Chapter 2

Sequences

In this chapter we have our first encounter with the concept of limit – the concept that lies at the heart of the calculus. We first study limits of sequences of real numbers. Limits of functions will be studied in the next chapter.

2.1. Limits of Sequences

Limits make sense in any context in which we have a notion of distance between objects. Thus, we begin with a discussion of the notion of distance between two real numbers.

Distance and Absolute Value. Recall that the absolute value $|x|$ of a number x is defined by

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0, \\ -x & \text{if} \quad x < 0. \end{cases}$$

Thus, $|x|$ is always a non-negative number. It can be thought of as the distance from x to 0. For example,

$$|3| = |-3| = 3$$

just means that the distance from 3 to 0 and the distance from -3 to 0 are the same, namely 3. More generally, if x and y are any two real numbers, the distance from x to y is $|x - y|$.

We will often need to specify that a number x is close to another number a. However, this doesn't mean anything unless we specify how close. If ϵ is a positive number, then the statement "x is within ϵ of a" does have meaning. It means that the distance between x and a is less than ϵ – that is,

$$|x - a| < \epsilon.$$

This statement also means that x is in the open interval of radius ϵ, centered at a, as pointed out in part (b) of the following theorem.

Theorem 2.1.1. *If x, y, a, and ϵ are real numbers with $\epsilon > 0$, then*

(a) $|y| < \epsilon$ *if and only if* $-\epsilon < y < \epsilon$;

(b) $|x - a| < \epsilon$ *if and only if* $a - \epsilon < x < a + \epsilon$.

These statements remain true if "$<$" is replaced by "$\leq$".

Proof. To prove (a), we consider two cases:

(1) Suppose $y \geq 0$. Then $|y| = y$, and so $|y| < \epsilon$ if and only if $y < \epsilon$. The latter statement means the same as $-\epsilon < y < \epsilon$, because $-\epsilon < y$ is automatically true in this case.

(2) Suppose $y < 0$. Then $|y| = -y$, and so $|y| < \epsilon$ if and only if $-y < \epsilon$. This is true if and only if $-\epsilon < y$, which is true if and only if $-\epsilon < y < \epsilon$, because $y < \epsilon$ is automatically true in this case.

Part (b) follows from part (a). That is, if we apply part (a) with $y = x - a$, then we conclude that $|x - a| < \epsilon$ if and only if $-\epsilon < x - a < \epsilon$, and this is true if and only if $a - \epsilon < x < a + \epsilon$.

If "$<$" is replaced by "$\leq$", the proofs of (a) and (b) remain the same. □

The following theorem will be used extensively throughout the text.

Theorem 2.1.2 (Triangle Inequality). *If a and b are real numbers, then*

(a) $|a + b| \leq |a| + |b|$ *and*

(b) $||a| - |b|| \leq |a - b|$.

Proof. For part (a), we observe that $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. If we add these inequalities, the result is

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

By the preceding theorem (with "$<$" replaced by "$\leq$"), this is equivalent to $|a+b| \leq |a| + |b|$. This proves part (a).

For part (b), we note that part (a) implies $|a| = |b + (a - b)| \leq |b| + |a - b|$ and this yields

$$|a| - |b| \leq |a - b| \tag{2.1.1}$$

when we subtract $|b|$ from both sides. If we interchange b and a, then the right side of this inequality stays the same and the left side becomes $|b| - |a|$. Thus, the inequality

$$|b| - |a| \leq |b| + |a - b|$$

also holds. This and (2.1.1) together imply part (b). □

Sequences. A sequence of real numbers is a function from the natural numbers to the real numbers. That is, it is an assignment of a real number a_n to each natural number n. Traditionally, we use the notation

$$\{a_n\}_{n=1}^{\infty} \quad \text{or simply} \quad \{a_n\}$$

to denote a sequence, rather than using standard function notation. Alternatively, we may describe a sequence by writing out its first few terms and possibly its nth term:

$$a_1, a_2, a_3, \dots \quad \text{or} \quad a_1, a_2, a_3, \dots, a_n, \dots .$$

Example 2.1.3. Write each of the following sequences in the form

$$a_1, a_2, a_3, \dots, a_n, \dots :$$

(a) the sequence $\{(-1)^n 1/n\}$;

(b) the sequence of positive even integers;

(c) the sequence defined inductively by $a_1 = 2$ and $a_{n+1} = \dfrac{a_n + 1}{2}$.

Solution: The answers are

(a) $-1, 1/2, -1/3, \dots, (-1)^n 1/n, \dots$;

(b) $2, 4, 6, \dots, 2n, \dots$;

(c) $2, 3/2, 5/4, \dots, 1 + 1/2^{n-1}, \dots$.

The first two are obvious. For (c), we prove that $a_n = 1 + 1/2^{n-1}$ by induction. This is certainly true for $n = 1$. If it is true for an integer n, then $a_n = 1 + 1/2^{n-1}$ and so

$$a_{n+1} = (a_n + 1)/2 = (1 + 1/2^{n-1} + 1)/2 = 1 + 1/2^n.$$

Thus, our formula for a_n is true for $n+1$ if it is true for n. By induction, it is true for all natural numbers.

It is sometimes convenient to begin the indexing of a sequence at some integer k other than 1. For example, the sequence

$$1, 2, 4, 8, \dots, 2^n, \dots$$

has description $n \to 2^{n-1}$ as a function from the natural numbers to the real numbers, or, using standard sequence notation, $\{2^{n-1}\}_{n=1}^{\infty}$, but it is usually more convenient to think of it as the function $n \to 2^n$ from the non-negative integers to the reals and to denote it $\{2^n\}_{n=0}^{\infty}$. Similarly, the sequence

$$8/3, 4, 32/5, 32/3, 128/7, \dots$$

can be described as the sequence $\left\{\dfrac{2^{n+2}}{n+2}\right\}_{n=1}^{\infty}$, but it may be more convenient to describe it as $\left\{\dfrac{2^n}{n}\right\}_{n=3}^{\infty}$. Passing from one notation to the other is a *change of variables* in the index – that is, n is replaced by $n - 2$ and the starting point for the sequence is changed from $n = 1$ to $n = 3$ (since $n - 2$ is 1 when n is 3).

Limits of Sequences. A sequence $\{a_n\}$ converges to a number a if the distance from a_n to a can be made less than any given positive number by insisting that n

be sufficiently large. More precisely:

Definition 2.1.4. A sequence $\{a_n\}$ of real numbers is said to *converge* to the number a, or have *limit* equal to a, if, for each $\epsilon > 0$, there is a real number N such that

$$|a_n - a| < \epsilon \quad \text{whenever} \quad n > N.$$

In this case, we will write $\lim_{n\to\infty} a_n = a$ or $\lim a_n = a$ or simply $a_n \to a$.

Remark 2.1.5. If we compare what would be required by the above definition for $\lim a_n = a$ and what would be required for $\lim |a_n - a| = 0$, then we find that the requirements are identical. Thus, $a_n \to a$ if and only if $|a_n - a| \to 0$.

The limit of a sequence (if it exists) is well defined – that is, a sequence cannot have more than one limit.

Theorem 2.1.6. *If $a_n \to a$ and $a_n \to b$, then $a = b$.*

Proof. If $a_n \to a$ and $a_n \to b$, then, for each $\epsilon > 0$ there are numbers N_1 and N_2 such that

$$n > N_1 \quad \text{implies} \quad |a_n - a| < \epsilon/2 \text{ and}$$
$$n > N_2 \quad \text{implies} \quad |a_n - b| < \epsilon/2.$$

If n is an integer larger than both N_1 and N_2, then

$$|b - a| = |(a_n - a) + (b - a_n)| \le |a_n - a| + |b - a_n| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This implies that $|b - a|$ is smaller than every positive number ϵ. Since $|b - a| \ge 0$, this is possible only if $|b - a| = 0$ – that is, only if $a = b$. (In this argument we used an important property of the real number system without comment. In Exercise 2.1.12 you are asked to figure out what property that is.) □

Finding the limit of a sequence often involves two steps: (1) make a good intuitive guess as to what the limit should be and (2) prove that your guess is correct by using the above definition or theorems that have been proved using it. The following example illustrates the first of these steps.

Example 2.1.7. Make an educated guess as to what the limits are for the following sequences:

(a) $\{1/n\}$;

(b) $\left\{\dfrac{n}{2n+1}\right\}$;

(c) $\{(-1)^n\}$;

(d) $\{\sqrt{4 + 1/n}\}$.

Solution: (a) The larger n becomes, the smaller $1/n$ becomes. Thus, it appears that $\lim 1/n = 0$.

(b) If we divide the numerator and denominator of $\dfrac{n}{2n+1}$ by n, the result is $\dfrac{1}{2+1/n}$. If $1/n \to 0$, then it should be the case that $\dfrac{1}{2+1/n} \to 1/2$. Thus, we choose $1/2$ as our guess.

(c) Since the sequence $\{(-1)^n\}$ alternates between -1 and 1, it does not appear to converge to any one number. Thus, we guess that it does not converge.

(d) If $1/n \to 0$, then it should be the case that $\sqrt{4+1/n} \to \sqrt{4} = 2$. Thus, our guess is 2.

Example 2.1.8. Use the definition of limit to verify that the guesses in the preceding example are correct:

Solution: (a) Given $\epsilon > 0$, we must show that there is an N such that $n > N$ implies $1/n < \epsilon$. However, since $1/n < \epsilon$ if and only if $n > 1/\epsilon$, if we choose $N = 1/\epsilon$, then, indeed, $n > N$ implies $1/n < \epsilon$.

(b) Given $\epsilon > 0$, we must show that there is an N such that

$$n > N \quad \text{implies} \quad \left|\frac{n}{2n+1} - 1/2\right| < \epsilon.$$

Some work with the expression in absolute values shows us how to do this:

$$\left|\frac{n}{2n+1} - 1/2\right| = \left|\frac{2n-2n+1}{4n+2}\right| = \frac{1}{4n+2} < \frac{1}{4n}.$$

Thus, $\left|\frac{n}{2n+1} - 1/2\right| < \epsilon$ whenever $\frac{1}{4n} < \epsilon$ – that is, whenever $n > \frac{1}{4\epsilon}$. Thus, it suffices to choose $N = \frac{1}{4\epsilon}$.

(c) We will show that there is no number a which satisfies the definition of the statement $\lim(-1)^n = a$. Let a be any real number and choose $\epsilon = 1/2$. If $\lim(-1)^n = a$, then there must be an N such that

$$n > N \quad \text{implies} \quad |(-1)^n - a| < 1/2.$$

Since there are both even and odd integers $n > N$, this means that

$$|1-a| < 1/2 \quad \text{and} \quad |-1-a| < 1/2.$$

Then the triangle inequality (Theorem 2.1.2(a)) implies

$$2 = |1-a+1+a| \le |1-a| + |1+a| = |1-a| + |-1-a| < 1/2 + 1/2 = 1.$$

Since it is not true that $2 < 1$, our assumption that $\lim(-1)^n = a$ must be false. Since this is the case no matter what real number we choose for a, we conclude that $\{(-1)^n\}$ has no limit. (Once again, as in the proof of Theorem 2.1.6, we used here, without comment, a special property of the real number system. Exercise 2.1.12 asks you to state what property that is.)

(d) Given $\epsilon > 0$, we must show there is an N such that

$$n > N \quad \text{implies} \quad |\sqrt{4+1/n} - 2| < \epsilon.$$

We simplify this problem by rationalizing the positive expression $\sqrt{4+1/n} - 2$:

$$\begin{aligned} |\sqrt{4+1/n} - 2| = \sqrt{4+1/n} - 2 &= \frac{(\sqrt{4+1/n} - 2)(\sqrt{4+1/n} + 2)}{\sqrt{4+1/n} + 2} \\ &= \frac{4+1/n-4}{\sqrt{4+1/n}+2} < \frac{1/n}{\sqrt{4}+2} = \frac{1}{4n}. \end{aligned} \tag{2.1.2}$$

Thus, if $N = 1/(4\epsilon)$, then $n > N$ implies $|\sqrt{4+1/n} - 2| < \epsilon$.

Exercise Set 2.1

1. Show that
   (a) if $|x-5|<1$, then x is a number greater than 4 and less than 6;
   (b) if $|x-3|<1/2$ and $|y-3|<1/2$, then $|x-y|<1$;
   (c) if $|x-a|<1/2$ and $|y-b|<1/2$, then $|x+y-(a+b)|<1$.
2. Use the triangle inequality to prove that there is no number x which satisfies both $|x-1|<1/2$ and $|x-2|<1/2$.
3. Put each of the following sequences in the form $a_1, a_2, a_3, \dots, a_n, \dots$. This requires that you compute the first 3 terms and find an expression for the nth term.
   (a) The sequence of positive odd integers.
   (b) The sequence defined inductively by $a_1 = 1$ and $a_{n+1} = -\dfrac{a_n}{2}$.
   (c) The sequence defined inductively by $a_1 = 1$ and $a_{n+1} = \dfrac{a_n}{n+1}$.

In each of the next five exercises, first make an educated guess as to what you think the limit is. Then use the definition of limit to prove that your guess is correct.

4. $\lim 1/n^2$.
5. $\lim \dfrac{2n-1}{3n+1}$.
6. $\lim(-1)^n/n$.
7. $\lim \dfrac{n}{n^3+4}$.
8. $\lim\{\sqrt{n+1}-\sqrt{n}\}$.
9. Prove that $\lim(1/n+(-1)^n/n^2)=0$.
10. Prove that $\lim 2^{-n} = 0$. Hint: Prove first that $2^n \ge n$ for all natural numbers n.
11. Prove that if $a_n \to 0$ and k is any constant, then $ka_n \to 0$.
12. In the proof of Theorem 2.1.6 we failed to point out that one step is true only because we are working in the real number system and not some other ordered field. What special property of the real number system makes this argument work? This same property is also used without comment in Example 2.1.8, solution, part (c).

2.2. Using the Definition of Limit

It is important that mathematics students become comfortable with the notion of limit of a sequence. Unfortunately, it is a difficult concept to grasp. Students almost always have difficulty with it at first and learn to understand it only through repeated exposure and extensive practice in its use. This section is designed to provide some of this practice.

Using Identities and Inequalities. In each of the following examples, we wish to show that a certain sequence $\{a_n\}$ has limit a. The strategy for doing this, in each case, is to use identities and inequalities on the expression $|a_n - a|$ until we can show that it is less than or equal to some much simpler expression in n that can clearly be made less than any given ϵ by choosing n large enough.

Example 2.2.1. Prove that $\lim \dfrac{n}{2n-3} = 1/2$.

Solution: We have

$$\left|\frac{n}{2n-3} - 1/2\right| = \left|\frac{2n-2n+3}{4n-6}\right| = \left|\frac{3}{4n-6}\right|.$$

Now $4n - 6 = n + (3n - 6) \geq n$ whenever $n > 1$. Thus,

$$\left|\frac{n}{2n-3} - 1/2\right| \leq \frac{3}{4n-6} \leq \frac{3}{n}$$

provided $n > 1$. Given $\epsilon > 0$, if we choose $N = \max\{1, 3/\epsilon\}$, then

$$\left|\frac{n}{2n-3} - 1/2\right| \leq \frac{3}{n} < \epsilon \quad \text{whenever} \quad n > N.$$

This completes the proof that $\lim \dfrac{n}{2n-1} = 1/2$.

Example 2.2.2. Prove that $\lim(2 + 1/n)^2 = 4$.

Solution: We have

$$|(2+1/n)^2 - 4| = |2+1/n+2||2+1/n-2| = \frac{4+1/n}{n} \leq \frac{5}{n}.$$

Thus, given $\epsilon > 0$, if we set $N = 5/\epsilon$, we have

$$|(2+1/n)^2 - 4| \leq \frac{5}{n} < \epsilon \quad \text{whenever} \quad n > N.$$

This proves that $\lim(2 + 1/n)^2 = 4$.

Using Information About a Limit. Knowing that a sequence converges or that it converges to a specific number always provides a great deal of other information. We give some examples below.

Theorem 2.2.3. *If* $\lim a_n = a$ *and* $a < c$*, then there exists an* N *such that*

$$a_n < c \quad \textit{for all} \quad n > N.$$

Similarly, if $b < a$*, then there is an* N *such that*

$$b < a_n \quad \textit{for all} \quad n > N.$$

Proof. If $a < c$, then $c - a > 0$. Since $\lim a_n = a$, for each $\epsilon > 0$, there is an N such that

$$|a_n - a| < \epsilon \quad \text{whenever} \quad n > N.$$

If we use this in the case where $\epsilon = c - a$, it tells us there is an N such that

$$|a_n - a| < c - a \quad \text{whenever} \quad n > N.$$

This implies

$$a - c + a < a_n < a + c - a \quad \text{whenever} \quad n > N,$$

by Theorem 2.1.1(b). Thus, $a_n < c$ for all $n > N$.

The second statement of the theorem is proved in the same way. □

A sequence $\{a_n\}$ is bounded above (or below) if the set of numbers which appear as terms of $\{a_n\}$ is bounded above (or below) as a set of numbers. A sequence which is bounded above and bounded below is simply said to be bounded.

The following corollary follows directly from the preceding theorem. We leave the details to the exercises.

Corollary 2.2.4. *If a sequence $\{a_n\}$ converges, then it is bounded.*

Theorem 2.2.5. *If $\{a_n\}$ is a sequence and $\lim a_n = a$, then $\lim |a_n| = |a|$.*

Proof. We use the second form of the triangle inequality (Theorem 2.1.2(b)) to write

$$||a_n| - |a|| \leq |a_n - a|. \tag{2.2.1}$$

Since $\lim a_n = a$, given $\epsilon > 0$, there is an N such that

$$|a_n - a| < \epsilon \quad \text{whenever} \quad n > N.$$

Then, by (2.2.1), it is also true that

$$||a_n| - |a|| < \epsilon \quad \text{whenever} \quad n > N.$$

Thus, $\lim |a_n| = |a|$. □

Example 2.2.6. For a sequence $\{a_n\}$ with $\lim a_n = a$, prove $\lim a_n^2 = a^2$.

Solution: We first note that

$$|a_n^2 - a^2| = |a + a_n||a_n - a| \leq (|a_n| + |a|)|a_n - a|. \tag{2.2.2}$$

We know that $\lim |a_n| = |a|$ by the previous theorem. Since $|a| < |a| + 1$, Theorem 2.2.3 implies that there is an N_1 such that $|a_n| < |a| + 1$ for all $n > N_1$. This and (2.2.2) together imply that

$$|a_n^2 - a^2| < (2|a| + 1)|a_n - a| \quad \text{whenever} \quad n > N_1.$$

Given $\epsilon > 0$, we choose N_2 such that $|a_n - a| < \dfrac{\epsilon}{2|a| + 1}$ whenever $n > N_2$. We can do this because $\lim a_n = a$. If we set $N = \max(N_1, N_2)$, then

$$|a_n^2 - a^2| < \epsilon \quad \text{whenever} \quad n > N.$$

Hence, $\lim a_n^2 = a^2$.

An Equivalent Definition of Limit. The following theorem rephrases the definition of limit in a way that may provide some additional insight.

Theorem 2.2.7. *A sequence $\{a_n\}$ converges to a if and only if, for each $\epsilon > 0$, there are only finitely many n for which $|a_n - a| \geq \epsilon$.*

Proof. Given $\epsilon > 0$, set

$$A_\epsilon = \{n \in \mathbb{N} : |a_n - a| \geq \epsilon\}.$$

If $\lim a_n = a$ and $\epsilon > 0$, there is an N such that $|a_n - a| < \epsilon$ whenever $n > N$. This means that A_ϵ is contained in the set $\{1, 2, \ldots, N\}$ and, hence, is finite.

Conversely, suppose that, for each $\epsilon > 0$, the set A_ϵ is finite. Then given $\epsilon > 0$, the set A_ϵ has a largest element N. This means $n \notin A_\epsilon$ if $n > N$ – that is, $|a_n - a| < \epsilon$ if $n > N$. This implies that $\lim a_n = a$. □

Negating the Limit Definition. What does it mean for it not to be true that $\lim a_n = a$? That is, what is the *negation* of the statement "for each $\epsilon > 0$ there is an N such that $|a_n - a| < \epsilon$ whenever $n > N$"? If it is not true that for each $\epsilon > 0$, there is an N such that ..., then for some $\epsilon > 0$, there is no N such that .... If we fill in the dots, we get the following statement:

The sequence $\{a_n\}$ does not converge to a if and only if for some $\epsilon > 0$ there is no N such that $|a_n - a| < \epsilon$ for all $n > N$.

We may rephrase the second half of this statement to obtain:

The sequence $\{a_n\}$ does not converge to a if and only if for some $\epsilon > 0$ and for every N there is an $n > N$ such that $|a_n - a| \geq \epsilon$.

Negating the equivalent definition of limit given in Theorem 2.2.7 yields a somewhat simpler statement:

The sequence $\{a_n\}$ does not converge to a if and only if for some $\epsilon > 0$ there are infinitely many $n \in \mathbb{N}$ for which $|a_n - a| \geq \epsilon$.

Example 2.2.8. Show that the sequence $\{2^{-n}+(1+(-1)^n)2^{-50}\}$ does not converge to 0.

Solution: Try computing a few terms of this sequence on a calculator. It appears to be converging to 0. However, if we choose $\epsilon = 2^{-49}$, then for every even $n \in N$

$$|2^{-n} + (1 + (-1)^n)2^{-50} - 0| = 2^{-n} + 2 \cdot 2^{-50} \geq 2^{-49}.$$

Since this inequality holds for infinitely many n, the sequence does not converge to 0.

Exercise Set 2.2

In each of the following six exercises, first make an educated guess as to what you think the limit is. Then use the definition of limit to prove that your guess is correct.

1. $\lim \dfrac{3n^2 - 2}{n^2 + 1}$.
2. $\lim \dfrac{n}{n^2 + 2}$.
3. $\lim \dfrac{1}{\sqrt{n}}$.
4. $\lim \left(\dfrac{n}{n+1}\right)^2$.
5. $\lim(\sqrt{n^2 + n} - n)$.
6. $\lim(1 + 1/n)^3$.
7. Prove Corollary 2.2.4.
8. Prove that if $\lim a_n = a$, then $\lim a_n^3 = a^3$.
9. Does the sequence $\{\cos(n\pi/3)\}$ have a limit? Justify your answer.
10. Give an example of a sequence $\{a_n\}$ which does not converge but for which the sequence $\{|a_n|\}$ does converge.
11. Prove that if $\{a_n\}$ and $\{b_n\}$ are sequences with $|a_n| \le b_n$ for all n and if $\lim b_n = 0$, then $\lim a_n = 0$ also.
12. Prove the following partial converse to Theorem 2.2.3: Suppose $\{a_n\}$ is a convergent sequence. If there is an N such that $a_n \le c$ for all $n > N$, then $\lim a_n \le c$. Also, if there is an N such that $b \le a_n$ for all $n > N$, then $b \le \lim a_n$.
13. Use the result of the preceding exercise to prove that an interval I is closed if and only if each sequence in I that converges actually converges to a point of I.
14. Prove Corollary 2.2.4. That is, prove that a convergent sequence is bounded.
15. For a certain sequence $\{a_n\}$ there is an $\epsilon > 0$ such that every millionth term of the sequence $\{a_n\}$ is greater than ϵ. Can such a sequence converge to 0? Justify your answer.

2.3. Limit Theorems

We reiterate that the strategy to use in proving a statement of the form

$$\lim a_n = a$$

directly from the definition is to use a string of identities and inequalities to conclude that $|a_n - a|$ is less than or equal to a simpler expression in n that we can easily force to be less than ϵ by making n sufficiently large. This strategy was used throughout the previous two sections. The following theorem formalizes this strategy in a way that will lead us to use the right approach to many limit proofs.

Theorem 2.3.1. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of real numbers and suppose* $\lim b_n = 0$. *If* $a \in \mathbb{R}$ *and if there is an* N_1 *such that*

$$|a_n - a| \le b_n \quad \text{for all} \quad n > N_1, \tag{2.3.1}$$

then $\lim a_n = a$.

Proof. Since $\lim b_n = 0$, given any $\epsilon > 0$, there is an N_2 such that

$$b_n = |b_n - 0| < \epsilon \quad \text{whenever} \quad n > N_2.$$

It now follows from (2.3.1) that

$$|a_n - a| < \epsilon \quad \text{whenever} \quad n > N = \max\{N_1, N_2\}.$$

Thus, $\lim a_n = a$. □

Of course, to prove that $\lim a_n = a$ using this theorem one must establish an inequality of the form (2.3.1), where $\{b_n\}$ is a sequence of non-negative terms that we know converges to 0. The proof of the next theorem uses this technique. The proof is easy and is left to the exercises.

A sequence $\{b_n\}$ for which there is a number k such that $b_n \le k$ for all n is said to be *bounded above*. If there is a number m such that $m \le b_n$ for all n, then the sequence is said to be *bounded below*. A sequence which is bounded above and below is simply said to be *bounded*. Note that a sequence $\{b_n\}$ is bounded if and only if $\{|b_n|\}$ is bounded above (Exercise 2.3.6). Recall from Corollary 2.2.4 that convergent sequences are bounded.

Theorem 2.3.2. *Let* $\{a_n\}$ *be a sequence of real numbers such that* $\lim a_n = 0$, *and let* $\{b_n\}$ *be a bounded sequence. Then* $\lim a_n b_n = 0$.

The following theorem is often called the *squeeze principle*.

Theorem 2.3.3. *If* $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *are sequences for which there is a number* K *such that*

$$b_n \le a_n \le c_n \quad \text{for all} \quad n > K$$

and if $b_n \to a$ *and* $c_n \to a$, *then* $a_n \to a$.

Proof. Since $b_n \to a$ and $c_n \to a$, given $\epsilon > 0$, there are numbers N_1 and N_2 such that

$$\begin{aligned} a - \epsilon < b_n < a + \epsilon \quad &\text{for all} \quad n > N_1 \text{ and} \\ a - \epsilon < c_n < a + \epsilon \quad &\text{for all} \quad n > N_2. \end{aligned} \tag{2.3.2}$$

Then for $n > N = \max\{N_1, N_2, K\}$ we have

$$a - \epsilon < b_n \le a_n \le c_n < a + \epsilon.$$

This implies $|a_n - a| < \epsilon$. Thus, $\lim a_n = a$. □

Example 2.3.4. Prove that if $\{a_n\}$ is a sequence of positive numbers converging to a positive number a, then $\lim \sqrt{a_n} = \sqrt{a}$.

Solution: We will use Theorem 2.3.1. Rationalizing the numerator gives us

$$|\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} < \frac{1}{\sqrt{a}}|a_n - a|.$$

Since $a_n \to a$, Remark 2.1.5 implies $|a_n - a| \to 0$. Then Theorems 2.3.2 and 2.3.1 imply $\sqrt{a_n} \to \sqrt{a}$.

Example 2.3.5. Prove that if $|a| < 1$, then $\lim a^n = 0$.

Solution: The result is trivial in the case $a = 0$. If $a \neq 0$, we set $b = |a|^{-1} - 1$. Then $b > 0$ and $|a|^{-1} = 1 + b$. We use the Binomial Theorem (Theorem 1.2.12) to expand $|a|^{-n} = (1+b)^n$:

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2}b^2 + \cdots + b^n.$$

Since all the terms involved are positive, it follows that $|a|^{-n} = (1+b)^n \geq nb$. Inverting this yields

$$|a^n| \leq \frac{1}{nb} = \frac{1}{b}\frac{1}{n}.$$

Since $1/n \to 0$, it follows from Theorems 2.3.2 and 2.3.1 that $a^n \to 0$.

The Main Limit Theorem. This is the theorem that tells us that the limit concept behaves well with regard to the usual algebraic operations.

Theorem 2.3.6. *Suppose $a_n \to a$, $b_n \to b$, c is a real number, and k is a natural number. Then*

(a) $ca_n \to ca$;

(b) $a_n + b_n \to a + b$;

(c) $a_n b_n \to ab$;

(d) $a_n/b_n \to a/b$ *if* $b \neq 0$ *and* $b_n \neq 0$ *for all* n;

(e) $a_n^k \to a^k$;

(f) $a_n^{1/k} \to a^{1/k}$ *if* $a_n \geq 0$ *for all* n.

Proof. Part (a) follows immediately from Theorem 2.3.2 applied to the sequence $\{c(a_n - a)\}$. We will prove (c) and (e) and leave (b), (d), and (f) to the exercises.

(c) We use the strategy suggested by Theorem 2.3.1. We have

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n - a||b_n| + |a||b_n - b|,$$

by the triangle inequality. Furthermore, we have that $\{b_n\}$ is bounded by Corollary 2.2.4, and so $\{|b_n|\}$ is bounded above. We also have $|a_n - a| \to 0$, by Remark 2.1.5. Therefore, by Theorem 2.3.2, $|a_n - a||b_n| \to 0$. By part (a), $|a||b_n - b| \to 0$. By part (b) the sum $|a_n - a||b_n| + |a||b_n - b|$ converges to 0 and, hence, $a_n b_n \to ab$ by Theorem 2.3.1.

(e) We use the identity

$$a_n^k - a^k = (a_n - a)(a_n^{k-1} + a_n^{k-2}a + a_n^{k-3}a^2 + \cdots + a^{k-1}) = (a_n - a)b_n,$$

where

$$b_n = a_n^{k-1} + a_n^{k-2}a + a_n^{k-3}a^2 + \cdots + a^{k-1}.$$

Now, because the sequence $\{a_n\}$ converges, it is bounded and, hence, $\{|a_n|\}$ is bounded above. We choose an upper bound m for $\{|a_n|\}$ which also satisfies $|a| \le m$. Then

$$|b_n| \le km^k.$$

Since k and m are fixed, the sequence $\{|b_n|\}$ is bounded above.

We conclude from Theorem 2.3.2 that $|a_n - a||b_n| \to 0$ and from Theorem 2.3.1 that $a_n^k \to a^k$. □

Example 2.3.7. Use the Main Limit Theorem to find $\lim \dfrac{n^2 + 3n + 1}{3n^2 - 7n + 2}$.

Solution: In a problem of this type, we divide the numerator and denominator by the highest power of n that appears in either one. In this case, that is the second power. The result is

$$\frac{1 + 3/n + 1/n^2}{3 - 7/n + 2/n^2}.$$

The Main Limit Theorem then tells us that

$$\begin{aligned}
&\lim \frac{1 + 3/n + 1/n^2}{3 - 7/n + 2/n^2} = \frac{\lim(1 + 3(1/n) + 2(1/n)^2)}{\lim(3 - 7(1/n) + 2(1/n)^2)} \\
&= \frac{1 + 3\lim(1/n) + 2\lim(1/n)^2}{3 - 7\lim(1/n) + 2\lim(1/n)^2} = \frac{1 + 3\lim(1/n) + 2(\lim 1/n)^2}{3 - 7\lim(1/n) + 2(\lim 1/n)^2} \qquad (2.3.3) \\
&= \frac{1 + 3 \cdot 0 + 2(0)^2}{3 - 7 \cdot 0 + 2(0)^2} = 1/3.
\end{aligned}$$

Here, we didn't explicitly refer to the parts of the Main Limit Theorem as we used them, but it is clear that the first equality uses (d), the second (a) and (b), the third (e), and the fourth the fact that $\lim 1/n = 0$ (Example 2.1.8).

Theorem 2.3.8. *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences converging to a and b, respectively, and if there is a number K such that $a_n \le b_n$ whenever $n > K$, then $a \le b$.*

Proof. The sequence $c_n = b_n - a_n$ is a sequence with $b - a$ as its limit and with terms that are non-negative for $n > K$. If $b - a$ were negative, then Theorem 2.2.3 would imply $b_n - a_n < 0$ for all sufficiently large n. Since this is not the case, we conclude that $a \le b$. □

Exercise Set 2.3

1. Use the Main Limit Theorem to find $\lim \dfrac{2n^3 - n + 1}{3n^3 + n^2 + 6}$.
2. Use the Main Limit Theorem to find $\lim \dfrac{n^2 - 5}{n^3 + 2n^2 + 5}$.
3. Use the Main Limit Theorem to find $\lim \dfrac{2^n}{2^n + 1}$.

4. Prove that $\lim \frac{\sin n}{n} = 0$.
5. Prove Theorem 2.3.2.
6. Prove that a sequence $\{a_n\}$ is both bounded above and bounded below if and only if its sequence of absolute values $\{|a_n|\}$ is bounded above.
7. Prove part (b) of Theorem 2.3.6.
8. Prove that if $\{b_n\}$ is a sequence of positive terms and $b_n \to b > 0$, then there is a number $m > 0$ such that $b_n \geq m$ for all n.
9. Prove part (d) of Theorem 2.3.6. Hint: Use the previous exercise.
10. Prove part (f) of Theorem 2.3.6. Hint: Use the identity
$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + y^{k-1})$$
with $x = a_n^{1/k}$ and $y = a^{1/k}$.
11. For each natural number n, let $b_n = n^{1/n} - 1$. Then b_n is positive and $n = (1+b_n)^n$. Use the Binomial Theorem (Theorem 1.2.12) to prove that $n \geq \frac{n(n-1)}{2} b_n^2$ and, hence, that $b_n \leq \sqrt{\frac{2}{n-1}}$.
12. Prove that $\lim n^{1/n} = 1$. Hint: Use the result of the previous exercise.
13. Prove that if $a > 0$, then $\lim a^{1/n} = 1$. Hint: Do this first for $a \geq 1$; use the result of the previous exercise and the squeeze principle.

2.4. Monotone Sequences

A sequence of real numbers $\{a_n\}$ is said to be *non-decreasing* if $a_n \leq a_{n+1}$ for each n. The sequence is said to be *non-increasing* if $a_n \geq a_{n+1}$ for each n. If it is one or the other (either non-decreasing or non-increasing), the sequence is said to be *monotone*.

Convergence of Monotone Sequences. In this section and the next, we will develop powerful tools for proving that a sequence converges. These tools work even in situations where we have no idea what the limit might be. It is the completeness axiom for the real number system that makes these results possible.

Theorem 2.4.1 (Monotone Convergence Theorem). *Each bounded monotone sequence converges.*

Proof. A non-decreasing sequence $\{a_n\}$ is bounded if and only if it is bounded above, since it is automatically bounded below by a_1. Similarly, a non-increasing sequence is bounded if and only if it is bounded below.

We will prove that every non-decreasing sequence that is bounded above converges. The proof that every non-increasing sequence that is bounded below converges is the same but with all the inequalities reversed.

Thus, suppose $\{a_n\}$ is non-decreasing and bounded above. Then the set
$$A = \{a_n : n \in \mathbb{N}\}$$

is a non-empty set which is bounded above. By the completeness axiom **C**, this set has a least upper bound a. That is,

$$\sup_n a_n = \sup A = a$$

is finite. We will show that a is the limit of the sequence $\{a_n\}$.

Given $\epsilon > 0$, the number $a - \epsilon$ is less than a and so it is not an upper bound for A. This means there is some natural number N such that $a - \epsilon < a_N$. If $n > N$, then $a_N \leq a_n$ since $\{a_n\}$ is a non-decreasing sequence. This implies $a - \epsilon < a_n$. We also have $a_n \leq a < a + \epsilon$, since a is an upper bound for $\{a_n\}$. Combining these inequalities yields

$$a - \epsilon < a_n < a + \epsilon \quad \text{for all} \quad n > N.$$

By Theorem 2.1.1(b), this is equivalent to

$$|a_n - a| < \epsilon \quad \text{for all} \quad n > N.$$

We conclude that $\lim a_n = a$. □

Example 2.4.2. Let a sequence be defined inductively by $a_1 = 0$ and

$$a_{n+1} = \frac{a_n + 1}{2}. \tag{2.4.1}$$

Prove that this sequence converges and find its limit.

Solution: This is a non-decreasing sequence (Exercise 1.2.13). Also, a simple induction argument shows that it is bounded above by 1. Therefore it is a bounded monotone sequence, and it converges by the previous theorem. Let $\lim a_n = a$. If we take the limit of both sides of (2.4.1), the result is $a = (a+1)/2$, or $a/2 = 1/2$. Thus, $a = 1$.

A less trivial example is the following:

Example 2.4.3. Let a sequence $\{a_n\}$ be defined inductively by $a_1 = 2$ and

$$a_{n+1} = \frac{a_n^2 + 2}{2a_n}. \tag{2.4.2}$$

Prove that this sequence converges and then find its limit.

Solution: We first note that a trivial induction argument shows that $a_n > 0$ for all n. This is true when $n = 1$ and it is true for $n + 1$ whenever it is true for n by (2.4.2).

We will prove that the sequence is non-increasing. To show that $a_n \geq a_{n+1}$, we must show that $a_n \geq \dfrac{a_n^2 + 2}{2a_n}$. If we assume that $a_n > 0$, then we may multiply this inequality by $2a_n$ to obtain the equivalent inequality

$$2a_n^2 \geq a_n^2 + 2 \quad \text{or} \quad a_n^2 \geq 2.$$

We conclude that $a_n \geq a_{n+1}$ as long as a_n is positive and $a_n^2 \geq 2$ – that is, as long as $a_n \geq \sqrt{2}$. Now $a_1 = 2$ and so the sequence starts out with a number greater than or equal to $\sqrt{2}$. Every other number in this sequence has the form

$$\frac{x^2 + 2}{2x}$$

for some positive x. We claim that every such number is greater than or equal to $\sqrt{2}$. In fact

$$0 \le (x - \sqrt{2})^2 = x^2 - 2\sqrt{2}\,x + 2, \quad \text{and so} \quad 2\sqrt{2}\,x \le x^2 + 2.$$

This implies $\sqrt{2} \le \dfrac{x^2+2}{2x}$. Thus every a_n is greater than or equal to $\sqrt{2}$.

We now know that the sequence $\{a_n\}$ is non-increasing and bounded below by $\sqrt{2}$. Thus, it is a bounded monotone sequence and has a limit by the previous theorem. Call the limit a. By (2.4.2), we have

$$2a_n a_{n+1} = a_n^2 + 2.$$

If we take the limit of both sides of this equation and note that $\lim a_n = \lim a_{n+1} = a$, then the result is

$$2a^2 = a^2 + 2 \quad \text{or} \quad a^2 = 2.$$

Thus, $a = \sqrt{2}$.

Infinite Limits.

Definition 2.4.4. If $\{a_n\}$ is a sequence of real numbers, then we say $\lim a_n = \infty$ if, for every real number M, there is a number N such that

$$a_n > M \quad \text{whenever} \quad n > N.$$

Similarly, we say $\lim a_n = -\infty$ if for every real number M there is an N such that

$$a_n < M \quad \text{whenever} \quad n > N.$$

Example 2.4.5. If $r > 0$, prove that $\lim n^r = \infty$.

Solution: To prove that $\lim n^r = \infty$, we must show that for every M there is an N such that

$$n^r > M \quad \text{whenever} \quad n > N.$$

Clearly, we need only choose N to be $M^{1/r}$.

With $+\infty$ and $-\infty$ as possible limits of a sequence, we can now assert:

Theorem 2.4.6. *Every monotone sequence has a limit.*

The proof of this is left to the exercises.

Note that we must now make a distinction between a sequence *converging* and a sequence *having a limit*. A sequence may have a limit which is infinite, but a sequence which converges must have a finite limit.

Theorem 2.4.7. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Then*

(a) *if $a_n > 0$ for all n, then $\lim a_n = \infty$ if and only if $\lim 1/a_n = 0$;*

(b) *if $\{b_n\}$ is bounded below, then $\lim a_n = \infty$ implies $\lim(a_n + b_n) = \infty$;*

(c) $\lim a_n = \infty$ *if and only if* $\lim(-a_n) = -\infty$*;*

(d) *if* $a_n \le b_n$ *for all* n*, then* $\lim a_n = \infty$ *implies* $\lim b_n = \infty$*;*

(e) *if there is a positive constant* k *such that* $k \le b_n$ *for all* n*, then* $\lim a_n = \infty$ *implies* $\lim a_n b_n = \infty$*.*

Proof. We will prove (a) and (b) and leave (c), (d), and (e) to the exercises.

(a) If we are given an ϵ, we will set $M = 1/\epsilon$. Conversely, if we are given an M, we will set $\epsilon = 1/M$. Then the statements

$$|1/a_n| < \epsilon \quad \text{and} \quad a_n > M$$

mean the same thing (since a_n is positive) so that, if there is an N such that one of these statements is true for all $n > N$, then the other statement is also true for all $n > N$. Thus, $\lim 1/a_n = 0$ if and only if $\lim a_n = \infty$.

(b) Let b_n be bounded below by, say, K. Assuming $\lim a_n = \infty$, we wish to show that $\lim(a_n + b_n) = \infty$. Given $M \in \mathbb{R}$, the number $M - K$ is also in $\mathbb{R}$ and so, by our assumption that $\lim a_n = \infty$, we know there is an N such that

$$a_n > M - K \quad \text{whenever} \quad n > N.$$

Then

$$a_n + b_n > M - K + K = M \quad \text{whenever} \quad n > N.$$

Thus, $\lim(a_n + b_n) = \infty$. □

Example 2.4.8. Find the following limits:

(a) $\lim \dfrac{2n^2+3}{n+1}$;

(b) $\lim a^n$ for $a > 1$;

(c) $\lim(\sqrt{n} + (-1)^n)$.

Solution: (a) We factor the largest power of n that occurs out of each of the denominator and the numerator. The result is

$$\frac{2n^2+3}{n+1} = \frac{n^2(2+3/n^2)}{n(1+1/n)} = n\frac{2+3/n^2}{1+1/n}.$$

Now $\lim n = \infty$ and $\dfrac{2+3/n^2}{1+1/n} \ge 1$ for all n. Thus,

$$\lim \frac{2n^2+3}{n+1} = \infty,$$

by Theorem 2.4.7(e).

(b) Since $|1/a| < 1$, it follows from Example 2.3.5 that $\lim 1/a^n = 0$. Then $\lim a^n = +\infty$ by Theorem 2.4.7(a). Another proof of this fact is suggested in Exercise 2.4.7.

(c) Since $\sqrt{n} = n^{1/2}$, Example 2.4.5 implies that $\lim \sqrt{n} = \infty$. Then Theorem 2.4.7(b) implies that $\lim(\sqrt{n} + (-1)^n) = \infty$.

Exercise Set 2.4

1. Tell which of these sequences are non-increasing, non-decreasing, bounded? Justify your answers.
   (a) $\{n^2\}$.
   (b) $\left\{\frac{1}{\sqrt{n}}\right\}$.
   (c) $\left\{\frac{(-1)^n}{n}\right\}$.
   (d) $\left\{\frac{n}{2^n}\right\}$.
   (e) $\left\{\frac{n}{n+1}\right\}$.
2. Prove that the sequence of Example 1.2.11 converges and decide what number it converges to.
3. If $a_1 = 1$ and $a_{n+1} = (1 - 2^{-n})a_n$, prove that $\{a_n\}$ converges.
4. Let $\{d_n\}$ be a sequence of 0's and 1's and define a sequence of numbers $\{a_n\}$ by
   $$a_n = d_1 2^{-1} + d_2 2^{-2} + \cdots + d_n 2^{-n}.$$
   Prove that this sequence converges to a number between 0 and 1.
5. Let $\{s_n\}$ be the sequence of partial sums of a series with positive terms. That is,
   $$s_n = \sum_{k=1}^{n} a_k \quad \text{with all} \quad a_k \geq 0.$$
   Prove that $\lim s_n$ exists (though it may not be finite).
6. Give an alternate proof to the result of Example 2.3.5 that does not use the Binomial Theorem. Instead, first show that $\{|a|^n\}$ is a non-increasing sequence. Then show that 0 is the only possible value for the limit.
7. Give an alternate proof of the result of Example 2.4.8(b) that does not use Example 2.3.5. Use the method of the previous exercise.
8. Prove that $\lim \dfrac{n^5 + 3n^3 + 2}{n^4 - n + 1} = \infty$.
9. Prove that $\lim \dfrac{2^n}{n} = \infty$.
10. Prove Theorem 2.4.6.
11. Prove part (c) of Theorem 2.4.7.
12. Prove part (d) of Theorem 2.4.7.
13. Prove part (e) of Theorem 2.4.7.
14. Suppose $\{a_n\}$ and $\{b_n\}$ are non-decreasing sequences that are interlaced in the sense that each term of the sequence $\{a_n\}$ is less than or equal to some term of the sequence $\{b_n\}$ and vice versa. Prove that $\lim a_n = \lim b_n$.

2.5. Cauchy Sequences

In this section we will prove two of the most important theorems about convergence of sequences. The proofs are based on the Nested Interval Property, which we describe below.

Nested Intervals. A *nested sequence of closed bounded intervals* is a sequence

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

in which each I_n is a closed bounded interval and each interval in the sequence contains the next one. Thus, each of the intervals I_n has the form $[a_n, b_n]$ for real numbers $a_n < b_n$. The nested condition means that $I_n \supset I_{n+1}$ for each n – that is,

$$a_n \le a_{n+1} < b_{n+1} \le b_n$$

for each n.

Theorem 2.5.1 (Nested Interval Property). *If $I_1 \supset I_2 \supset I_3 \supset \cdots$ is a nested sequence of closed bounded intervals, then $\bigcap_n I_n \neq \emptyset$. That is, there is at least one point x that is in all the intervals I_n.*

Proof. Let $I_n = [a_n, b_n]$, as above. Then the sequence $\{a_n\}$ of left endpoints is a non-decreasing sequence which is bounded above (by b_1), and the sequence $\{b_n\}$ of right endpoints is a non-increasing sequence which is bounded below (by a_1). The Monotone Convergence Theorem (Theorem 2.4.1) implies that both sequences converge.

If $a = \lim a_n$ and $b = \lim b_n$, then $a \le b$ by Theorem 2.3.8. In fact,

$$a_n \le a \le b \le b_n$$

for each n. This means that $[a, b] \subset I_n$ for every n and, hence, that $[a, b] \subset \bigcap_n I_n$.

The set $[a, b]$ is a closed interval if $a < b$ and a single point if $a = b$. In either case, it is non-empty. □

We leave to the exercises the problem of showing that this theorem is false if we don't insist that the intervals are closed or if we don't insist that they are bounded.

The Bolzano-Weierstrass Theorem. A sequence $\{b_k\}$ is a *subsequence* of the sequence $\{a_n\}$ if it is made up of some of the terms of $\{a_n\}$, taken in the order that they appear in $\{a_n\}$. More precisely:

Definition 2.5.2. A sequence $\{b_k\}$ is a *subsequence* of the sequence $\{a_n\}$ if there is a strictly increasing sequence of natural numbers $\{n_k\}$ such that $b_k = a_{n_k}$.

Example 2.5.3. Give three examples of subsequences of the sequence

$$0,\ 3/2,\ -2/3,\ 5/4,\ -4/5,\ 7/6,\ -6/7,\ 9/8, \dots, (-1)^n + 1/n, \dots.$$

Does the original sequence converge? How about the three subsequences?

Solution:

(a) $3/2,\ 5/4,\ 7/6, \dots, 1 + 1/(2k), \dots$.

(b) $0,\ -2/3,\ -4/5, \dots, -1 + 1/(2k-1), \dots$.

(c) $3/2,\ 5/4,\ 9/8, \dots, 1 + 1/2^k, \dots$.

The original sequence clearly does not converge, but sequence (a) converges to 1, (b) converges to -1, and (c) converges to 1.

Theorem 2.5.4. *If $\{a_n\}$ has a limit (possibly infinite), then each of its subsequences has the same limit.*

Proof. We will prove this in the case of a finite limit. The other cases are similar and are covered in the exercises.

Suppose $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. Then $\{n_k\}$ is an increasing sequence of natural numbers, and this implies that $n_k \geq k$ for all k (Exercise 2.5.4).

Now suppose $\lim a_n = a$. Given $\epsilon > 0$, there is an $\mathbb{N}$ such that

$$|a_n - a| < \epsilon \quad \text{whenever} \quad n > N.$$

Then $k > N$ implies $n_k > N$, since $n_k \geq k$. Thus,

$$|a_{n_k} - a| < \epsilon \quad \text{whenever} \quad k > N.$$

By definition, this means that $\lim a_{n_k} = a$. □

Theorem 2.5.5 (Bolzano-Weierstrass Theorem). *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. If $\{a_n\}$ is a bounded sequence, then it has an upper bound M and a lower bound m. This means that every a_n is contained in the interval $I_1 = [m, M]$. We will construct a nested sequence of closed bounded intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots \tag{2.5.1}$$

such that I_k contains infinitely many of the terms of $\{a_n\}$ for each k and the length of I_k is $(M - m)/2^{k-1}$.

The first term of our sequence is I_1. Certainly I_1 contains infinitely many terms of $\{a_n\}$ – in fact, it contains all of them – and its length is $M - m$. The recursion relation for our inductive definition is as follows: if $I_1 \supset I_2 \supset \cdots \supset I_k$ have been chosen in such a way that I_k contains infinitely many terms of $\{a_n\}$ and the length of I_k is $(M - m)/2^{k-1}$, then we choose I_{k+1} as follows. We cut I_k into two closed intervals by dividing it at its midpoint. One of the two halves must contain infinitely many terms of $\{a_n\}$ since I_k does. Let I_{k+1} be the right half if it has this property; otherwise let it be the left half. Then I_{k+1} is contained in I_k, has length $(M-m)/2^k$, and contains infinitely many terms of $\{a_n\}$. By induction, there exists an infinite sequence (2.5.1) with the required properties.

By the Nested Interval Theorem, there is a point a that is in every one of the intervals I_k. Also, each interval I_k contains infinitely many terms of the sequence $\{a_n\}$. We will inductively define a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ with the property that $a_{n_k} \in I_k$ for each k. We choose $n_1 = 1$ and define n_{k+1} in terms of n_k by the rule that n_{k+1} is the first integer greater than n_k such that $a_{n_{k+1}} \in I_{k+1}$. This is the basis for an inductive definition of the sequence we seek. Once this sequence of integers has been chosen, then $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. We will show that this subsequence converges to a.

For each k, a and a_{n_k} both belong to I_k. This means the distance between them can be no greater than the length of I_k, which is $(M-m)2^{1-k}$. That is,

$$|a_{n_k} - a| \leq \frac{M-m}{2^{k-1}}.$$

Since $\dfrac{M-m}{2^{k-1}} \to 0$, Theorem 2.3.1 implies that $\lim a_{n_k} = a$. □

Example 2.5.6. Construct a sequence $\{a_n\}$ as follows: for each n let a_n be the number obtained by replacing by 0 all digits to the left of the decimal point in the decimal expansion of $10^n\pi$. Does this sequence have a convergent subsequence?

Solution: This is a crazy sequence and it certainly does not appear to converge. However, each number in this sequence lies between 0 and 1 and so it is a bounded sequence. By the Bolzano-Weierstrass Theorem it has a convergent subsequence.

Cauchy Sequences.

Definition 2.5.7. A sequence $\{a_n\}$ is said to be a *Cauchy sequence* if, for every $\epsilon > 0$, there is an N such that

$$|a_n - a_m| < \epsilon \quad \text{whenever} \quad n, m > N.$$

Intuitively, this means we can make the terms of the sequence arbitrarily close to each other by going far enough out in the sequence. It is by no means obvious that this means that the sequence converges, but it does.

Theorem 2.5.8. *A sequence of real numbers $\{a_n\}$ is a Cauchy sequence if and only if it converges.*

Proof. There are two things to prove here – the "if" and the "only if" parts. First we do the "if" part – that is, we will prove that a sequence is Cauchy if it converges.

Assume $\{a_n\}$ converges to a number a. Then, given $\epsilon > 0$, there is an N such that

$$|a_n - a| < \epsilon/2 \quad \text{whenever} \quad n > N.$$

If $n, m > N$, then

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore, $\{a_n\}$ is Cauchy.

Now for the "only if" part. Suppose $\{a_n\}$ is Cauchy. We first prove that $\{a_n\}$ is bounded. In fact, there is an N such that

$$|a_n - a_m| < 1 \quad \text{whenever} \quad n, m > N.$$

In particular, $|a_n - a_{N+1}| < 1$ for all $n > N$. This implies that

$$a_{N+1} - 1 < a_n < a_{N+1} + 1 \quad \text{whenever} \quad n > N.$$

Then $\max\{a_1, \dots, a_N, a_{N+1} + 1\}$ is an upper bound for $\{a_n\}$. Similarly, we have that $\min\{a_1, \dots, a_N, a_{N+1} - 1\}$ is a lower bound for $\{a_n\}$. Thus, $\{a_n\}$ is a bounded sequence.

We next use the Bolzano-Weierstrass Theorem to conclude there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ which converges to a number a. Finally, we use the definition of Cauchy sequence and what it means for a_{n_k} to converge to a. Given $\epsilon > 0$, there are numbers N_1 and N_2 such that

$$|a_n - a_m| < \epsilon/2 \quad \text{whenever} \quad n > N_1$$

and

$$|a_{n_k} - a| < \epsilon/2 \quad \text{whenever} \quad k > N_2.$$

If $n > N_1$ and if we choose a $k > \max\{N_1, N_2\}$, then

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \le |a_n - a_{n_k}| + |a_{n_k} - a| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This completes the proof that every Cauchy sequence is convergent. □

Example 2.5.9. Show that the sequence of partial sums of the series $\sum_{k=1}^{\infty}(-1)^k \frac{k}{4^k}$ converges.

Solution: We have $s_n = \sum_{k=1}^{n}(-1)^k \frac{k}{4^k}$ and so, for $m > n$,

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} (-1)^k \frac{k}{4^k} \right| \le \sum_{k=n+1}^{m} \frac{1}{2^k} \le \frac{1}{2^{n+1}} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}.$$

Here we have used the fact that $k \le 2^k$ for all k and the fact that the geometric series $\sum_{k=0}^{\infty} 2^{-k}$ has sum $\frac{1}{1-1/2} = 2$.

Since $\lim 1/2^n = 0$, by Example 2.3.5, given $\epsilon > 0$, there is an N such that $n > N$ implies $1/2^n < \epsilon$. Then $|s_m - s_n| < \epsilon$ for all n, m with $m > n > N$. This means that $\{s_n\}$ is Cauchy and, hence, converges.

Exercise Set 2.5

1. Give an example of a nested sequence of bounded open intervals that does not have a point in its intersection.
2. Give an example of a nested sequence of closed but unbounded intervals which does not have a point in its intersection.
3. Prove that if I is a closed, bounded interval which is contained in the union of some collection of open intervals, then I is contained in the union of some finite subcollection of these open intervals.
4. Prove by induction that if $\{n_k\}$ is an increasing sequence of natural numbers, then $n_k \ge k$ for all k.
5. Which of the following sequences $\{a_n\}$ have a convergent subsequence? Justify your answer.
   (a) $a_n = (-2)^n$.
   (b) $a_n = \dfrac{5 + (-1)^n n}{2 + 3n}$.
   (c) $a_n = 2^{(-1)^n}$.

6. For each of the following sequences $\{a_n\}$, find a subsequence which converges. Justify your answer.
   (a) $a_n = (-1)^n$.
   (b) $a_n = \sin n\pi/4$.
   (c) $a_n = \dfrac{n}{2^{k_n}} - 1$ with k_n the largest integer k so that $2^k \le n$.
7. For each of the following sequences, determine how many different limits of subsequences there are. Justify your answer.
   (a) $\{1 + (-1)^n\}$.
   (b) $\{\cos n\pi/3\}$.
   (c) $1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, 1, 1/2, 1/3, 1/4, 1/5, \ldots$.
8. Does the sequence $\sin n$ have a convergent subsequence? Why?
9. Prove that a sequence which satisfies $|a_{n+1} - a_n| < 2^{-n}$ for all n is a Cauchy sequence.
10. Suppose a sequence $\{a_n\}$ has the property that for every $\epsilon > 0$, there is an N such that
$$|a_{n+1} - a_n| < \epsilon \quad \text{whenever} \quad n > N.$$
Is $\{a_n\}$ necessarily Cauchy? Prove it is or give an example where it is not.
11. Let $s_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k2^k}$ be the sequence of partial sums of the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k}$. Prove that $\{s_n\}$ converges. Hint: Show that it is a Cauchy sequence.
12. Given a series $\displaystyle\sum_{k=1}^{\infty} a_k$, set $s_n = \displaystyle\sum_{k=1}^{n} a_k$ and $t_n = \displaystyle\sum_{k=1}^{n} |a_k|$. Prove that $\{s_n\}$ converges if $\{t_n\}$ is bounded.

2.6. lim inf and lim sup

According to the Bolzano-Weierstrass Theorem, a bounded sequence has a convergent subsequence. In fact, a bounded sequence has many convergent subsequences and these may converge to many different limits, as is illustrated by some of the exercises in the previous section. Here we will show that there is a smallest closed interval that contains all of these limits. The endpoints of this interval are the lim inf and the lim sup of the sequence.

Given a sequence $\{a_n\}$, we construct two monotone sequences $\{i_n\}$ and $\{s_n\}$ with $\{a_n\}$ trapped in between. They are defined as follows:

$$\begin{aligned} i_n &= \inf\{a_k : k \ge n\}, \\ s_n &= \sup\{a_k : k \ge n\}. \end{aligned} \tag{2.6.1}$$

Note that the i_n will all be $-\infty$ if $\{a_n\}$ is not bounded below and the s_n will all be $+\infty$ if $\{a_n\}$ is not bounded above. However, if $\{a_n\}$ is bounded, say $m \le a_n \le M$ for all n, then $m \le i_n \le s_n \le M$ for each n. Hence, in this case, the numbers i_n and s_n are all finite and $\{i_n\}$ and $\{s_n\}$ are bounded sequences.

Theorem 2.6.1. *Given a bounded sequence $\{a_n\}$, if $\{i_n\}$ and $\{s_n\}$ are defined as above, then*

(a) *$\{i_n\}$ is a non-decreasing sequence;*

(b) *$\{s_n\}$ is a non-increasing sequence;*

(c) *$i_n \le a_n \le s_n$ for all n.*

Proof. If $A_n = \{a_k : k \ge n\}$, then $A_{n+1} \subset A_n$ for each n. It follows from Theorem 1.5.7(e) that, for all n,

$$\begin{aligned} s_{n+1} &= \sup A_{n+1} \le \sup A_n = s_n \text{ and} \\ i_{n+1} &= \inf A_{n+1} \ge \inf A_n = i_n. \end{aligned} \tag{2.6.2}$$

Also, since $a_n \in A_n$, $i_n = \inf A_n \le a_n \le \sup A_n = s_n$. □

Since the sequences $\{i_n\}$ and $\{s_n\}$ are monotone, their limits exist.

Definition 2.6.2. If $\{a_n\}$ is a sequence and $\{i_n\}$ and $\{s_n\}$ are defined as above, then we set

$$\begin{aligned} \liminf a_n &= \lim i_n, \\ \limsup a_n &= \lim s_n. \end{aligned} \tag{2.6.3}$$

Note that if $\{a_n\}$ is not bounded below, then $\liminf a_n = -\infty$, while if $\{a_n\}$ is not bounded above, then $\limsup a_n = +\infty$.

Example 2.6.3. Find $\liminf a_n$ and $\limsup a_n$ if $a_n = (-1)^n + 1/n$.

Solution: As before, we let $i_n = \inf\{a_k : k \ge n\}$ and $s_n = \sup\{a_k : k \ge n\}$.

We claim $i_n = -1$ for all n. In fact,

$$-1 \le (-1)^k + 1/k \quad \text{for all} \quad k$$

implies

$$i_k = \inf\{(-1)^k + 1/k : k \ge n\} \ge -1.$$

Furthermore, $(-1)^k + 1/k$ approaches -1 for large odd k, so no number greater than -1 is a lower bound for $\{a_k : k \ge n\}$. Thus, $i_n = -1$, as claimed. This implies that $\liminf a_n = \lim i_n = -1$.

We claim that $1 \le s_n \le 1+1/n$. In fact, the set $\{(-1)^k + 1/k : k \ge n\}$ contains numbers greater than 1 no matter what n is, and so

$$s_n = \sup\{(-1)^k + 1/k : k \ge n\} \ge 1.$$

Furthermore, $(-1)^k + 1/k \le 1 + 1/n$ if $k \ge n$. Thus, $1 \le s_n \le 1 + 1/n$. This implies that $\limsup a_n = \lim s_n = 1$.

Subsequential Limits. If $\{a_n\}$ is a sequence, then by a *subsequential limit* of $\{a_n\}$ we mean a number which is the limit of some subsequence of $\{a_n\}$.

Theorem 2.6.4. *Every subsequential limit of $\{a_n\}$ lies between $\liminf a_n$ and $\limsup a_n$.*

Proof. If $\{a_{n_k}\}$ is a convergent subsequence of $\{a_n\}$, Theorem 2.6.1(c) implies

$$i_{n_k} \le a_{n_k} \le s_{n_k},$$

where $i_n = \inf\{a_k : k \ge n\}$ and $s_n = \sup\{a_k : k \ge n\}$. The sequences $\{i_{n_k}\}$ and $\{s_{n_k}\}$ are subsequences of $\{i_n\}$ and $\{s_n\}$, respectively, and, hence, have the same limits, namely $\liminf a_n$ and $\limsup a_n$, by Theorem 2.5.4. It follows from Theorem 2.3.8 and the above inequalities that

$$\liminf a_n \le \lim a_{n_k} \le \limsup a_n. \qquad \square$$

Theorem 2.6.5. *If $\{a_n\}$ is a sequence, then $\limsup a_n$ and $\liminf a_n$ are subsequential limits of $\{a_n\}$.*

Proof. We will show that $\limsup a_n$ is a subsequential limit of $\{a_n\}$. The same statement for lim inf has a similar proof. We will assume that $\limsup a_n$ is a finite number s. The case where $\limsup a_n = \infty$ is left as an exercise.

We must show that there is some subsequence of $\{a_n\}$ which converges to $s = \limsup a_n$. We will construct such a sequence inductively. As before, we let $s_n = \sup\{a_k : k \ge n\}$. For each $\epsilon > 0$, the number $s_n - \epsilon$ is less than s_n and so it is not an upper bound for $\{a_k : k \ge n\}$. This means there is an element of $\{a_k : k \ge n\}$ which is greater than $s_n - \epsilon$ but less than or equal to s_n. We will choose a sequence of such elements by induction.

We choose n_1 such that $s_1 - 1 < a_{n_1} \le s_1$. Suppose $n_1 < n_2 < \cdots < n_m$ have been chosen so that

$$s_j - 1/j < a_{n_j} \le s_j \quad \text{for} \quad j = 1, \dots, m. \tag{2.6.4}$$

We may then choose $n_{m+1} > n_m$ such that $s_{n_{m+1}} - 1/(m+1) < a_{n_{m+1}} \le s_{n_{m+1}}$. However, $n_{m+1} \ge m+1$ and so $s_{n_{m+1}} \le s_{m+1}$. In other words (2.6.4) holds with m replaced by $m+1$. This completes the induction step and proves that there is an increasing sequence of natural numbers $\{n_j\}$ such that (2.6.4) holds for all j.

Since both $s_j - 1/j \to s$ and $s_j \to s$, the subsequence $\{a_{n_j}\}$ also converges to s by the squeeze principle. $\square$

A Criterion for Convergence.

Theorem 2.6.6. *A sequence $\{a_n\}$ has limit a if and only if*

$$\limsup a_n = \liminf a_n = a.$$

Proof. We first prove that if $\limsup a_n = \liminf a_n = a$, then $\lim a_n$ exists and equals a. By Theorem 2.6.1(c),

$$i_n \le a_n \le s_n,$$

where i_n and s_n are as before. Since $\lim i_n = \lim s_n = a$, it follows from the squeeze principle that $\lim a_n = a$.

Next we assume $\lim a_n = a$. By Theorem 2.5.4 each subsequence of $\{a_n\}$ also has limit a. Since $\limsup a_n$ and $\liminf a_n$ are subsequential limits of $\{a_n\}$, they must both be equal to a. This completes the proof. $\square$

Exercise Set 2.6

1. Find $\limsup a_n$ and $\liminf a_n$ for the following sequences:
   (a) $a_n = (-1)^n$;
   (b) $a_n = (-1/n)^n$;
   (c) $a_n = \sin n\pi/3$.
2. Find lim inf and lim sup for the sequence of Exercise 2.5.6(c).
3. Find lim inf and lim sup for the sequence of Exercise 2.5.7(c).
4. If $\limsup a_n$ and $\limsup b_n$ are finite, prove that
$$\limsup\,(a_n + b_n) \le \limsup a_n + \limsup b_n.$$
5. If $\limsup a_n$ is finite, prove that $\liminf(-a_n) = -\limsup a_n$.
6. If $k \ge 0$ and $\limsup a_n$ is finite, prove that $\limsup ka_n = k \limsup a_n$.
7. If $a_n \ge 0$ and $b_n \ge 0$, prove that $\limsup a_n b_n \le (\limsup a_n)(\limsup b_n)$.
8. If $\{a_n\}$ and$\{b_n\}$ are non-negative sequences and $\{b_n\}$ converges, prove that $\limsup a_n b_n = (\limsup a_n)(\lim b_n)$.
9. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers – that is, a sequence of rational numbers in which each rational number appears exactly once. That such a thing exists is proved in the appendix. Show that, for each $x \in \mathbb{R}$, there is a subsequence of this sequence which converges to x. Hint: Use Exercise 1.4.7.
10. Prove Theorem 2.6.5 for lim sup in the case where $\limsup a_n = +\infty$.
11. Prove that c is $\limsup a_n$ if and only if there is a subsequence of $\{a_n\}$ which converges to c but there is no subsequence of $\{a_n\}$ which converges to a number greater than c.
12. Which numbers do you think are subsequential limits of $\{\sin n\}_{n=1}^{\infty}$? Can you prove that your guess is correct?