Preface

While such subjects as number theory and probability theory are commonly offered to undergraduates, it seems that Fourier analysis is rarely found, which is shocking when one considers the value of this subject not just within mathematics but also in the physical sciences and engineering. The author hopes that this book will encourage the view that Fourier analysis can be fruitfully presented not just to undergraduates, but even to younger undergraduates with no more experience than three or four terms of calculus. Such students will find a gentle introduction to the art of writing proofs and will be better prepared for advanced calculus and complex variables.

A student who has taken a course in advanced calculus may wonder what can be done with that machinery. The answer is: harmonic analysis (among other things). Paul Halmos is reported to have said words to the effect that the tragedy of Fourier series is that they were invented (in 1807) before convergence. The wonderful thing is that analysts such as Cauchy, Dirichlet, Riemann, and Weierstrass were motivated to develop the foundations of real analysis in order to make sense of Fourier series. In particular, Riemann defined his integral in order to provide a more rigorous basis for the discussion of Fourier series.

This book could be used for a capstone course of an undergraduate program or for beginning graduate students as a way to motivate the study of the Lebesgue integral. Since it is hoped that this book will be useful at a wide range of levels, it contains far more material than would ever be used in a single one term course. The author will be happy to provide suggestions adjusted to the instuctor's purpose.

We study Fourier analysis in three important settings. First we consider the Discrete Fourier Transform, which has to do with the use of roots of unity to describe periodic sequences. The results in this setting are easily obtained, and they form a framework for our endeavors in the more difficult subsequent settings. The point is that in the discrete setting there is no issue of convergence, but with Fourier Series we discover that convergence is a delicate matter. With Fourier Transforms of functions defined on the real line, matters are similar, but with additional difficulties. In the two latter situations we encounter points in our arguments where a detail is needed from advanced calculus or Lebegue measure theory. On such occasions, we simply quote the needed result and move on.

On the subject of Fourier Series, some authors use $\cos nx$ and $\sin nx$, so that all functions have period 2π. The consequence of this prescription is that most formulas have a $1/(2\pi)$ or $1/\pi$. Our contention is that the subject is more elegant when one works with functions with period 1, so that the basic building blocks are $\cos 2\pi nx$ and $\sin 2\pi nx$. But $\cos 2\pi nx + i \sin 2\pi nx = e^{2\pi inx}$ (a fact that will be a subject of discussion in Chapter 1), and it is more elegant still to use the complex exponential rather than sines and cosines. Of course, to proceed in this way, one must first become more comfortable with complex numbers. Hence that is the topic of Chapter 1. In general, when we are faced with a function with some strange period, we make a linear change of variable so that everything is translated into issues of functions with period 1. If sines and cosines are involved, we may convert to complex exponentials. When we resolve whatever is at issue, we may convert back to sines and cosines, if we wish. This is a little reminiscent of a problem expressed in terms of pounds and feet, which we would convert to grams and meters, and then convert back after the calculation is done.

Fourier analysis has links to many other branches of mathematics. We occasionally make remarks relating to such topics as linear algebra, probability theory, or number theory. Such digressions may be safely ignored by readers who are unfamiliar with the related subject in question.

Among the following chapters, sections, and appendices are found several valuable topics that are rarely found in the undergraduate (and sometimes even the graduate) curriculum. These include linear recurrences (in §F.4), summability theory (in §4.3), Bernoulli polynomials and Euler–Maclaurin summation (in §9.5), uniform distribution (in §9.6), Chebyshev polynomials (in Appendix C), and inequalities (in Appendix I).

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Hugh L. Montgomery
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