname{Res}(f, b + iw).$$