Preface

This is a book for undergraduates. To be more precise, it is designed for students who have learned the basic principles of analysis, as taught to undergraduates in advanced calculus courses, and are prepared to explore substantial topics in classical analysis. And there is much to explore: Fourier series, orthogonal polynomials, Stirling's formula, the gamma function, Bernoulli numbers, elliptic integrals, Bessel functions, Tauberian theorems, etc. Yet the modern undergraduate curriculum typically does not encompass such topics, except perhaps by way of physical applications. In effect the student struggles to master abstract concepts and general theorems of analysis, then is left wondering what to do with them.

It was not always so. Around 1950 the typical advanced calculus course in American colleges contained a selection of concrete topics such as those just mentioned. However, the development could not be entirely rigorous because the underlying theory of calculus had been deferred to graduate courses. To remedy this unsatisfactory state of affairs, the theory of calculus was moved to the undergraduate level. Textbooks by Walter Rudin and Creighton Buck helped transform advanced calculus to a theoretical study of basic principles. Certainly much was gained in the process, but also much was lost. Various concrete topics, natural sequels to the abstract theory, were crowded out of the curriculum.

The purpose of this book is to recover the lost topics and introduce others, making them accessible at the undergraduate level by building on the theoretical foundation provided in modern advanced calculus courses. My aim has been to develop the mathematics in a rigorous way while holding the prerequisites to a minimum. The exposition probes rather deeply into each topic and is at times intellectually demanding, but every effort has been made to include the background necessary for full comprehension. It is hoped that undergraduate students (and other readers) will find the material exciting and will be inspired to make further studies in the realm of classical analysis.

The book evolved from a collection of notes prepared over the years for students in my own advanced calculus courses, as supplements to the text, presenting concrete topics such as the Wallis product formula, Stirling's formula, the Weierstrass approximation theorem, Euler's sum, and the convergence of Fourier series. Similar handouts for students in differential equations treated the method of Frobenius and the Sturm comparison theorem, both with applications to Bessel functions. Versions of the chapters on infinite products and the gamma function served as background material for students in my complex analysis courses. Feedback from students in all of those courses helped to improve the presentation.

Although the reader is assumed to have acquired a good command of basic principles, a preliminary chapter is included by way of review, and to provide a convenient reference when the principles are invoked in later chapters. The second and third chapters also contain some standard material. The book is designed for individual study, but may serve also as a text for a second-semester course in advanced calculus. There is some progression of material throughout the book, but each chapter is largely self-contained.

The desire to maintain an elementary level, excluding techniques of complex analysis and Lebesgue integration, sometimes led to awkward problems of exposition. In the end I am especially impressed by the power of Fourier analysis. The convergence theorem for Fourier series often turns out to be a viable substitute for the more systematic methods of complex analysis. On the other hand, some topics do call for more advanced techniques. For instance, Lebesgue integrals are the natural setting for Fourier transforms, and the functional equation for the Riemann zeta function can be fully understood only in the context of complex function theory. Nevertheless, I felt compelled to discuss both topics in the book.

Classical analysis is a vast area that defies comprehensive treatment. Any book on the subject must make a selection of topics. With its focus on functions of a single real variable, this book automatically excludes a great many attractive topics. Otherwise, the choices partly reflect my own background and interests.

Many exercises are offered. Mathematics is not a spectator sport; it is learned by participation. My conviction too is that abstract principles are best appreciated when applied to good problems. Therefore, while some exercises are straightforward, I have tried to make others more interesting, more challenging, and consequently more rewarding. Hints are often provided. A few of the problems, not really exercises, come with an asterisk and an invitation to consult the literature. References for each topic are grouped at the end of the relevant chapter.

Historical notes are sprinkled throughout the book. To put a human face on the mathematics, the book includes capsule scientific biographies of the major players and a gallery of portraits. Some historical notes also shed light on the origin and evolution of mathematical ideas. A few discussions of physical applications serve the same purpose.

Many friends and colleagues helped to shape the book. I am especially indebted to Dick Askey, Martin Chuaqui, Dima Khavinson, Jeff Lagarias, Hugh Montgomery, Brad Osgood, Bill Ross, and Harold Shapiro for mathematical suggestions and encouragement as the writing progressed. Alex Lapanowski, a student in the undergraduate honors program at Michigan, read large parts of the manuscript and made valuable suggestions. Dragan Vukotić and his students Irina Arévalo and Diana Giraldo also read portions of the manuscript and spotted a number of small errors. I am enormously grateful to David Ullrich, who read the manuscript carefully, checked many of the exercises, pointed out errors and inaccuracies, and suggested important improvements in the exposition. In particular, he devised the relatively simple proof of Hilbert's inequality presented in Chapter 4. Thanks to all of these readers the book is better, but any remaining faults are the author's responsibility.

In the final stages of preparation the AMS production staff made expert contributions to the book. Special thanks go to the editor Ina Mette for her continual encouragement and willingness, for better or worse, to accommodate my peculiar wishes in regard to content and format. Finally, I must acknowledge the important role of my wife Gay. Without her support the book would not have been written.

Peter Duren