Chapter 7
Tauberian Theorems

It makes no sense to speak of the sum of a divergent infinite series. Nevertheless, a series that is not "too badly divergent" can be assigned a generalized sum in a variety of natural ways. We have already encountered this notion of summability in connection with Abel's theorem, which asserts that every convergent series is Abel summable to its ordinary sum. More generally, an Abelian theorem is any statement to the effect that a method of summability assigns to each convergent series its ordinary sum. A Tauberian theorem goes in the opposite direction and asserts that every summable series which is not too badly divergent is actually convergent. In this chapter we develop some elegant and important Tauberian theorems that will find application to other topics later in the book.

7.1. Summation of divergent series

According to Abel's theorem, if an infinite series $\sum_{n=0}^{\infty} a_n$ converges and has sum s, then the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for $|x| < 1$ and $f(x) \to s$ as $x \to 1-$. The theorem was originally viewed as a device for evaluating the sums of convergent series such as

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2 \qquad \text{and}$$
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}\,.$$

However, the possible existence of the Abel limit when a series diverges suggests a natural way to assign it a generalized sum. For example, the series $\sum_{n=0}^{\infty}(-1)^n$ is divergent, but

$$f(x) = \sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x} \to \frac{1}{2} \qquad \text{as} \quad x \to 1-\,,$$

so we may say that the series is *Abel summable* to the sum $\frac{1}{2}$. Observe that the extended notion of sum retains its linearity. In other words, if $\sum a_n$ is Abel summable to A and $\sum b_n$ is Abel summable to B, then $\sum(a_n + b_n)$ is summable to $A + B$ and $\sum ca_n$ is summable to cA for any constant c. Also, Abel's theorem guarantees that the Abel sum of a convergent series exists and is equal to the ordinary sum.

A similar technique for summation of divergent series can be based on the averages of partial sums. Let

$$s_n = a_0 + a_1 + \cdots + a_n$$

denote the partial sums of the series, and let

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}$$

denote their arithmetic means, also known as the *Cesàro means*. The series $\sum a_n$ is said to be *Cesàro summable* to the sum σ if if $\sigma_n \to \sigma$ as $n \to \infty$. Recall (*cf.* Chapter 1, Exercise 13) that $\sigma_n \to s$ whenever $s_n \to s$, but the sequence $\{\sigma_n\}$ may converge when $\{s_n\}$ does not. For instance, the divergent series $\sum_{n=0}^{\infty}(-1)^n$ has partial sums $s_n = 1$ when n is even and $s_n = 0$ when n is odd, so it is Cesàro summable to the sum $\frac{1}{2}$, which is the same as its Abel sum. The series $\sum_{n=1}^{\infty}(-1)^{n+1}n$ is not Cesàro summable, but is Abel summable to $\frac{1}{4}$. (See Exercise 1.)

With this meager evidence in hand, one may suspect that Abel summation is more powerful than Cesàro summation, which is indeed the case. In 1880, Georg Frobenius [**2**] strengthened Abel's theorem by showing that a series is Abel summable if it is Cesàro summable. Here is a more precise statement, in notation introduced above.

Frobenius' Theorem. *If $\sigma_n \to \sigma$ as $n \to \infty$, then the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ and $f(x) \to \sigma$ as $x \to 1-$.*

Proof. Two summations by parts give

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty}(s_n - s_{n-1})x^n = (1-x)\sum_{n=0}^{\infty} s_n x^n$$

$$= (1-x)^2 \sum_{n=0}^{\infty}(s_0 + s_1 + \cdots + s_n)x^n = (1-x)^2 \sum_{n=0}^{\infty}(n+1)\sigma_n x^n\,.$$

In view of the identity

$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}\,,$$

this may be written as

$$\sum_{n=0}^{\infty} a_n x^n - \sigma = (1-x)^2 \sum_{n=0}^{\infty}(n+1)(\sigma_n - \sigma)x^n\,.$$

But $\sigma_n \to \sigma$, so for each $\varepsilon > 0$ there is an integer N such that

$$|\sigma_n - \sigma| < \varepsilon \qquad \text{for all} \ \ n > N\,.$$

Then for $0 < x < 1$,

$$\begin{aligned}\left|\sum_{n=0}^{\infty}(n+1)(\sigma_n-\sigma)x^n\right| &\le \sum_{n=0}^{N}(n+1)|\sigma_n-\sigma|x^n + \sum_{n=N+1}^{\infty}(n+1)|\sigma_n-\sigma|x^n \\ &\le \sum_{n=0}^{N}(n+1)|\sigma_n-\sigma| + \frac{\varepsilon}{(1-x)^2}\,,\end{aligned}$$

and so

$$\left|\sum_{n=0}^{\infty} a_n x^n - \sigma\right| \le (1-x)^2 \sum_{n=0}^{N}(n+1)|\sigma_n-\sigma| + \varepsilon < 2\varepsilon$$

when x is sufficiently close to 1. In other words, $f(x) \to \sigma$ as $x \to 1-$. □

Many other summation procedures have been introduced, and the corresponding "Abelian theorems" proved, asserting that whenever a series is convergent or summable by some method, it must be summable to the same sum by another more powerful method. For instance, a series is said to be *Borel summable* if the limit

$$\lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} \frac{1}{n!}\, s_n x^n$$

exists. The method is named for Émile Borel, who introduced it in 1899 and pointed out the corresponding Abelian theorem, that a convergent series is Borel summable to its ordinary sum (*cf.* Exercise 2). Borel summability arises naturally in complex function theory, especially in problems of analytic continuation.

To give one more example, a series is said to be *Lambert summable* if the limit

$$\lim_{x\to 1-} \sum_{n=0}^{\infty} a_n L(x^n)$$

exists, where

$$L(x) = \frac{x\log(1/x)}{1-x} \qquad \text{for} \quad 0 < x < 1\,, \quad \text{and} \quad L(1) = 1\,,$$

is known as the *Lambert kernel.* Note that $L(x) \to 1$ as $x \to 1-$. Lambert summability has important applications to number theory and is involved in proofs of the prime number theorem.

7.2. Tauber's theorem

An infinite series may be summable by various methods and yet fail to converge. We have seen elementary examples of divergent series that are Cesàro or Abel summable. In the converse direction, however, there is a general principle that if a series is summable by some method and is not "too badly divergent", then it is actually convergent. The first result of this type was found in 1897 by Alfred Tauber (1866–1942), an Austrian mathematician who later specialized in actuarial mathematics. Tauber [**12**] showed that the converse of Abel's theorem is valid under the additional hypothesis that $na_n \to 0$.

Tauber's Theorem. *Suppose $na_n \to 0$ as $n \to \infty$, so that the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges in the interval $(-1,1)$. If $f(x) \to A$ as $x \to 1-$, then the infinite series $\sum_{n=0}^{\infty} a_n$ converges to the sum A.

Briefly, Tauber's theorem asserts that if a series is Abel summable and its terms satisfy the additional condition $na_n \to 0$, then the series is convergent. The theorem uncovers a remarkable phenomenon. Not only does the Abel method fail to sum series that diverge too rapidly, but it also fails to sum series whose divergence is too slow. For example, if $|a_n| \le 1/(n\log n)$, Tauber's theorem tells us that the series $\sum a_n$ is not Abel summable unless it is convergent. The same is true if $|a_n| \le 1/n$, but the proof is based on a stronger form of Tauber's theorem requiring only that the sequence $\{na_n\}$ be bounded. More about this later.

Proof of Tauber's theorem. With the notation $s_n = \sum_{k=0}^{n} a_k$ for the partial sums, we can write

$$s_n - f(x) = \sum_{k=1}^{n} a_k(1 - x^k) - \sum_{k=n+1}^{\infty} a_k x^k .$$

Taking $0 < x < 1$, we apply the inequality

$$1 - x^k = (1 - x)(1 + x + \cdots + x^{k-1}) \leq k(1 - x)$$

to conclude that

$$|s_n - f(x)| \leq (1 - x) \sum_{k=1}^{n} k|a_k| + \sum_{k=n+1}^{\infty} |a_k| x^k .$$

Now let $\varepsilon > 0$ be given. By hypothesis, we may choose n large enough that $k|a_k| < \varepsilon$ for all $k > n$. Then

$$\sum_{k=n+1}^{\infty} |a_k| x^k < \varepsilon \sum_{k=n+1}^{\infty} \frac{1}{k} x^k < \frac{\varepsilon}{n} \sum_{0}^{\infty} x^k = \frac{\varepsilon}{n(1-x)} .$$

Now choose $x = x_n = 1 - \frac{1}{n}$, so that $1 - x_n = \frac{1}{n}$ and $x_n \to 1$ as $n \to \infty$. This gives the estimate

$$|s_n - f(x_n)| \leq \frac{1}{n} \sum_{k=1}^{n} k|a_k| + \varepsilon$$

for n sufficiently large. But since $k|a_k| \to 0$, it follows that the arithmetic means

$$\frac{1}{n} \sum_{k=1}^{n} k|a_k|$$

also tend to 0 as $n \to \infty$. Therefore, $|s_n - f(x_n)| < 2\varepsilon$ for all n sufficiently large. In view of the hypothesis that $f(x) \to A$ as $x \to 1-$, this proves that $s_n \to A$. □

7.3. Theorems of Hardy and Littlewood

Beginning around 1910, the two British analysts G. H. Hardy (1877–1947) and J. E. Littlewood (1885–1977) began a series of investigations initially inspired by Tauber's theorem. First Hardy [**4**] proved an analogue of Tauber's theorem by postulating the more restrictive Cesàro summability instead of Abel summability but requiring only that the sequence $\{na_n\}$ be bounded. His result can be stated as follows.

Hardy's Theorem. *If the infinite series $\sum_{n=0}^{\infty} a_n$ is Cesàro summable and $\{na_n\}$ is bounded, then the series is convergent.*

Hardy conjectured that Tauber's theorem remains valid if the condition $na_n \to 0$ is weakened to the boundedness of $\{na_n\}$. His Cambridge colleague Littlewood [**11**] then verified the conjecture by proving the following theorem, which strengthens the theorems of both Tauber and Hardy.

Littlewood's Theorem. *Suppose the sequence $\{na_n\}$ is bounded, so that the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges in the interval $(-1, 1)$. If $f(x) \to A$ as $x \to 1-$, then the infinite series $\sum_{n=0}^{\infty} a_n$ converges to the sum A.

This work marked the beginning of the famous Hardy–Littlewood collaboration, which extended over a period of 35 years and was one of the most fruitful collaborations in the history of mathematics. Together these two mathematicians made seminal contributions to a variety of subjects including number theory, Fourier series, and functions of a complex variable. They also developed a series of results analogous to Tauber's theorem, which they called "Tauberian theorems". A *Tauberian theorem* is the converse of an Abelian theorem under a supplementary growth condition called a *Tauberian condition*. For instance, Littlewood's Tauberian theorem asserts that the converse of Abel's theorem is true under the Tauberian condition that $\{na_n\}$ is bounded.

In 1914, Hardy and Littlewood together found a Tauberian theorem that goes from Abel to Cesàro summability and therefore provides a partial converse of Frobenius' theorem.

Hardy–Littlewood Theorem. *If $f(x) = \sum_{n=0}^{\infty} a_n x^n \to A$ as $x \to 1-$ and $s_n \geq 0$ for all n, then $\sigma_n \to A$ as $n \to \infty$.*

It is easy to see that the Hardy–Littlewood theorem remains valid more generally if $s_n \geq -C$ for some constant C, which is certainly true if the partial sums s_n are bounded. Hardy and Littlewood also found that the Tauberian condition in Littlewood's theorem, the boundedness of $\{na_n\}$, can be relaxed to a one-sided condition $a_n \geq -C/n$.

The first proofs of Littlewood's theorem and of the Hardy–Littlewood theorem were similar and were much more difficult than those of Tauber's or Hardy's theorem. For a long time no simplifications were found and these results were considered to be quite deep. Then in 1930, Karamata [**8**] found a clever and surprisingly simple approach based on the Weierstrass approximation theorem. Karamata's elegant proof will be discussed in the next section. His strategy was to prove the Hardy–Littlewood theorem, then to deduce Littlewood's theorem from it. Littlewood's original approach was to prove Cesàro summability and then appeal to Hardy's theorem to conclude that the series converges. Some years later, Wielandt [**14**] found a simple refinement of Karamata's method that avoids the detour through Cesàro summability and proves Littlewood's theorem directly. The details are described in the next section.

7.4. Karamata's proof

We now turn to Karamata's proof of the Hardy–Littlewood and Littlewood Tauberian theorems. Our treatment is adapted from the books of Titchmarsh [**13**] and Korevaar [**10**]. Karamata's method is based on the Weierstrass approximation theorem, which can be invoked to establish the following lemma.

Lemma. *Let $g(x)$ be continuous on the interval $[0,1]$ except for a possible jump-discontinuity at a point $c \in (0,1)$. Then for each $\varepsilon > 0$ there exist polynomials $P(x)$ and $Q(x)$ such that $P(x) < g(x) < Q(x)$ and*

$$\int_0^1 \big[g(x) - P(x)\big]\,dx < \varepsilon \qquad \int_0^1 \big[Q(x) - g(x)\big]\,dx < \varepsilon\,.$$

Proof. Suppose that $g(x)$ has a jump-discontinuity at $c \in (0,1)$, and let $g(c+)$ and $g(c-)$ denote the right- and left-hand limits. Suppose without loss of generality that $g(c-) \le g(c+)$. For fixed $\delta \in (0,c)$, let $\ell(x)$ be the linear function such that

$$\ell(c-\delta) = g(c-\delta) + \varepsilon/2\,, \qquad \ell(c) = g(c+) + \varepsilon/2\,.$$

Define

$$\phi(x) = \begin{cases} g(x) + \varepsilon/2 & \text{if } \ 0 \le x < c-\delta \ \text{ or } \ c < x \le 1 \\ \max\{\ell(x),\, g(x)+\varepsilon/2\} & \text{if } \ c-\delta \le x \le c\,. \end{cases}$$

Then $\phi(x)$ is continuous in $[0,1]$ and $\phi(x) \ge g(x)+\varepsilon/2$. Choose a polynomial $Q(x)$ such that $|Q(x) - \phi(x)| < \varepsilon/2$ for all $x \in [0,1]$. Then $Q(x) > g(x)$ and

$$\int_0^1 \big[Q(x) - g(x)\big]\,dx < \varepsilon$$

if δ is sufficiently small. A similar construction produces a polynomial $P(x) < g(x)$ with $\int_0^1 [g(x) - P(x)]\,dx < \varepsilon$. □

With the lemma in hand, we first prove the Hardy–Littlewood theorem, which is a somewhat easier than Littlewood's theorem.

Proof of Hardy–Littlewood theorem. By hypothesis,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} s_n x^n \to A \qquad \text{as } x \to 1- \ .$$

We claim that this implies, more generally, that

$$\lim_{x\to 1-} (1-x) \sum_{n=0}^{\infty} s_n\, x^n\, p(x^n) = A \int_0^1 p(t)\, dt \tag{1}$$

for every polynomial $p(x)$. To see this, it suffices to consider monomials $p(x) = x^k$. Then the left-hand side is

$$\begin{aligned}(1-x) \sum_{n=0}^{\infty} s_n\, x^{n+kn} &= \frac{1-x}{1-x^{k+1}} \left\{ (1-x^{k+1}) \sum_{n=0}^{\infty} s_n (x^{k+1})^n \right\} \\ &\to \frac{A}{k+1} = A \int_0^1 p(t)\, dt \qquad \text{as } x \to 1- \,,\end{aligned}$$

as claimed. Invoking the lemma, we can now infer from (1) that

$$\lim_{x\to 1-} (1-x) \sum_{n=0}^{\infty} s_n\, x^n\, g(x^n) = A \int_0^1 g(t)\, dt \tag{2}$$

if g is continuous in $[0,1]$ except for a possible jump-discontinuity. It is here that the hypothesis $s_n \ge 0$ is needed. To carry out the details, let $\varepsilon > 0$ be given and choose polynomials P and Q as in the lemma. Then

$$\sum_{n=0}^{\infty} s_n\, x^n\, P(x^n) \le \sum_{n=0}^{\infty} s_n\, x^n\, g(x^n) \le \sum_{n=0}^{\infty} s_n\, x^n\, Q(x^n) \,, \qquad 0 < x < 1 \,.$$

Since $s_n \ge 0$ implies $A \ge 0$, it follows that

$$\begin{aligned}\limsup_{x\to 1-} (1-x) \sum_{n=0}^{\infty} s_n\, x^n\, g(x^n) &\le \lim_{x\to 1-} (1-x) \sum_{n=0}^{\infty} s_n\, x^n\, Q(x^n) \\ &= A \int_0^1 Q(t)\, dt \le A \int_0^1 g(t)\, dt + A\varepsilon \,.\end{aligned}$$

Letting $\varepsilon \to 0$, we conclude that

$$\limsup_{x\to 1-} (1-x) \sum_{n=0}^{\infty} s_n\, x^n\, g(x^n) \le A \int_0^1 g(t)\, dt \,.$$

In a similar way we find that

$$\liminf_{x\to 1-} (1-x)\sum_{n=0}^{\infty} s_n\, x^n\, g(x^n) \geq A\int_0^1 g(t)\,dt\,.$$

Combining the last two inequalities, we arrive at a proof of (2).

Now choose

$$g(t) = \begin{cases} 0\,, & 0 \leq t < 1/e \\ 1/t\,, & 1/e \leq t \leq 1\,. \end{cases}$$

Then

$$\int_0^1 g(t)\,dt = \int_{1/e}^1 1/t\,dt = 1\,.$$

Let $x_N = e^{-1/N}$ and observe that $x_N^n \geq 1/e$ if and only if $n \leq N$, so that

$$\sum_{n=0}^{\infty} s_n\, x_N^n\, g(x_N^n) = \sum_{n=0}^{N} s_n = (N+1)\sigma_N\,.$$

The relation (2) now shows that

$$(N+1)(1-x_N)\sigma_N = (1-x_N)\sum_{n=0}^{\infty} s_n\, x_N^n\, g(x_N^n) \to A\int_0^1 g(t)\,dt = A$$

as $N\to\infty$, since $x_N \to 1$. But

$$(N+1)(1-x_N) = (N+1)(1-e^{-1/N}) \to 1 \qquad \text{as } N\to\infty\,,$$

so this proves that $\sigma_N \to A$. □

Proof of Littlewood's theorem. For each polynomial

$$P(x) = \sum_{k=1}^{m} b_k x^k \qquad \text{with } \ P(0) = 0\,,$$

the hypothesis that $f(x)\to A$ implies

$$\sum_{n=0}^{\infty} a_n P(x^n) = \sum_{k=1}^{m} b_k \sum_{n=0}^{\infty} a_n x^{kn} \to A\sum_{k=1}^{m} b_k = P(1)A \tag{3}$$

as $x\to 1-$. Now choose the "cutoff function"

$$g(t) = \begin{cases} 0\,, & 0 \leq t < 1/e \\ 1\,, & 1/e \leq t \leq 1\,, \end{cases}$$

so that

$$s_N = \sum_{n=0}^{N} a_n = \sum_{n=0}^{\infty} a_n\, g(x_N^n)\,, \qquad \text{where} \;\; x_N = e^{-1/N}\,.$$

To prove that $s_N \to A$ as $N \to \infty$, it will suffice to show that

$$\sum_{n=0}^{\infty} a_n\, g(x^n) \to g(1)A = A \qquad \text{as} \;\; x \to 1- \;.$$

In order to show that

$$\limsup_{x\to 1-} \sum_{n=0}^{\infty} a_n\, g(x^n) \le A\,, \tag{4}$$

we will apply the relation (3) to a polynomial P with the properties $P(0) = 0$, $P(1) = 1$, and $P(t) \ge g(t)$ for $0 \le t \le 1$. For this purpose we choose a polynomial Q with

$$Q(t) \ge \frac{g(t) - t}{t(1-t)} = h(t)\,, \qquad 0 < t < 1\,,$$

which gives a polynomial $P(t) = t + t(1-t)Q(t)$ with the required properties. The Tauberian condition $|na_n| \le C$ implies that $na_n \ge -C$, so that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n g(x^n) - \sum_{n=0}^{\infty} a_n P(x^n) = -\sum_{n=1}^{\infty} a_n \big[P(x^n) - g(x^n)\big] \\
&\le C \sum_{n=1}^{\infty} \frac{1}{n}\big[P(x^n) - g(x^n)\big] \;\le\; C \sum_{n=1}^{\infty} \frac{1-x}{1-x^n}\big[P(x^n) - g(x^n)\big] \\
&= C(1-x)\sum_{n=1}^{\infty} \phi(x^n)\,, \qquad \text{where} \;\; \phi(t) = \frac{P(t) - g(t)}{1-t}\,.
\end{aligned}$$

Here we have used the elementary inequality

$$\frac{1-x^n}{1-x} = 1 + x + \cdots + x^{n-1} \le n \qquad \text{for} \quad 0 \le x < 1\,.$$

We claim now that

$$\lim_{x\to 1-} (1-x)\sum_{n=1}^{\infty} \phi(x^n) = \int_0^1 \frac{\phi(t)}{t}\, dt\,,$$

since $\phi(t)$ has a continuous extension to the interval $[0,1]$ except for the jump-discontinuity at $t = 1/e$. Indeed, for any fixed $x \in (0,1)$, the integral is approximated by its Riemann sum

$$\sum_{n=1}^{\infty} \frac{\phi(x^n)}{x^n}(x^n - x^{n+1}) = (1-x)\sum_{n=1}^{\infty} \phi(x^n)\,,$$

which tends to the integral as $x \to 1-$. But

$$\int_0^1 \frac{\phi(t)}{t}\,dt = \int_0^1 \frac{\big[P(t)-t\big]-\big[g(t)-t\big]}{t(1-t)}\,dt = \int_0^1 \big[Q(t)-h(t)\big]\,dt\,.$$

Since $Q(t) \geq h(t)$, the last integral is positive, and in view of the lemma it can be made arbitrarily small by suitable choice of the polynomial Q. Putting everything together and recalling from (3) that

$$\lim_{x\to 1-} \sum_{n=0}^{\infty} a_n P(x^n) = A\,,$$

we conclude that (4) holds.

A similar argument shows that

$$\liminf_{x\to 1-} \sum_{n=0}^{\infty} a_n g(x^n) \geq A\,. \tag{5}$$

For this we choose $P(t) = t + t(1-t)Q(t)$, where Q is a polynomial with

$$Q(t) \leq h(t) = \frac{g(t)-t}{t(1-t)} \qquad \text{for} \quad 0 < t < 1\,.$$

Then $P(0) = 0$, $P(1) = 1$, and $P(t) \leq g(t)$ for $0 \leq t \leq 1$. Hence the Tauberian condition $na_n \geq -C$ gives

$$\sum_{n=0}^{\infty} a_n g(x^n) - \sum_{n=0}^{\infty} a_n P(x^n) \geq -C\sum_{n=1}^{\infty} \frac{1}{n}\big[g(x^n)-P(x^n)\big] \geq C(1-x)\sum_{n=1}^{\infty}\phi(x^n)\,,$$

since $1 - x^n \leq n(1-x)$ and now

$$\phi(t) = \frac{P(t)-g(t)}{1-t} \leq 0 \qquad \text{for} \quad 0 < t < 1\,.$$

Consequently, we may conclude as before that

$$\liminf_{x\to 1-} \sum_{n=0}^{\infty} a_n\, g(x^n) \geq A + C\int_0^1 \big[Q(t)-h(t)\big]\,dt\,.$$

Another appeal to the lemma shows that the last integral can be made arbitrarily close to zero with suitable choice of the polynomial Q, which gives (5). Combining (4) and (5), we see that

$$\lim_{x\to 1-} \sum_{n=0}^{\infty} a_n\, g(x^n) = A\,.$$

As previously noted, this implies that $s_n \to A$, and so the theorem is proved. It should be observed that the proof used only the weaker Tauberian condition $na_n \geq -C$. □

Jovan Karamata (1902–1967) was a Yugoslavian, born in Zagreb (Croatia) and trained in mathematics at the University of Novi Sad (Serbia) and at Belgrade University, where he received a doctoral degree in 1926. Four years later his two-page paper [**8**] appeared, outlining his totally new, relatively simple approach to Tauberian theorems. The paper created an immediate sensation and brought him international recognition. Later he made other significant contributions to Tauberian theory (*cf.* Korevaar [**10**]). He was a professor at Belgrade University until 1951, then spent the rest of his career in Switzerland at the University of Geneva.

7.5. Hardy's power series

The following problem appeared in the puzzles column of *Emissary*, a newsletter of the Mathematical Sciences Research Institute [**1**].

Problem. *For positive real x less than 1, define*

$$f(x) = x - x^2 + x^4 - x^8 + x^{16} - \dots\,.$$

Does $f(x)$ have a limit as x approaches 1 from below? If so, what is the limit?

Before considering the problem in earnest, let us make a few simple observations. First of all, by pairing terms one sees that

$$f(x) = (x - x^2) + (x^4 - x^8) + \cdots > 0$$

and

$$f(x) = x - (x^2 - x^4) - (x^8 - x^{16}) - \cdots < x$$

for $0 < x < 1$. In particular, $f(x)$ is bounded in the interval $[0, 1]$. The identity $f(x) = x - f(x^2)$ shows that if $f(x)$ has a limit as $x \to 1-$, the limit must be $\frac{1}{2}$. Iteration of the identity shows that $f(x^4) < f(x)$, which suggests that $f(x)$ is an increasing function in the interval $0 < x < 1$. If so, then by the monotone boundedness theorem $f(x)$ does have a limit, and $f(x) \to \frac{1}{2}$ as $x \to 1-$. Figure 1 displays the graph of

$$\sum_{k=0}^{30} (-1)^k x^{2^k}, \qquad 0 < x < 1,$$

as generated by *Mathematica*. The figure appears to confirm that $f(x)$ increases to $\frac{1}{2}$ as $x \to 1-$.

However, the limit does not exist! Hardy [**3**] studied the series in 1907 and found by intricate analysis that as x tends to 1, the function $f(x)$ undergoes miniscule oscillations about the value $\frac{1}{2}$ and does not converge. A closer inspection of Figure 1 actually reveals the small oscillations. Figure 2 displays a graph of the same partial sum (up to $k = 30$), magnified to clarify the oscillations near the point $(1, \frac{1}{2})$.

In fact, the nonexistence of the limit is a direct consequence of a result of Hardy and Littlewood [**7**] known as the "high-indices" theorem, a remarkable Tauberian theorem for lacunary power series. Here is the statement.

High-Indices Theorem. *If $f(x) = \sum_{k=1}^{\infty} a_k x^{n_k}$ for $0 < x < 1$, where the exponents n_k are positive integers satisfying a condition of the form $n_{k+1}/n_k \geq q > 1$, and if $f(x) \to A$ as $x \to 1-$, then the infinite series $\sum_{k=1}^{\infty} a_k$ converges to A.*

Hardy and Littlewood obtained this result in 1926, relatively late in the development of basic Tauberian theory. Their discovery appears to have been inspired in part by the special power series (discussed above)

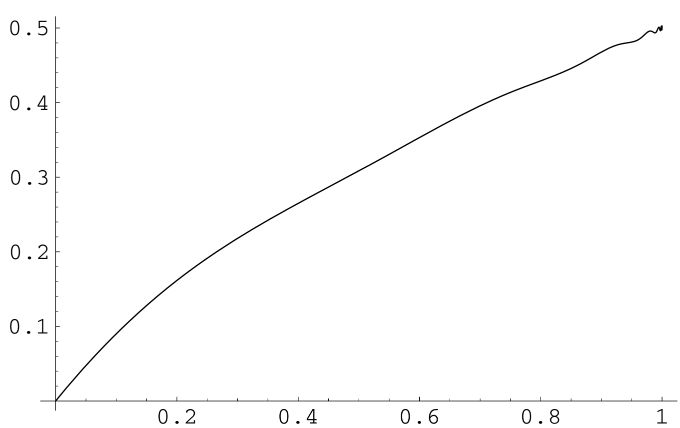

Figure 1. Hardy's sum for $0 \leq x \leq 1$.

that Hardy had studied in 1907. The most astounding feature of the high-indices theorem is that lacunarity alone serves as a Tauberian condition. For Hardy's power series we see that $n_k = 2^k$ is sufficiently lacunary, and the partial sums of the series $\sum a_k$ are alternately equal to 0 and 1, so the high-indices theorem shows that $f(x)$ cannot tend to a limit as $x \to 1-$.

A proof of the high-indices theorem is beyond the scope of this book, but can be found for instance in the book by Korevaar [**10**], p. 50 ff. Alternatively, we can appeal to the more elementary theorem of Hardy and Littlewood, as stated at the end of Section 7.3, to show that Hardy's sum does not tend to a limit as $x \to 1-$. For the function

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{k=0}^{\infty} (-1)^k x^{2^k},$$

it is seen that all partial sums $s_n = a_1 + \cdots + a_n$ are either 0 or 1, so the hypothesis $s_n \geq 0$ of the Hardy-Littlewood theorem is clearly satisfied. Therefore, if we can show that the sequence $\{\sigma_n\}$ of Cesàro means does not converge, it will follow that $f(x)$ does not converge as $x \to 1-$. In fact, a direct calculation (*cf.* Exercise 3) reveals that

$$\liminf_{n\to\infty} \sigma_n = \tfrac{1}{3} \qquad \text{and} \qquad \limsup_{n\to\infty} \sigma_n = \tfrac{2}{3},$$

so the series $\sum a_n$ is not Cesàro summable. Thus by the Hardy-Littlewood theorem, it is not Abel summable. In other words, $f(x)$ does not tend to a limit as $x \to 1-$.

A recent paper by Keating and Reade [**9**] offers further analysis of Hardy's power series and similar examples. There the authors apply the

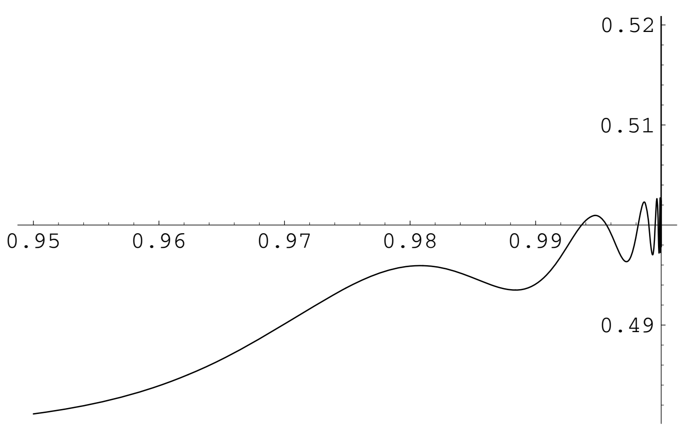

Figure 2. Hardy's sum for $0.95 \leq x \leq 1$.

Poisson summation formula of Fourier analysis (see Chapter 8) to prove that as $x \to 1-$, Hardy's function undergoes persistent oscillations about the value $\frac{1}{2}$ with approximate amplitude 0.00275, which is consistent with Figure 2.

Hardy's classic book [**5**] contains a wealth of information about Abelian and Tauberian theorems. Korevaar's more recent book [**10**] is a comprehensive reference for Tauberian theorems and their applications.

Exercises

1. Show that the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}n = 1 - 2 + 3 - 4 + \dots$$

is not Cesàro summable, but is Abel summable to $\frac{1}{4}$.

2. Show that every convergent series is Borel summable to its ordinary sum. Specifically, show that if $s_n \to s$, then

$$e^{-x}\sum_{n=0}^{\infty}\frac{1}{n!}\,s_n\,x^n \to s \qquad \text{as} \quad x \to \infty\,.$$

3. Consider Hardy's power series

$$\sum_{n=1}^{\infty} a_n\,x^n = \sum_{k=0}^{\infty}(-1)^k x^{2^k}\,.$$

Thus $a_n = (-1)^k$ if $n = 2^k$ and $a_n = 0$ otherwise. Show directly that the series $\sum a_n$ is not Cesàro summable by computing

$$\liminf_{n\to\infty}\sigma_n = \tfrac{1}{3} \qquad \text{and} \qquad \limsup_{n\to\infty}\sigma_n = \tfrac{2}{3}\,.$$

4. Show that

$$\sum_{k=0}^{\infty}(-1)^k x^{k^2} \to \frac{1}{2} \qquad \text{as} \quad x \to 1-\,.$$

Suggestion. Show that the series $\sum a_n$ is Cesàro summable, where $a_n = (-1)^k$ if $n = k^2$ and $a_n = 0$ otherwise.

5. (a) Show directly, without appeal to a Tauberian theorem, that the infinite series $\sum_{n=1}^{\infty}\frac{1}{n}$ is not Cesàro summable.

(b) Show directly that the same series is not Abel summable.

6. Show that for $a_n \geq 0$, the series $\sum a_n$ is Abel summable only if it is convergent.

7. (*a*) Prove the Abelian theorem for Lambert summability. In other words, show that if

$$\sum_{n=0}^{\infty} a_n = s\,, \qquad \text{then} \qquad \lim_{x\to 1-} \sum_{n=0}^{\infty} a_n L(x^n) = s\,,$$

where $L(x)$ is the Lambert kernel (*cf.* Section 7.1).

Hint. Apply a summation by parts and observe that $L(x)$ increases from 0 to 1 in the interval $[0, 1]$.

(*b*) Make the change of variables $x = e^{-t}$ to recast the conclusion in the form

$$a_0 + \lim_{t\to 0+} \sum_{n=1}^{\infty} \frac{n a_n t}{e^{nt} - 1} = s\,.$$

8. Show that the series

$$\sum_{n=1}^{\infty} \cos n\theta \qquad \text{and} \qquad \sum_{n=1}^{\infty} \sin n\theta\,, \qquad 0 < \theta < 2\pi\,,$$

are Abel summable to $-\frac{1}{2}$ and $\frac{1}{2}\cot\frac{\theta}{2}$, respectively.

Hint. Consider real and imaginary parts of the geometric series $\sum_{n=1}^{\infty} z^n$, where $z = re^{i\theta}$.

References

[1] E. R. Berlekamp and J. P. Buhler (editors), Puzzles Column, *Emissary*, Mathematical Sciences Research Institute, Berkeley, Fall 2004 and Spring 2005.

[2] G. Frobenius, "Ueber die Leibnitzsche Reihe", *J. Reine Angew. Math.* **89** (1880), 262–264.

[3] G. H. Hardy, "On certain oscillating series", *Quart. J. Math.* **38** (1907), 269–288.

[4] G. H. Hardy, "Theorems relating to the summability and convergence of slowly oscillating series", *Proc. London Math. Soc.* **8** (1910), 301–320.

[5] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[6] G. H. Hardy and J. E. Littlewood, "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *Proc. London Math. Soc.* **13** (1914), 174–191.

[7] G. H. Hardy and J. E. Littlewood, "A further note on the converse of Abel's theorem", *Proc. London Math. Soc.* **25** (1926), 219–236.

[8] J. Karamata, "Über die Hardy–Littlewoodschen Umkehrung des Abelschen Stetigkeitssatzes", *Math. Zeitschrift* **32** (1930), 319–320.

[9] J. P. Keating and J. B. Reade, "Summability of alternating gap series", *Proc. Edinburgh Math. Soc.* **43** (2000), 95–101.

[10] J. Korevaar, *Tauberian Theory: A Century of Developments*, Springer-Verlag, Berlin–Heidelberg, 2004.

[11] J. E. Littlewood, "The converse of Abel's theorem on power series", *Proc. London Math. Soc.* **9** (1911), 434–448.

[12] A. Tauber, "Ein Satz aus der Theorie der unendlichen Reihen", *Monatsh. Math. u. Phys.* **8** (1897), 273–277.

[13] E. C. Titchmarsh, *The Theory of Functions*, Second edition, Oxford University Press, London, 1939.

[14] H. Wielandt, "Zur Umkehrung des Abelschen Stetigkeitssatzes", *Math. Zeitschrift* **56** (1952), 206–207.

Chapter 9
The Gamma Function

The gamma function is a continuous extension of the factorial function. It is an important function with many remarkable properties. It can be represented as an integral and as an infinite product. In this chapter we develop the basic theory of the gamma function and its relation to the beta function. By way of background we begin with an integral that plays a central role in probability theory.

9.1. Probability integral

The integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

is important in probability theory because of its connection with the Gaussian or normal distribution. It is evaluated by a well known device that makes use of area integrals. Since the integrand is an even function, it is equivalent to show that

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\,. \tag{1}$$

To evaluate the integral in (1), we consider the function

$$f(x,y) = e^{-x^2} e^{-y^2}$$

and the three regions of integration shown in Figure 1. With the notation D_1 and D_2 for for the quarter-disks with respective radii R and $\sqrt{2}R$, and

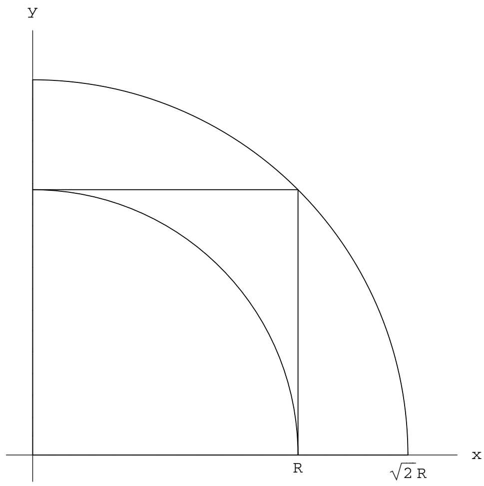

Figure 1. Three regions of integration.

S for the square with side-length R, we observe that $D_1 \subset S \subset D_2$. Since $f(x,y) > 0$, it is therefore clear that

$$\iint_{D_1} f(x,y)\,dA \le \iint_S f(x,y)\,dA \le \iint_{D_2} f(x,y)\,dA\,.$$

Now introduce polar coordinates in the regions D_1 and D_2 through the substitutions $x = r\cos\theta$ and $y = r\sin\theta$, and iterate all three integrals to conclude that

$$\int_0^{\pi/2}\int_0^R e^{-r^2}\,r\,dr\,d\theta \le \int_0^R\int_0^R e^{-x^2}e^{-y^2}\,dx\,dy \le \int_0^{\pi/2}\int_0^{\sqrt{2}R} e^{-r^2}\,r\,dr\,d\theta\,.$$

The advantage of the area-integral approach is that the polar-coordinate integrals can be calculated explicitly. These calculations reduce the inequalities to

$$\frac{\pi}{4}\left(1 - e^{-R^2}\right) \le \left(\int_0^R e^{-x^2}dx\right)^2 \le \frac{\pi}{4}\left(1 - e^{-2R^2}\right)\,.$$

Now let $R \to \infty$ to obtain

$$\left(\int_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4}\,,$$

which is equivalent to (1).

9.2. Gamma function

In the early development of calculus, there were various efforts to analyze the factorial function $n! = 1 \cdot 2 \cdot 3 \cdots n$ and the related binomial coefficients. Stirling's formula

$$n! \sim n^n e^{-n} \sqrt{2\pi n}\,, \qquad n \to \infty\,,$$

was published in 1730. Around the same time, Euler had been searching for a natural extension of the factorial function. The problem was to find a function $f(x)$, defined by an analytic formula for all $x > 0$, to interpolate the values $f(n) = n!$ at the positive integers. In 1729, Euler sketched a suitable construction in a letter to Christian Goldbach (1690–1764), who had proposed the problem along with Daniel Bernoulli (1700–1784), the son of Euler's teacher Johann Bernoulli. In fact, this letter was the first in a long series of mathematical correspondence between Euler and Goldbach, a very talented amateur mathematician who asked thoughtful questions. ("Goldbach's conjecture", that every even number larger than 2 is the sum of two primes, remains unsettled.)

Euler published the details of his construction, but soon found that his solution could be expressed in the equivalent integral form

$$f(x) = \int_0^1 (-\log t)^x \, dt\,.$$

Focusing on the integral representation, he then studied the function and developed its basic properties. Only later did Adrien-Marie Legendre (1752–1833) propose the notation $\Gamma(x)$ and the now standard definition of the gamma function:

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt\,, \qquad x > 0\,. \tag{2}$$

A change of variables shows that $\Gamma(x+1)$ is the same as Euler's function $f(x)$.

One may well wonder what is so special about the gamma function. There are many functions $g(x)$, infinitely differentiable for $x > 0$, with the property $g(n) = n!$ for all integers $n = 1, 2, \dots$. Why does the gamma function provide the "right" extension of the factorial? One response is that the gamma function not only has the values $\Gamma(n+1) = n!$, but as we shall see presently, it also satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. Further justification comes from the Bohr–Mollerup theorem, to be discussed later, but the real case for the gamma function is its frequent appearance in formulas of mathematical analysis.

Integration by parts shows that

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^x \, dt = \Big[-e^{-t} t^x\Big]_0^\infty + x \int_0^\infty e^{-t}\, t^{x-1}\, dt = x\Gamma(x)\,.$$

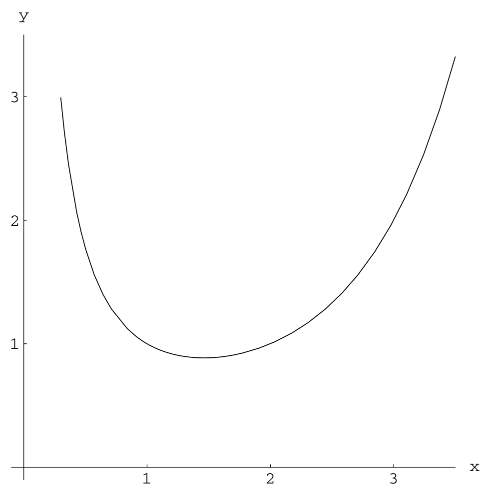

Figure 2. Graph of $y = \Gamma(x)$.

Since $\Gamma(1) = 1$, the formula $\Gamma(x+1) = x\Gamma(x)$ can be iterated to give $\Gamma(2) = 1$, $\Gamma(3) = 2$, and in general $\Gamma(n+1) = n!$ for $n = 1, 2, \ldots$. It is clear that $\Gamma(x) > 0$ for all $x > 0$, because the integrand is positive. A graph of the gamma function is shown in Figure 2.

The graph suggests that $\Gamma(x)$ descends to a minimum value at a point x_0 in the interval $1 < x_0 < 2$, and is increasing for $x > x_0$. It is not difficult to verify this (see Exercise 3). Gauss was interested in the number $x_0 = 1.4616\ldots$ and calculated it to many decimal places. He also calculated $\Gamma(x_0) = 0.8856\ldots$. Another feature suggested by the graph is that $\Gamma(x)$ is a convex function. In fact, we will see later that $\log\Gamma(x)$ is convex, a stronger property. The graph of $\log\Gamma(x)$ is displayed in Figure 3.

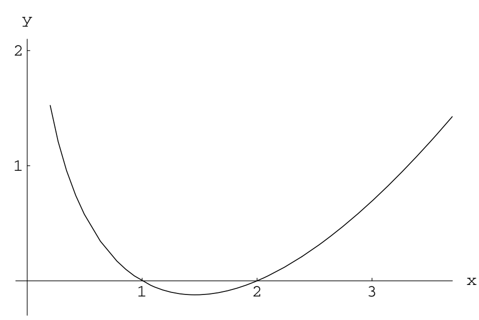

Figure 3. Graph of $y = \log\Gamma(x)$.

With the substitution $t = u^2$, the gamma function takes the form

$$\Gamma(x) = 2\int_0^\infty e^{-u^2} u^{2x-1}\,du\,, \qquad x > 0\,. \tag{3}$$

In view of the formula (1), this shows that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

The functional equation $\Gamma(x+1) = x\Gamma(x)$ can be iterated to show more generally that

$$\Gamma(x+n) = (x+n-1)(x+n-2)\cdots(x+1)x\Gamma(x)\,, \qquad x > 0\,,$$

for $n = 1, 2, \ldots$, or

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}\,, \qquad x > 0\,.$$

The last formula allows the definition of $\Gamma(x)$ to be extended to the interval $x > -n$, provided that x is not a negative integer or 0. Since n is an arbitrary positive integer, this extends the definition of $\Gamma(x)$ to the entire real line, excluding the singular points $x = 0, -1, -2, \ldots$.

9.3. Beta function

The *beta function* $B(x, y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt\,, \qquad x > 0\,,\ y > 0\,. \tag{4}$$

The change of variables $t = 1 - s$ reveals the symmetry property $B(x, y) = B(y, x)$. However, the most important property of the beta function is its expression in terms of the gamma function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \tag{5}$$

The proof of (5) makes use of the alternate expression

$$B(x,y) = 2\int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\,d\theta\,, \tag{6}$$

which results from the substitution $t = \cos^2\theta$. The argument is very similar to the calculation of the probability integral (1). The proof begins with a comparison of integrals of the positive function

$$g(u,v) = u^{2x-1}\,v^{2y-1}\,e^{-(u^2+v^2)}$$

over the same three regions as before, as displayed in Figure 1. Here, however, x and y are fixed positive parameters, whereas u and v are the variables of integration. After passing to polar coordinates with the substitutions $u = r\cos\theta$ and $v = r\sin\theta$, we arrive at the formulas

$$\begin{aligned} &\int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\,d\theta \int_0^R e^{-r^2}\, r^{2x+2y-1}\,dr \\ \le &\int_0^R e^{-u^2}\, u^{2x-1}\,du \int_0^R e^{-v^2}\, v^{2y-1}\,dv \\ \le &\int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\,d\theta \int_0^{\sqrt{2}R} e^{-r^2}\, r^{2x+2y-1}\,dr\,. \end{aligned}$$

Letting $R \to \infty$ and referring to (3) and (6), we conclude that

$$B(x,y)\Gamma(x+y) \le \Gamma(x)\Gamma(y) \le B(x,y)\Gamma(x+y)\,,$$

and (5) follows. Another proof is outlined in the exercises.

9.4. Legendre's duplication formula

Legendre's duplication formula for the gamma function expresses $\Gamma(2x)$ in terms of $\Gamma(x)$ and $\Gamma(x+\frac{1}{2})$. It is

$$\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\Gamma(x)\,\Gamma(x+\tfrac{1}{2})\,, \qquad x > 0\,. \tag{7}$$

Many proofs have been found. We will give a particularly elementary proof that uses the basic connection (5) between the beta and gamma functions, plus the special formula

$$B(x,x) = 2^{1-2x}B(x,\tfrac{1}{2})\,, \qquad x > 0\,. \tag{8}$$

To prove (8), observe first that because of the symmetry of the integrand in (4) when $x = y$, we can write

$$B(x,x) = 2\int_0^{1/2} \big[t(1-t)\big]^{x-1}\,dt\,, \qquad x > 0\,.$$

Next observe that $u = 4t(1-t)$ increases from 0 to 1 as t increases from 0 to $\frac{1}{2}$. Make this substitution in the integral, noting that

$$du = 4(1-2t)\,dt \qquad \text{and} \qquad 1-u = (1-2t)^2\,,$$

to arrive at the formula

$$B(x,x) = 2^{1-2x}\int_0^1 u^{x-1}(1-u)^{-1/2}\,du = 2^{1-2x}B(x,\tfrac{1}{2})\,,$$

which proves (8).

Legendre's formula (7) is now deduced from (8) through the relation (5). When (5) is substituted into (8), the formula reduces to

$$\frac{\Gamma(x)^2}{\Gamma(2x)} = 2^{1-2x}\frac{\Gamma(x)\,\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})}\,.$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, this is the Legendre duplication formula.

It may be observed that the special formula (8) is actually a disguised expression of Legendre's formula.

The name of Adrien-Marie Legendre (1752–1833) occurs frequently in topics of classical analysis. In Chapter 13 of this book we will encounter the Legendre differential equation and the associated Legendre polynomials, which are orthogonal over the interval $[-1, 1]$. Chapter 14 contains a discussion of the Legendre relation for elliptic integrals, a field that Legendre cultivated for many years. Yet some of Legendre's greatest contributions lay in celestial mechanics and in number theory. For instance, he developed and applied (before Gauss) the method of least squares for determination of astronomical orbits. He formulated the prime number theorem (*cf.* Chapter 10), the theorem that every arithmetic progression without common factor contains infinitely many primes (proved later by Dirichlet), and the law of quadratic reciprocity, verified by Gauss after Legendre had found a partial proof. Little is known about Legendre's personal life. A portrait traditionally believed to be that of Adrien-Marie Legendre has now been discredited and replaced by a recently discovered caricature portrait, the only true image of Legendre known to exist. An article by Duren [**5**] tells the story of the two portraits.

9.5. Euler's reflection formula

The important relation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}\,, \qquad 0 < x < 1\,, \tag{9}$$

known as *Euler's reflection formula*, links the gamma function with the sine function. It lies deeper than other properties of the gamma function, and there is no simple derivation. We will base a proof on the formula

$$\cos cx = \frac{2c}{\pi}\sin c\pi \left\{ \frac{1}{2c^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{c^2-n^2}\cos nx \right\}, \qquad -\pi \le x \le \pi\,, \tag{10}$$

which we derived in Section 8.5 from the convergence theorem for Fourier series. Set $x = 0$ in (10) to obtain the expression

$$\frac{\pi}{\sin c\pi} = 2c \left\{ \frac{1}{2c^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{c^2-n^2} \right\}, \qquad c \ne 0, \pm 1, \pm 2, \dots\,. \tag{11}$$

On the other hand, since $\Gamma(1) = 1$, the formula

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \qquad x > 0\,,\; y > 0$$

for the beta function gives

$$\Gamma(c)\Gamma(1-c) = B(c,1-c) = \int_0^1 t^{c-1}(1-t)^{-c}\,dt = \int_0^\infty \frac{x^{c-1}}{1+x}\,dx\,, \quad 0 < c < 1\,,$$

with the substitution $x = \frac{t}{1-t}$. But the substitution $u = 1/x$ reduces part of the last integral to

$$\int_1^\infty \frac{x^{c-1}}{1+x}\,dx = \int_0^1 \frac{u^{-c}}{1+u}\,du\,,$$

so we arrive at the expression

$$\Gamma(c)\Gamma(1-c) = \int_0^1 \frac{x^{c-1}}{1+x}\,dx + \int_0^1 \frac{x^{-c}}{1+x}\,dx\,, \qquad 0 < c < 1\,.$$

Now introduce the identity

$$\frac{1}{1+x} = 1 - \frac{x}{1+x}$$

into the first integral to obtain

$$\begin{aligned}
\Gamma(c)\Gamma(1-c) &= \int_0^1 x^{c-1}\,dx + \int_0^1 \frac{x^{-c}-x^c}{1+x}\,dx \\
&= \frac{1}{c} + \int_0^1 (x^{-c}-x^c)(1-x+x^2-x^3+\dots)\,dx \\
&= \frac{1}{c} + \sum_{n=0}^\infty (-1)^n \int_0^1 (x^{-c}-x^c)x^n\,dx \\
&= \frac{1}{c} + \sum_{n=1}^\infty (-1)^n \frac{2c}{c^2-n^2} = \frac{\pi}{\sin c\pi}\,, \qquad 0 < c < 1\,,
\end{aligned}$$

in view of (11). The term-by-term integration is justified by showing that the tail of the integrated series tends to zero:

$$\lim_{n\to\infty} \int_0^1 \frac{x^{n+1}}{1+x}(x^{-c}-x^c)\,dx = 0\,. \tag{12}$$

Indeed, the factor

$$\frac{x^{1-c}-x^{1+c}}{1+x}$$

is bounded in the interval $[0,1]$, while

$$\int_0^1 x^n\,dx = \frac{1}{n+1} \to 0 \qquad \text{as } n \to \infty\,.$$

This proves (12) and completes the completes the proof of the Euler reflection formula (9).

With the choice $x = \frac{1}{2}$, the reflection formula says that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Another proof of the reflection formula can be based on the infinite product representation of the gamma function, as developed in the next section (see Exercise 16).

9.6. Infinite product representation

Before turning to a representation of the gamma function as an infinite product, we will derive the formula

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\,n^x}{x(x+1)\cdots(x+n)}\,, \qquad x > 0\,. \tag{13}$$

The existence of the limit is not obvious and is part of the assertion. The result is called the *Gauss product formula* after Carl Friedrich Gauss (1777–1855), although it was known to Euler. In fact, this was Euler's original construction in 1729, but Gauss rediscovered the formula and recognized its importance.

One approach to (13) is to observe that

$$\begin{aligned}\int_0^1 t^{x-1}(1-t)^n\,dt = B(x,n+1) &= \frac{\Gamma(x)\Gamma(n+1)}{\Gamma(x+n+1)} \\ &= \frac{n!\,\Gamma(x)}{(x+n)(x+n-1)\cdots x\Gamma(x)} = \frac{n!}{x(x+1)\ldots(x+n)}\,.\end{aligned}$$

The substitution of t/n for t then leads to the representation

$$\frac{n!\,n^x}{x(x+1)\cdots(x+n)} = \int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^n dt = \int_0^\infty g_n(t)t^{x-1}\,dt\,,$$

where

$$g_n(t) = \begin{cases} \left(1-\frac{t}{n}\right)^n, & \text{for } 0 \le t \le n \\ 0\,, & \text{for } n < t < \infty\,. \end{cases}$$

Because $g_n(t)$ increases to the limit e^{-t} as $n \to \infty$, Dini's theorem (see Section 1.8) ensures that the integrals converge and

$$\lim_{n\to\infty} \frac{n!\,n^x}{x(x+1)\cdots(x+n)} = \int_0^\infty e^{-t}t^{x-1}\,dt = \Gamma(x)\,.$$

This concludes the proof of the Gauss product formula.

It is now a short step to the infinite product representation of the gamma function.

Theorem. *The formula*

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n}$$

holds for all $x \in \mathbb{R}$ *with* $x \neq 0, -1, -2, \ldots$. *Here*

$$\gamma = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{1}{k} - \log n \right\}$$

is Euler's constant.

The infinite product converges for each $x \in \mathbb{R}$ because

$$\begin{aligned}\left(1 + \frac{x}{n}\right) e^{-x/n} &= \left(1 + \frac{x}{n}\right)\left(1 - \frac{x}{n} + \frac{x^2}{2n^2} + \cdots\right) \\ &= 1 - \frac{x^2}{2n^2} + O\left(\frac{1}{n^3}\right)\end{aligned}$$

and the series $\sum 1/n^2$ converges. In fact, the product can be shown to converge uniformly on each bounded subset of $\mathbb{R}$, so that it represents a continuous function on $\mathbb{R}$. Note that the product is equal to zero at the points $x = 0, -1, -2, \ldots$ where the gamma function is infinite.

Proof of theorem. The Gauss product formula (13) allows us to write

$$\begin{aligned}\frac{1}{\Gamma(x)} &= \lim_{n\to\infty} \frac{x(x+1)\cdots(x+n)}{n!\, n^x} \\ &= \lim_{n\to\infty} x\left(1 + \frac{x}{1}\right)\left(1 + \frac{x}{2}\right)\cdots\left(1 + \frac{x}{n}\right) e^{-x\log n} \\ &= \lim_{n\to\infty} xe^{x(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)} \prod_{k=1}^{n}\left(1 + \frac{x}{k}\right) e^{-x/k} \\ &= xe^{\gamma x} \prod_{k=1}^{\infty}\left(1 + \frac{x}{k}\right) e^{-x/k}.\end{aligned}$$

□

The proof shows that the Gauss product formula (13) is essentially the same as the infinite product representation of $1/\Gamma(x)$.

9.7. Generalization of Stirling's formula

Implicit in the preceding calculations is an important generalization of Stirling's formula

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}, \qquad n \to \infty,$$

to the gamma function. Because $\Gamma(n) = (n-1)!$ and $\left(1 - \frac{1}{n}\right)^n \to 1/e$, an equivalent form is

$$\Gamma(n) \sim \sqrt{2\pi}\, n^{n-\frac{1}{2}}\, e^{-n}, \qquad n \to \infty.$$

But the result can be strengthened to

$$\Gamma(x) \sim \sqrt{2\pi}\, x^{x-\frac{1}{2}}\, e^{-x}, \qquad x \to \infty, \tag{14}$$

where $x \to \infty$ unrestrictedly. This is Stirling's formula for the gamma function.

Stirling's formula (14) is easily deduced from the relation (13), which can be rewritten as

$$\lim_{n\to\infty} \frac{\Gamma(n+a+1)}{n!\, n^a} = 1, \qquad a > 0,$$

where the convergence is uniform for $0 < a \le 1$, as is evident from the proof of (13). In view of Stirling's formula for $n!$, the last limit implies that

$$\lim_{n\to\infty} \frac{\Gamma(n+a)}{n^{n+a-\frac{1}{2}}\, e^{-n}} = \sqrt{2\pi},$$

or

$$\lim_{n\to\infty} \frac{\Gamma(n+a)}{(n+a)^{n+a-\frac{1}{2}}\, e^{-(n+a)}} = \sqrt{2\pi}, \qquad 0 < a \le 1,$$

since $\left(1 + \frac{a}{n}\right)^n \to e^a$ uniformly in the interval $0 < a \le 1$. This proves Stirling's formula for the gamma function.

9.8. Bohr–Mollerup theorem

Finally, we come to the remarkable fact that the gamma function is completely determined by the relation $\Gamma(x+1) = x\Gamma(x)$, the normalization $\Gamma(1) = 1$, and the fact that $\log\Gamma(x)$ is a convex function. Recall that a function f defined in an interval $I \subset \mathbb{R}$ is *convex* if for each pair of points $a, b \in I$ it satisfies the inequality

$$f(ra + (1-r)b) \le r f(a) + (1-r) f(b), \qquad 0 < r < 1.$$

Geometrically, this says that the graph of f lies beneath every chord.

We begin by proving that $\log \Gamma(x)$ is convex, a property evident from its graph in Figure 3. Suppose $0 < a < b$ and $0 < r < 1$. Then by definition

$$\begin{aligned}\Gamma(ra+(1-r)b) &= \int_0^\infty e^{-t} t^{ra+(1-r)b-1}\, dt \\ &= \int_0^\infty \left\{e^{-t}t^{a-1}\right\}^r \left\{e^{-t}t^{b-1}\right\}^{1-r} dt \\ &\le \left\{\int_0^\infty e^{-t}t^{a-1}\, dt\right\}^r \left\{\int_0^\infty e^{-t}t^{b-1}\, dt\right\}^{1-r} = \Gamma(a)^r\, \Gamma(b)^{1-r},\end{aligned}$$

where Hölder's inequality has been applied with the conjugate indices $p = 1/r$ and $q = 1/(1-r)$. Taking logarithms, we conclude that

$$\log \Gamma(ra+(1-r)b) \le r \log \Gamma(a) + (1-r)\log \Gamma(b), \qquad 0 < r < 1,$$

which shows that $\log \Gamma(x)$ is a convex function. (See also Exercise 26.)

Harald Bohr and Johannes Mollerup [**3**] discovered that the gamma function is actually characterized by its logarithmic convexity. Emil Artin [**2**] gave an elegant presentation of their argument and clarified the role of logarithmic convexity.

Bohr–Mollerup Theorem. *Let $G(x)$ be a positive function on the positive real axis $x > 0$ with the properties $G(x+1) = xG(x)$ and $G(1) = 1$. Suppose further that $\log G(x)$ is a convex function. Then $G(x) \equiv \Gamma(x)$.*

Proof. The hypotheses $G(x+1) = xG(x)$ and $G(1) = 1$ imply that $G(n+1) = n!$ for $n = 1, 2, \dots$. For any positive integer n and for $0 < x \le 1$, we express

$$n + x = (1-x)n + x(n+1)$$

as a convex combination of n and $n+1$. Then by the convexity hypothesis,

$$\log G(n+x) \le (1-x)\log G(n) + x \log G(n+1),$$

or

$$G(n+x) \le G(n)^{1-x} G(n+1)^x = n!\, n^{x-1}.$$

In a similar way, the convex combination

$$n+1 = x(n+x) + (1-x)(n+x+1)$$

produces the inequality

$$n! = G(n+1) \le G(n+x)^x G(n+x+1)^{1-x} = G(n+x)(n+x)^{1-x},$$

since $G(n+x+1) = (n+x)G(n+x)$. When the two inequalities are combined, we find that

$$n!(n+x)^{x-1} \le G(n+x) \le n!\, n^{x-1}\,. \tag{15}$$

But the property $G(x+1) = xG(x)$ can be iterated to give

$$G(x+n) = x(x+1)\cdots(x+n-1)G(x)\,,$$

so the inequality (15) takes the form

$$\frac{n!(n+x)^x}{x(x+1)\cdots(x+n)} \le G(x) \le \frac{n!\, n^x}{x(x+1)\cdots(x+n-1)n}\,. \tag{16}$$

We now let $n \to \infty$ and appeal to the Gauss product formula (13) to conclude from (16) that

$$\Gamma(x) \le G(x) \le \Gamma(x)\,,$$

or $G(x) = \Gamma(x)$ for $0 < x \le 1$. The extension to all $x > 0$ now results from the two identities $G(x+1) = xG(x)$ and $\Gamma(x+1) = x\Gamma(x)$. □

Alternatively, the inequality (16) can be used to establish the existence of the limit in (13) and to show that it is equal to $G(x)$. Since the reasoning applies equally well to $\Gamma(x)$, it then follows that $G(x) = \Gamma(x)$, which gives an independent proof of the Gauss product formula.

Aside from its aesthetic appeal, the Bohr–Mollerup theorem offers an effective way to verify formulas involving the gamma function. As an illustration, we now apply it to establish the *Gauss multiplication formula*

$$\Gamma(mx) = \frac{m^{mx-1/2}}{(2\pi)^{(m-1)/2}}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\cdots\Gamma\left(x+\frac{m-1}{m}\right) \tag{17}$$

for $x > 0$ and $m = 2, 3, \dots$. This is a generalization of Legendre's duplication formula (7), which corresponds to the special case $m = 2$. Replacing mx by x in (17), we want to show that the function

$$G(x) = \frac{m^{x-1/2}}{(2\pi)^{(m-1)/2}}\,\Gamma\left(\frac{x}{m}\right)\Gamma\left(\frac{x+1}{m}\right)\cdots\Gamma\left(\frac{x+m-1}{m}\right)$$

is equal to $\Gamma(x)$. In view of the Bohr–Mollerup theorem, it will suffice to show that $\log G(x)$ is convex, $G(x+1) = xG(x)$, and $G(1) = 1$. The logarithmic convexity follows at once from that of the gamma function. To see that $G(x+1) = xG(x)$, it is convenient to write $G(x) = \alpha_m\, g_m(x)$, where

$$\alpha_m = \frac{m^{-1/2}}{(2\pi)^{(m-1)/2}}$$

and

$$g_m(x) = m^x\,\Gamma\left(\frac{x}{m}\right)\Gamma\left(\frac{x+1}{m}\right)\cdots\Gamma\left(\frac{x+m-1}{m}\right).$$

Now observe that

$$g_m(x+1) = m\left[\Gamma\left(\frac{x+m}{m}\right)\Big/\Gamma\left(\frac{x}{m}\right)\right]g_m(x) = xg_m(x)\,,$$

since $\Gamma\left(\frac{x+m}{m}\right) = \Gamma\left(\frac{x}{m}+1\right) = \frac{x}{m}\Gamma\left(\frac{x}{m}\right)$. Thus $G(x+1) = xG(x)$.

The difficulty is to show that $G(1) = 1$. In other words, it is to be shown that

$$\alpha_m\, m\,\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\cdots\Gamma\left(\frac{m}{m}\right) = 1\,. \tag{18}$$

For this purpose we apply the Gauss product formula (13) to write

$$\Gamma\left(\frac{k}{m}\right) = \lim_{n\to\infty}\frac{n!\,n^{k/m}\,m^{n+1}}{k(k+m)(k+2m)\cdots(k+nm)}\,,\qquad k = 1,2,\dots,m\,.$$

The product of these limits is

$$\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\cdots\Gamma\left(\frac{m}{m}\right) = \lim_{n\to\infty}\frac{(n!)^m\,n^{(m+1)/2}\,m^{mn+m}}{(m+nm)!}\,. \tag{19}$$

But

$$\frac{(m+nm)!}{(nm)!(nm)^m} = \left(1+\frac{1}{nm}\right)\left(1+\frac{2}{nm}\right)\cdots\left(1+\frac{m}{nm}\right)\to 1$$

as $n\to\infty$, and so the expression (19) reduces to

$$\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\cdots\Gamma\left(\frac{m}{m}\right) = \lim_{n\to\infty}\frac{(n!)^m\,m^{mn}}{(nm)!\,n^{(m-1)/2}}\,.$$

Now Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$ can be applied to give

$$\frac{(n!)^m}{(nm)!}\sim\frac{(2\pi)^{(m-1)/2}\,n^{(m-1)/2}}{m^{mn+1/2}}\qquad\text{as}\quad n\to\infty\,,$$

so that

$$\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\cdots\Gamma\left(\frac{m}{m}\right) = \frac{(2\pi)^{(m-1)/2}}{m^{1/2}} = \frac{1}{m\alpha_m}\,.$$

This proves (18), which shows that $G(1) = 1$. Hence it follows from the Bohr–Mollerup theorem that $G(x) = \Gamma(x)$, which completes the proof of the Gauss multiplication formula.

9.9. A special integral

We close this chapter with a calculation of the integral

$$\int_0^\infty x^{-a}\sin x\,dx = \frac{\pi}{2\,\Gamma(a)}\csc(\pi a/2)\,, \qquad 0 < a < 2\,. \tag{20}$$

This result will be applied in the next chapter to derive a functional equation for the Riemann zeta function. Although the integral can be calculated by contour integration in the complex plane, the method used here is entirely elementary and is based only on standard properties of the gamma function.

Recall first the expression for the Laplace transform

$$F(s) = \int_0^\infty e^{-st}\sin t\,dt = \frac{1}{1+s^2}\,, \qquad s > 0\,, \tag{21}$$

which can be derived via two integrations by parts. Specifically,

$$F(s) = \frac{1}{s}\int_0^\infty e^{-st}\cos t\,dt = \frac{1}{s}\left\{\frac{1}{s} - \frac{1}{s}\int_0^\infty e^{-st}\sin t\,dt\right\}\,,$$

so that $(1+s^2)F(s) = 1$.

Next make a change of variables to obtain

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt = x^a\int_0^\infty t^{a-1}e^{-xt}\,dt\,, \qquad x > 0\,,\ a > 0\,.$$

Take $\varepsilon > 0$ and use this form of the gamma function to write

$$\begin{aligned}\int_0^\infty e^{-\varepsilon x}x^{-a}\sin x\,dx &= \frac{1}{\Gamma(a)}\int_0^\infty e^{-\varepsilon x}\sin x\int_0^\infty t^{a-1}e^{-xt}\,dt\,dx\\ &= \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}\int_0^\infty e^{-(\varepsilon+t)x}\sin x\,dx\,dt\\ &= \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}\frac{1}{1+(\varepsilon+t)^2}\,dt\,,\end{aligned}$$

where the formula (21) has been invoked. To justify the interchange in order of integration, observe that the exponential factor $e^{-\varepsilon x}$ makes the integrand absolutely integrable over the first quadrant of the (x,t) plane.

Now appeal to the integral analogue of Abel's theorem (Section 3.2) to conclude that

$$\begin{aligned}\int_0^\infty x^{-a}\sin x\,dx &= \lim_{\varepsilon\to 0}\int_0^\infty e^{-\varepsilon x}x^{-a}\sin x\,dx\\ &= \frac{1}{\Gamma(a)}\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt\,.\end{aligned}$$

But the last integral is a beta function in disguise. Let $u = t^2$ and refer to Exercise 13 to write

$$\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt = \frac{1}{2}\int_0^\infty \frac{u^{\frac{a}{2}-1}}{1+u}\,du = \frac{1}{2}\,B(\tfrac{a}{2}, 1-\tfrac{a}{2})\,.$$

Then apply the Euler reflection formula to obtain

$$B(\tfrac{a}{2}, 1-\tfrac{a}{2}) = \Gamma(\tfrac{a}{2})\,\Gamma(1-\tfrac{a}{2}) = \pi\csc(\pi a/2)\,,$$

which verifies the formula (20).

The expression (20) can be recast in other forms. For $0 < a < 1$ it combines with the Euler reflection formula to give

$$\begin{aligned}\int_0^\infty x^{a-1}\sin x\,dx &= \frac{\pi}{2\,\Gamma(1-a)}\,\csc(\pi(1-a)/2)\\ &= \frac{1}{2}\,\Gamma(a)\,\frac{\sin\pi a}{\cos(\pi a/2)} = \Gamma(a)\sin(\pi a/2)\,.\end{aligned}$$

More generally,

$$\int_0^\infty x^{a-1}\sin bx\,dx = b^{-a}\Gamma(a)\sin(\pi a/2)\,, \qquad 0 < a < 1\,,\ b > 0\,.$$

Exercises

1. Prove that $\Gamma(x)$ is continuous for $x > 0$.

2. Show that $\lim_{x\to 0+} x\Gamma(x) = 1$.

3. (*a*) Prove that the function $\Gamma(x)$ has derivatives of all orders at every point $x > 0$, and that its nth derivative is given by

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t}t^{x-1}(\log t)^n\,dt\,, \qquad n = 1, 2, \dots\,.$$

(*b*) Show that $\Gamma''(x) > 0$ for $x > 0$, so that the curve $y = \Gamma(x)$ is convex.

(*c*) Show that $\Gamma(x)$ attains a minimum value for $x > 0$ at a point x_0 in the interval $1 < x_0 < 2$. Show further that $\Gamma(x)$ is decreasing in the interval $(0, x_0)$ and increasing in (x_0, ∞).

4. Show that

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}\,, \qquad n = 1, 2, \dots\,.$$

5. Verify that Euler's definition

$$\Gamma(x) = \int_0^1 \big(\log(1/t)\big)^{x-1}\, dt$$

is equivalent to the definition (2). Show more generally that

$$\int_0^1 t^{x-1}\big(\log(1/t)\big)^{y-1}\, dt = x^{-y}\,\Gamma(y)\,, \qquad x > 0\,,\ y > 0\,.$$

6. Calculate the integrals

$$(a) \qquad \int_0^1 \big(\log(1/t)\big)^{1/2}\, dt = \frac{\sqrt{\pi}}{2}$$

$$(b) \qquad \int_0^1 \big(\log(1/t)\big)^{-1/2}\, dt = \sqrt{\pi}$$

$$(c) \qquad \int_0^1 \big(t\log(1/t)\big)^{1/2}\, dt = \frac{\sqrt{2\pi}}{3\sqrt{3}}$$

$$(d) \qquad \int_0^1 \big(t\log(1/t)\big)^{-1/2}\, dt = \sqrt{2\pi}\,.$$

7. Calculate the integral

$$\int_0^1 \sqrt{t(1-t)}\, dt = \frac{\pi}{8}\,.$$

8. Express the function

$$f(x) = \int_0^{\pi/2} (\sin 2\theta)^x\, d\theta\,, \qquad x > -1\,,$$

in terms of the gamma function.

9. Show that

$$\Gamma(\tfrac{1}{6}) = \frac{\sqrt{3}}{\sqrt[3]{2}\,\sqrt{\pi}}\,\Gamma(\tfrac{1}{3})^2\,.$$

10. Calculate the integrals

$$(a) \qquad \int_0^1 \sqrt{1-t^3}\, dt = \frac{\sqrt{3}}{10\pi\sqrt[3]{2}}\,\Gamma(\tfrac{1}{3})^3$$

$$(b) \qquad \int_0^1 \sqrt{1-t^4}\, dt = \frac{1}{6\sqrt{2\pi}}\,\Gamma(\tfrac{1}{4})^2$$

$$(c) \qquad \int_0^1 \frac{1}{\sqrt{1-t^3}}\, dt = \frac{1}{2\pi\sqrt{3}\sqrt[3]{2}}\,\Gamma(\tfrac{1}{3})^3$$

$$(d) \qquad \int_0^1 \frac{1}{\sqrt{1-t^4}}\, dt = \frac{1}{4\sqrt{2\pi}}\,\Gamma(\tfrac{1}{4})^2$$

$$(e) \qquad \int_0^1 \frac{t^2}{\sqrt{1-t^4}}\, dt = \frac{1}{\sqrt{2\pi}}\,\Gamma(\tfrac{3}{4})^2\,.$$

11. Use the beta function to calculate

$$\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta = \frac{\sqrt{2}}{\sqrt{\pi}}\, \Gamma(\tfrac{3}{4})^2 .$$

Compare the result with that of Exercise 10(e).

12. Find the area of the region bounded by the hypocycloid $x^{2/3}+y^{2/3}=1$.

13. Use the substitution $t = \frac{u}{1+u}$ to derive the formula

$$B(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du .$$

14. Apply the formula of the preceding exercise to calculate the integral

$$\int_0^\infty \frac{x^3}{(1+x)^7}\, dx = \frac{1}{60} .$$

15. Calculate the integral

$$\int_0^\infty \frac{x^{-a}}{1+x}\, dx = \frac{\pi}{\sin \pi a}, \qquad 0 < a < 1 .$$

More generally, calculate

$$\int_0^\infty \frac{x^{m-1}}{1+x^n}\, dx = \frac{\pi}{n\, \sin(m\pi/n)}, \qquad 0 < m < n ,$$

a result known to Euler. (Here m and n need not be integers.)

16. Compare the infinite product representation of $\Gamma(x)$ with that of the sine function to obtain another proof of the Euler reflection formula.

17. Prove that $\Gamma'(1) = -\gamma$, where γ is Euler's constant. Conclude that

$$\gamma = -\int_0^\infty e^{-t} \log t\, dt .$$

Hint. Take logarithmic derivatives in the infinite product formula for $\Gamma(x)$. Justify the term-by-term differentiation.

18. Calculate the integral

$$\int_0^\infty \frac{\log x}{1+e^x}\, dx = -\gamma \log 2 + \sum_{n=1}^\infty (-1)^n \frac{\log n}{n} = -\frac{1}{2}(\log 2)^2 .$$

Hint. Expand $1/(1+e^x) = e^{-x}/(1+e^{-x})$ into geometric series and integrate term by term (justify), then refer to Exercise 13 of Chapter 2 for evaluation of the infinite series.

19. The *Laplace transform* of a function f is the function $F = \mathcal{L}(f)$ defined by the integral

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt\,, \qquad s > s_0\,,$$

which is assumed to converge for all real numbers s larger than some number s_0. For each exponent $a > -1$, show that the Laplace transform of $f(t) = t^a$ is

$$F(s) = \frac{\Gamma(a+1)}{s^{a+1}}\,, \qquad s > 0\,.$$

20. The *convolution* of two functions f and g defined on $(0, \infty)$ is the function $h = f * g$ given by

$$h(t) = \int_0^t f(u)g(t-u)\,du\,, \qquad 0 < t < \infty\,.$$

Show that $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$. In other words, show that the Laplace transform of a convolution is the product of transforms.

21. If $f(t) = t^{x-1}$ and $g(t) = t^{y-1}$ for some $x > 0$ and $y > 0$, show that the convolution $h = f * g$ has the form $h(t) = t^{x+y-1}B(x, y)$. By taking Laplace transforms, conclude that

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}\,.$$

22. Use Stirling's formula for the gamma function to show that

$$\lim_{n\to\infty} \frac{\Gamma(n+c-a)\,\Gamma(n+c-b)}{\Gamma(n+c)\,\Gamma(n+c-a-b)} = 1$$

for arbitrary real numbers a, b, and c.

23. Calculate the integral

$$\int_0^\infty x^{-a}\cos x\,dx = \frac{\pi}{2\,\Gamma(a)}\sec(\pi a/2)\,, \qquad 0 < a < 1\,.$$

24. (a) Calculate the integral

$$\int_{-1}^1 (1-x^2)^n\,dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}\,, \qquad n = 1, 2, \dots\,,$$

by making the substitution $t = 1 - x^2$ to transform it into a beta function.

(*b*) Deduce that

$$\int_0^{\pi/2} \sin^{2n+1}\theta\, d\theta = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

(*c*) Apply the formula (6) for the beta function to reprove (*b*) and to calculate

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{(2n)!\,\pi}{2^{2n+1}(n!)^2}.$$

(Compare with Exercise 15 in Chapter 2.)

25. Let $v_n(r)$ denote the volume of the n-dimensional sphere

$$B_n(r) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \le r^2\}$$

of radius r, where $n = 1, 2, \ldots$. Then $v_1(r) = 2r$, $v_2(r) = \pi r^2$, and $v_3(r) = \frac{4}{3}\pi r^3$. In general, it is easy to see that $v_n(r) = V_n r^n$, where V_n is the volume of the unit sphere $B_n(1)$. The numbers V_n can be calculated recursively by slicing:

$$V_{n+1} = \int_{-1}^{1} v_n\left(\sqrt{1-x^2}\right) dx = V_n \int_{-1}^{1} (1-x^2)^{n/2}\, dx.$$

Derive the formula

$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \qquad n = 1, 2, \ldots.$$

Show in particular that

$$v_4(r) = \tfrac{1}{2}\pi^2 r^4, \quad v_5(r) = \tfrac{8}{15}\pi^2 r^5, \quad v_6(r) = \tfrac{1}{6}\pi^3 r^6, \quad \text{and} \quad v_7(r) = \tfrac{16}{105}\pi^3 r^7.$$

26. The *digamma function* is $\psi(x) = \frac{d}{dx}\{\log\Gamma(x)\} = \Gamma'(x)/\Gamma(x)$.

(*a*) Apply the functional equation $\Gamma(x+1) = x\Gamma(x)$ to show that $\psi(x+1) = \psi(x) + \frac{1}{x}$, $x > 0$.

(*b*) Generalize the relation $\Gamma'(1) = -\gamma$ by proving that

$$\psi(n+1) = \left(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n}\right) - \gamma, \qquad n = 1, 2, \ldots.$$

(*c*) Use the infinite product formula for $1/\Gamma(x)$ to show that

$$\psi(x) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n-1}\right) - \gamma, \qquad x > 0.$$

(*d*) Deduce that

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}, \qquad x > 0,$$

and conclude that the function $\log\Gamma(x)$ is convex.

27. Apply Stirling's formula to verify the Gauss product formula (13) in the special case where x is a positive integer.

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, U.K., 1999.

[2] Emil Artin, *The Gamma Function*, English translation, Holt, Rinehart and Winston, New York, 1964.

[3] H. Bohr and J. Mollerup, *Lærebog i Matematisk Analyse*, Volume 3, J. Gjellerup, Copenhagen, 1922.

[4] Philip J. Davis, "Leonhard Euler's integral: A historical profile of the gamma function", *Amer. Math. Monthly* **66** (1959), 849–869.

[5] P. Duren, "Changing faces: The mistaken portrait of Legendre", *Notices Amer. Math. Soc.* **56** (2009), 1440–1443; *Ibid.* **57** (2010), 703.

[6] D. V. Widder, *Advanced Calculus*, Second edition, Prentice–Hall, Englewood Cliffs, N. J., 1961; reprinted by Dover Publications, Mineola, NY, 1989.