Historical Prologue

The theory of differential equations has a long history, beginning with Isaac Newton. From the early Greeks through Copernicus, Kepler, and Galileo, the motions of planets had been described directly in terms of their properties or characteristics, for example, that they moved on approximately elliptical paths (or in combinations of circular motions of different periods and amplitudes). Instead of this approach, Newton described the laws that determine the motion in terms of the forces acting on the planets. The effect of these forces can be expressed by differential equations. The basic law he discovered was that the motion is determined by the gravitational attraction between the bodies, which is proportional to the product of the two masses of the bodies and one over the square of the distance between the bodies. The motion of one planet around a sun obeying these laws can then be shown to lie on an ellipse. The attraction of the other planets could then explain the deviation of the motion of the planet from the elliptic orbit. This program was continued by Euler, Lagrange, Laplace, Legendre, Poisson, Hamilton, Jacobi, Liouville, and others.

By the end of the nineteenth century, researchers realized that many nonlinear equations did not have explicit solutions. Even the case of three masses moving under the laws of Newtonian attraction could exhibit very complicated behavior and an explicit solution was not possible (e.g., the motion of the sun, earth, and moon cannot be given explicitly in terms of known functions). Short term solutions could be given by power series, but these were not useful in determining long-term behavior. Poincaré, working from 1880 to 1910, shifted the focus from finding explicit solutions to discovering geometric properties of solutions. He introduced many of the ideas in specific examples, which we now group together under the heading of chaotic dynamical systems. In particular, he realized that a deterministic system (in which the outside forces are not varying and are not random) can exhibit behavior that is apparently random (i.e., it is chaotic).

In 1898, Hadamard produced a specific example of geodesics on a surface of constant negative curvature which had this property of chaos. G. D. Birkhoff continued the work of Poincaré and found many different types of long-term limiting behavior, including the α- and ω-limit sets introduced in Sections 4.1 and 11.1. His work resulted in the book [**Bir27**] from which the term "dynamical systems" comes.

During the first half of the twentieth century, much work was carried out on nonlinear oscillators, that is, equations modeling a collection of springs (or other physical forces such as electrical forces) for which the restoring force depends nonlinearly on the displacement from equilibrium. The stability of fixed points was studied by several people including Lyapunov. (See Sections 4.5 and 5.3.) The existence of a periodic orbit for certain self-excited systems was discovered by Van der Pol. (See Section 6.3.) Andronov and Pontryagin showed that a system of differential equations was structurally stable near an attracting fixed point, [**And37**] (i.e., the solutions for a small perturbation of the differential equation could be matched with the solutions for the original equations). Other people carried out research on nonlinear differential equations, including Bendixson, Cartwright, Bogoliubov, Krylov, Littlewood, Levinson, and Lefschetz. The types of solutions that could be analyzed were the ones which settled down to either (1) an equilibrium state (no motion), (2) periodic motion (such as the first approximations of the motion of the planets), or (3) quasiperiodic solutions which are combinations of several periodic terms with incommensurate frequencies. See Section 2.2.4. By 1950, Cartwright, Littlewood, and Levinson showed that a certain forced nonlinear oscillator had infinitely many different periods; that is, there were infinitely many different initial conditions for the same system of equations, each of which resulted in periodic motion in which the period was a multiple of the forcing frequency, but different initial conditions had different periods. This example contained a type of complexity not previously seen.

In the 1960s, Stephen Smale returned to using the topological and geometric perspective initiated by Poincaré to understand the properties of differential equations. He wrote a very influential survey article [**Sma67**] in 1967. In particular, Smale's "horseshoe" put the results of Cartwright, Littlewood, and Levinson in a general framework and extended their results to show that they were what was later called chaotic. A group of mathematicians worked in the United States and Europe to flesh out his ideas. At the same time, there was a group of mathematicians in Moscow lead by Anosov and Sinai investigating similar ideas. (Anosov generalized the work of Hadamard to geodesics on negatively curved manifolds with variable curvature.) The word "chaos" itself was introduced by T.Y. Li and J. Yorke in 1975 to designate systems that have aperiodic behavior more complicated than equilibrium, periodic, or quasiperiodic motion. (See [**Li,75**].) A related concept introduced by Ruelle and Takens was a *strange attractor*. It emphasized more the complicated geometry or topology of the attractor in phase space, than the complicated nature of the motion itself. See [**Rue71**]. The theoretical work by these mathematicians supplied many of the ideas and approaches that were later used in more applied situations in physics, celestial mechanics, chemistry, biology, and other fields.

The application of these ideas to physical systems really never stopped. One of these applications, which has been studied since earliest times, is the description and determination of the motion of the planets and stars. The study of the mathematical model for such motion is called *celestial mechanics*, and involves a finite number of bodies moving under the effects of gravitational attraction given by the Newtonian laws. Birkhoff, Siegel, Kolmogorov, Arnold, Moser, Herman, and many others investigated the ideas of stability and found complicated behavior for systems arising in celestial mechanics and other such physical systems, which could be described by what are called *Hamiltonian differential equations*. (These equations preserve energy and can be expressed in terms of partial derivatives of the energy function.) K. Sitnikov in [**Sit60**] introduced a situation in which three masses interacting by Newtonian attraction can exhibit chaotic oscillations. Later, Alekseev showed that this could be understood in terms of a "Smale horseshoe", [**Ale68a**], [**Ale68b**], and [**Ale69**]. The book by Moser, [**Mos73**], made this result available to many researchers and did much to further the applications of horseshoes to other physical situations. In the 1971 paper [**Rue71**] introducing strange attractors, Ruelle and Takens indicated how the ideas in nonlinear dynamics could be used to explain how turbulence developed in fluid flow. Further connections were made to physics, including the periodic doubling route to chaos discovered by Feigenbaum, [**Fei78**], and independently by P. Coullet and C. Tresser, [**Cou78**].

Relating to a completely different physical situation, starting with the work of Belousov and Zhabotinsky in the 1950s, certain mathematical models of chemical reactions that exhibit chaotic behavior were discovered. They discovered some systems of differential equations that not only did not tend to an equilibrium, but also did not even exhibit predictable oscillations. Eventually, this bizarre situation was understood in terms of chaos and strange attractors.

In the early 1920s, A.J. Lotka and V. Volterra independently showed how differential equations could be used to model the interaction of two populations of species, [**Lot25**] and [**Vol31**]. In the early 1970s, May showed how chaotic outcomes could arise in population dynamics. In the monograph [**May75**], he showed how simple nonlinear models could provide "mathematical metaphors for broad classes of phenomena." Starting in the 1970s, applications of nonlinear dynamics to mathematical models in biology have become widespread. The undergraduate books by Murray [**Mur89**] and Taubes [**Tau01**] afford good introductions to biological situations in which both oscillatory and chaotic differential equations arise. The books by Kaplan and Glass [**Kap95**] and Strogatz [**Str94**] include a large number of other applications.

Another phenomenon that has had a great impact on the study of nonlinear differential equations is the use of computers to find numerical solutions. There has certainly been much work done on deriving the most efficient algorithms for carrying out this study. Although we do discuss some of the simplest of these, our focus is more on the use of computer simulations to find the properties of solutions. E. Lorenz made an important contribution in 1963 when he used a computer to study nonlinear equations motivated by the turbulence of motion of the atmosphere. He discovered that a small change in initial conditions leads to very different outcomes in a relatively short time; this property is called *sensitive* *dependence on initial conditions* or, in more common language, the *butterfly effect*. Lorenz used the latter term because he interpreted the phenomenon to mean that a butterfly flapping its wings in Australia today could affect the weather in the United States a month later. We describe more of his work in Chapter 7. It was not until the 1970s that Lorenz's work became known to the more theoretical mathematical community. Since that time, much effort has gone into showing that Lorenz's basic ideas about these equations were correct. Recently, Warwick Tucker has shown, using a computer-assisted proof, that this system not only has sensitive dependence on initial conditions, but also has what is called a "chaotic attractor". (See Chapter 7.) About the same time as Lorenz, Ueda discovered that a periodically forced Van der Pol system (or other nonlinear oscillator) has what is now called a chaotic attractor. Systems of this type are also discussed in Chapter 7. (For a later publication by Ueda, see also [**Ued92**].)

Starting about 1970 and still continuing, there have been many other numerical studies of nonlinear equations using computers. Some of these studies were introduced as simple examples of certain phenomena. (See the discussion of the Rössler Attractor given in Section 7.4.) Others were models for specific situations in science, engineering, or other fields in which nonlinear differential equations are used for modeling. The book [**Enn97**] by Enns and McGuire presents many computer programs for investigation of nonlinear functions and differential equations that arise in physics and other scientific disciplines.

In sum, the last 40 years of the twentieth century saw the growing importance of nonlinearity in describing physical situations. Many of the ideas initiated by Poincaré a century ago are now much better understood in terms of the mathematics involved and the way in which they can be applied. One of the main contributions of the modern theory of dynamical systems to these applied fields has been the idea that erratic and complicated behavior can result from simple situations. Just because the outcome is chaotic, the basic environment does not need to contain stochastic or random perturbations. The simple forces themselves can cause chaotic outcomes.

There are three books of a nontechnical nature that discuss the history of the development of "chaos theory": the best seller *Chaos: Making a New Science* by James Gleick [**Gle87**], *Does God Play Dice?, The Mathematics of Chaos* by Ian Stewart [**Ste89**], and *Celestial Encounters* by Florin Diacu and Philip Holmes [**Dia96**]. Stewart's book puts a greater emphasis on the role of mathematicians in the development of the subject, while Gleick's book stresses the work of researchers making the connections with applications. Thus, the perspective of Stewart's book is closer to the one of this book, but Gleick's book is accessible to a broader audience and is more popular. The book by Diacu and Holmes has a good treatment of Poincaré's contribution and the developments in celestial mechanics up to today.