Chapter 2

Sequences and Series of Real Numbers

In the previous chapter we studied real numbers and their properties in detail. Among what we discussed was the Euclidean distance function d_e, introduced in Section 1.3. As we mentioned before, this notion of distance plays a crucial role in our analysis of real numbers. As a first manifestation of this fact, we show the way the distance can be used in the theory of real sequences and series. Sequences and series of real numbers appear in almost every calculus course. In this chapter we develop a solid theory for sequences and series and explain the way analysis helps us to remove the shortcomings of the theory one learns in calculus. To this end, we have to present the theory from its very beginning. This will help you to rebuild the building of sequences and series in your mind. To motivate the presentation, let us pose some questions.

(2.a) Why should we care about real sequences?

(2.b) What are the possible reasons for the divergence of a sequence?

(2.c) Does every sequence have a monotone subsequence? Does every sequence have a convergent subsequence?

(2.d) Is there a limit-free formulation of convergence? More precisely, can we prove that a sequence is convergent without knowing, or even guessing, the value of its limit?

(2.e) Why do we study infinite series?

(2.f) What happens if we change the order in which the terms of a given series appear? Does the new series have the same convergence situation as the original one?

These are some of the questions we will answer in this chapter. Question **(2.a)** will be answered in Section 2.1, where we show the necessity of considering sequences on the basis of what we learned in Chapter 1. To answer question **(2.b)**, we first find a necessary condition for convergence. This is *boundedness*: A sequence is *bounded* when its range (the set of all its terms) is a bounded subset of $\mathbb{R}$. With this definition, we then observe that any convergent sequence is bounded. A partial answer to **(2.b)** is obtained by the contrapositive law: Unboundedness is a possible reason for divergence. This is just a partial answer because, as we will see shortly, some bounded sequences are also divergent. This motivates us to pursue our study of divergent sequences in Section 2.2.

The second section begins with an important question: What makes a bounded sequence into a divergent one? To answer this question we study subsequences and the way they can be used to determine the convergence or divergence of a sequence. Our studies motivate the notions of *limit superior* and *limit inferior*. These concepts and their applications in the theory of sequences and series provide a good instance of the way analysis strengthens calculus. The questions posed in **(2.c)** will also be answered in this section. In fact, we will see that the first question can be answered in the affirmative, while the second one has a negative answer. It will be proved, however, that every *bounded* sequence of real numbers has a convergent subsequence. This result is known as the *Bolzano–Weierstrass theorem*.

We will answer question **(2.d)** in Section 2.3. To do so, we introduce *Cauchy sequences* and show that a real sequence is convergent if and only if it is Cauchy. Here the so-called *Cauchy's condition*, which determines Cauchy sequences, depends only on the terms of the sequence under study and has nothing to do to the limit. Thus it is the limit-free formulation of convergence we asked for in **(2.d)**. The third section is followed by the very short, but yet important, Section 2.4. This section contains some simple results about sequences in closed and bounded intervals.

The basic theory of infinite series of real numbers will be developed in Section 2.5, where we answer question **(2.e)** using an argument that justifies the mathematical meaning of a series. This section also includes some convergence tests, the most important of which are the strengthened versions of the ratio and root tests one learns in calculus. These last tests use the notions of limit superior and limit inferior, introduced in the second section, to strengthen their calculus versions.

We answer the questions posed in **(2.f)** in Section 2.6, which is devoted to a study of the rearrangements of series. It will be shown that every rearrangement of an absolutely convergent series converges to the same value as the original series, and that this is not the case for conditionally convergent series.

Finally in Section 2.7, we briefly discuss power series. The material presented in this section provides the necessary background for the study of Taylor series in Chapter 4.

2.1. Real Sequences, Their Convergence, and Boundedness

Intuitively, a sequence in some nonempty set X is an *ordered list* of the elements of X, something like $x_1, x_2, x_3, \ldots$. For example, arranging even natural numbers such as $2, 4, 6, \ldots$ allows us to think of the *sequence of even numbers* in $\mathbb{N}$. Formally, a *sequence* in X is a *function* from $\mathbb{N}$ into X. For instance, if we define a function x on $\mathbb{N}$ by $x(n) = 2n$, then the range of x is the subset of $\mathbb{N}$ whose only elements are even numbers. So, according to our intuitive understanding of the even numbers as a sequence in $\mathbb{N}$, we may think of x, or its range $\{x(n) : n \in \mathbb{N}\}$, as a sequence.

Now assume that x is a function from $\mathbb{N}$ into X, that is, a sequence in X. We use in place of $x(m)$ the simpler notation x_m and call this the mth term of the sequence $\{x_n\}$. In the first part of this book we are only concerned with real sequences, that is, ones whose range is a subset of $\mathbb{R}$. For this reason, by a sequence we always mean a real sequence.

To specify a sequence we may either

(1) present a formula for its nth term;

(2) list a few of its first terms so that the rule defining the nth term can be easily deduced; or

(3) define it by *recursion*.

Defining a sequence *recursively* means to present a couple of its first terms, and then give a formula that allows one to compute a term using its preceding one(s). For example, $x_n = n!$, $x_n :\ 1, 2!, 3!, 4!, \ldots$ and $x_1 = 1,\ x_{n+1} = (n+1)x_n$ for every $n \in \mathbb{N}$ all specify the same sequence.

Example 2.1. In each case find a formula that determines the nth term of the given sequence.

(1) $x_n :\ -1, 1, -1, 1, \ldots;$

(2) $y_n :\ -2, 5/2, -10/3, 17/4, \ldots.$

Solution.

(1) If we pay no attention to the signs, all terms include 1. So, to create the alternative appearance of the minus, we may write $x_n = (-1)^n$.

(2) We can write $-2 = -(1 + (1/1))$, $5/2 = 2 + (1/2)$, $-10/3 = -(3 + (1/3))$, and $17/4 = 4 + (1/4)$. Hence, $y_n = (-1)^n(n + (1/n))$ is the right formula.

Example 2.2. Find the first five terms of the following recursively defined sequences.

(1) $a_1 = 1,\ a_{n+1} = 3a_n;\ n \in \mathbb{N}.$

(2) $b_1 = 3,\ b_2 = 5,\ b_{n+2} = b_n + b_{n+1};\ n \in \mathbb{N}.$

Solution.

(1) We know that $a_1 = 1$. So $a_2 = 3a_1 = 3$, $a_3 = 3a_2 = 9$, $a_4 = 3a_3 = 27$, and $a_5 = 3a_4 = 81$. It is easy to find a formula for the nth term of this sequence. This is $a_n = 3^{n-1}$.

(2) Based on the recursive formula, $b_3 = 8$, $b_4 = 13$, and $b_5 = 21$.

Motivation: Why Do We Need Sequences? Before proceeding to the theory of sequences, it is appropriate to answer question **(2.a)**. Why should we care about real sequences? What makes us eager to learn so much about them? Perhaps you learned in your calculus courses that sequences can be used in natural sciences. An instance is the recursively defined *Fibonacci's sequence*

$$x_1 = 1,\ x_2 = 1,\ x_n = x_{n-1} + x_{n-2};\ n \geq 3,$$

which is important in biology. But, we are not going to use such applications to motivate our argument. This is because we are studying *analysis*, a discipline which requires a considerable amount of mathematical insight. So, it is better to find motivations within mathematics itself.

To begin with, consider a set $A \subset \mathbb{R}$ which is bounded from above, and let x be the supremum of A, which exists as a real number by the axiom of completeness. Then for every $n \in \mathbb{N}$, $x - 1/n$ is not an upper bound for A, and this gives us an element of A, say x_n, which is greater than $x - 1/n$. Since $x_n \leq x$, we find that $|x_n - x| < 1/n$ for every $n \in \mathbb{N}$.

Now, let $\epsilon > 0$ be given. By the Archimedean property we find $N \in \mathbb{N}$ such that $1/N < \epsilon$. Then for every $n \geq N$,

$$|x_n - x| < 1/n \leq 1/N < \epsilon.$$

In summary, given a set $A \subset \mathbb{R}$ with supremum x, we can find a sequence $\{x_n\}$ in A for which the following statement is true.

(SC) For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$.

(Here *SC* is used as the abbreviation of *sequential convergence*.)

What does *(SC)* say?

As we saw in Chapter 1, $d_e(x, y) = |x - y|$ is the distance between the points that represent x and y on the real line. Thus, the inequality $|x_n - x| < \epsilon$ says that the distance between x_n and x is less than ϵ. Since $\epsilon > 0$ is arbitrary and N is chosen in accordance with ϵ, this way of thinking allows us to interpret the statement *(SC)* as follows.

The terms x_n will be as close to x as we wish, provided that n is sufficiently large.

The truth of *(SC)* shows that the terms x_n of the sequence are *gathering* around x. In fact when *(SC)* is true, with the terminology of Chapter 1 every ϵ-neighborhood of x contains all terms of $\{x_n\}$ except perhaps a finite number of them.

Definition 2.3. If *(SC)* is true for a sequence $\{x_n\}$ and some $x \in \mathbb{R}$, we say that $\{x_n\}$ *converges* to x.

With this terminology, our above argument can be summed up into the following important result.

Theorem 2.4. *If x is the supremum of a set $A \subset \mathbb{R}$, then there exists a sequence $\{x_n\}$ in A that converges to x.*

What does Theorem 2.4 say?

Theorem 2.4 shows why sequences are so important in analysis. As we remember from Chapter 1, suprema and infima constitute a crucial part of the theory of real numbers, and the above theorem *ties* suprema to sequences. In this way, analysis strengthens our knowledge of sequences by demonstrating their relevance to the least upper bounds. Also, Theorem 2.4 tells us, in view of the above interpretation of *(SC)*, that the supremum of a set is in some sense *adhered* to it and cannot be of a considerable distance from its elements.

Example 2.5. Recall from Example 1.43(1) that $\sup(0,1) = 1$. Find a sequence in $(0,1)$ that converges to 1.

Solution. We claim that $\{1 - 1/(2n)\}$ is such a sequence. First, it is clear that $0 < 1 - 1/(2n) < 1$ for every n, so that this is a sequence in $(0,1)$. Next, if $\epsilon > 0$ is given, we may find $N \in \mathbb{N}$ such that $1/N < 2\epsilon$. Then, for every $n \geq N$,

$$\left|\left(1 - \frac{1}{2n}\right) - 1\right| = \frac{1}{2n} \leq \frac{1}{2N} < \epsilon.$$

This proves that $\{1 - 1/(2n)\}$ converges to 1.

Of course, Theorem 2.4 tells nothing about the uniqueness of the sequence that converges to the supremum, as the sequence is not unique in general. For example, $\{1 - 1/(3n)\}$ is another sequence in $(0,1)$ which converges to 1.

Exercise 2.6. In each case find the supremum of the given set, then find a sequence in the set that converges to the supremum.

(1) $A = [-1, 0]$.

(2) $B = \{-1, 1\}$.

Exercise 2.7. Let x be the infimum of a set $A \subset \mathbb{R}$ which is bounded from below. Prove that there exists a sequence $\{x_n\}$ in A that converges to x.

Now, let us consider another situation in which sequences appear naturally. As we saw in Theorem 1.49, between any two real numbers a rational number can be found. So if $x \in \mathbb{R}$ is arbitrary, then for every $n \in \mathbb{N}$ a rational number p_n can be found such that $x < p_n < x + 1/n$. Now if $\epsilon > 0$ is given, by choosing $N \in \mathbb{N}$ with $1/N < \epsilon$, we see that

$$|p_n - x| < 1/n \leq 1/N < \epsilon$$

for every $n \geq N$, that is, $\{p_n\}$ converges to x. Therefore, we proved the following interesting result.

Proposition 2.8. *If x is any real number, there is a sequence of rational numbers that converges to x.*

Of course when x is rational, $\{x - 1/(2n)\}$ is a sequence of rational numbers that converges to x. This can be proved in the same way as we showed the convergence of $\{1 - 1/(2n)\}$ to 1 in Example 2.5. The following example provides an illustration when x is irrational.

Example 2.9. Prove that $\{[2^n\sqrt{2}]/2^n\}$ converges to $\sqrt{2}$.

Solution. It is clear that this is a sequence of rational numbers. Since

$$[x] \le x < 1 + [x]$$

for each $x \in \mathbb{R}$, we observe that for every $n \in \mathbb{N}$,

$$\left| \frac{[2^n\sqrt{2}]}{2^n} - \sqrt{2} \right| = \sqrt{2} - \frac{[2^n\sqrt{2}]}{2^n} < \frac{1}{2^n}.$$

Hence, if we choose $N > \log_2^{1/\epsilon}$, then $|[2^n\sqrt{2}]/2^n - \sqrt{2}| \le 1/2^N < \epsilon$ for every $n \ge N$. This proves the desired result.

Exercise 2.10. Determine those real numbers x for which the sequence $\{[2^n x]/2^n\}$ converges to x.

Exercise 2.11. Show that for every real number x, there is a sequence of irrational numbers which converges to x.

More on Convergence: Basic Facts and Examples. We are now ready to start a careful study of sequences. When $\{x_n\}$ converges to x, we say that x is the *limit* of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$. The following proposition describes the reason we used "the limit" instead of "a limit".

Proposition 2.12. *If $\{x_n\}$ converges to x and y, then $x = y$.*

Proof. To prove $x = y$, it is enough to show that $|x - y| < \epsilon$ for every $\varepsilon > 0$. This is because 0 is the only nonnegative real number which is less than every positive number. So let $\epsilon > 0$ be given. Find N_1 and N_2 in $\mathbb{N}$ such that $|x_n - x| < \epsilon/2$ and $|x_n - y| < \epsilon/2$ for every $n \ge N_1$ and $n \ge N_2$, respectively. Now, if $N = \max\{N_1, N_2\}$, then by the triangle inequality

$$|x - y| = |x - x_N + x_N - y| \le |x - x_N| + |x_N - y| < \epsilon.$$ □

What does Proposition 2.12 say?

Proposition 2.12 says that every sequence can converge to only one limit. This is intuitively evident because when the terms of a sequence are gathering around some point x, the same cannot be true for another point y.

The above uniqueness result can be proved in an indirect way. Indeed, assuming $x \neq y$, we get $|x - y| > 0$. Now, for $\epsilon = (1/2)|x - y| > 0$, we may find some term x_n whose distance from both x and y is less than ϵ. Then, the triangle inequality yields

$$|x - y| \le |x - x_n| + |x_n - y| < 2\epsilon = |x - y|,$$

a contradiction.

Our next result shows that the convergence or divergence of a sequence cannot be affected by changing a finite number of its terms.

Proposition 2.13. *Assume that a sequence $\{x_n\}$ converges to x, and that $\{y_n\}$ is a sequence satisfying $y_n = x_n$ for all but finitely many $n \in \mathbb{N}$. Then $\{y_n\}$ also converges to x.*

Proof. Let $k \in \mathbb{N}$ be such that $y_n = x_n$ holds for every $n \geq k$. If $\epsilon > 0$ is given, find N so large that $n \geq N$ implies $|x_n - x| < \epsilon$. Then, $|y_n - x| < \epsilon$ for every $n \geq \max\{k, N\}$. This completes the proof. □

What does Proposition 2.13 say?

Proposition 2.13 shows that the convergence of a sequence is *stable* under changing a finite number of its terms. *Stability results* appear here and there in mathematics and in analysis in particular. *They determine the amount of change a system, situation, or condition may tolerate before loosing its original form.* In simpler words, Proposition 2.13 says that a convergent sequence remains convergent if we modify a finite number of its terms.

Exercise 2.14. Is the convergence of a sequence stable under changing an infinite number of its terms? Justify your answer.

Now that we are sure about the uniqueness of limits, it is time to consider some more examples.

Example 2.15. Verify the following equalities by the $\epsilon - N$ definition of limit.

(1) $\lim_{n\to\infty} 1/n = 0$.
(2) $\lim_{n\to\infty} (\sin^3(n+1))/n^4 = 0$.
(3) $\lim_{n\to\infty} (n^2 - 1)/(n^2 + 1) = 1$.
(4) $\lim_{n\to\infty} a^n = 0$, if a is any real number satisfying $|a| < 1$.
(5) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Solution. Let $\epsilon > 0$ be given.

(1) If we find N such that $1/N < \epsilon$, then for every $n \geq N$,

$$|1/n - 0| = 1/n \leq 1/N < \epsilon.$$

(2) Let N be such that $1/N < \sqrt[4]{\epsilon}$. Then for every $n \geq N$,

$$\left| \frac{\sin^3(n+1)}{n^4} - 0 \right| = \frac{|\sin^3(n+1)|}{n^4} \leq \frac{1}{n^4} \leq \frac{1}{N^4} < \epsilon.$$

Here, the first inequality follows from the fact that $|\sin x| \leq 1$ for every $x \in \mathbb{R}$.

(3) For every n, $|(n^2 - 1)/(n^2 + 1) - 1| = 2/(n^2 + 1) < 2/n^2$. So, if we find $N > \sqrt{2/\epsilon}$, then $|(n^2 - 1)/(n^2 + 1) - 1| < \epsilon$ for every $n \geq N$.

(4) If $a = 0$, then the desired equality holds obviously. So, assume that $0 < |a| < 1$. Let $b = 1/|a| - 1$. Then b is positive, and $|a| = 1/(1 + b)$. Now Bernoulli's inequality (1.9) of Example 1.31 yields for every $n \in \mathbb{N}$ that $(1 + b)^n \geq 1 + nb$. Hence,

$$|a|^n = 1/(1 + b)^n \leq 1/(1 + nb) < 1/(nb).$$

Thus, if we choose N such that $1/N < \epsilon b$, then for every $n \geq N$,

$$|a^n - 0| = |a|^n < 1/(nb) \leq 1/(Nb) < \epsilon.$$

(5) It is enough to show that $a_n = \sqrt[n]{n} - 1$ tends to zero as n tends to infinity. To do so, we should prove that for all sufficiently large n, $|a_n| = a_n < \epsilon$. Now notice that by the Binomial Theorem for every $n > 1$,

$$n = (1 + a_n)^n = 1 + na_n + \frac{1}{2}n(n-1)a_n^2 + \cdots + a_n^n \geq 1 + na_n + \frac{1}{2}n(n-1)a_n^2.$$

Thus, a simple calculation shows $a_n^2 \leq 2/n$. Therefore, $a_n \leq \sqrt{2/n}$ for every n, and if we choose a natural number $N > 2/\epsilon^2$, then $a_n < \epsilon$ for every $n \geq N$.

Exercise 2.16. If c is any positive real number, prove that $\lim_{n\to\infty} c^{1/n} = 1$. *Hint.* Consider two cases, $0 < c < 1$ and $c > 1$, and use Bernoulli's inequality (Example 1.31) in each case.

Exercise 2.17. If $0 < r < 1$ is arbitrary, prove that the sequence $\{\sqrt{n}r^n\}$ converges to 0.

Divergence: A First Glance. When a sequence is not convergent, we say that it is *divergent*.

Remark 2.18. To prove that a sequence $\{x_n\}$ is divergent, one way is to negate statement *(SC)* for every $x \in \mathbb{R}$. In fact, the divergence of $\{x_n\}$ can be proved by showing that the following statement is true.

Given any $x \in \mathbb{R}$ a positive number ϵ can be found such that for every $N \in \mathbb{N}$, $|x_n - x| \geq \epsilon$ holds for some $n \geq N$.

Example 2.19. Use the method of Remark 2.18 to prove that the sequence $\{\sqrt{n}\}$ is divergent.

Solution. Let $x \in \mathbb{R}$ be arbitrary, and consider $\epsilon = 1$. If $N \in \mathbb{N}$ is given, choose $n > \max\{N, (1+x)^2\}$. If $1 + x \leq 0$, then it is clear that

$$\sqrt{n} > 1 + x, \tag{2.1}$$

because this holds for every natural number n. Otherwise, (2.1) follows from our choice of n. In either case, (2.1) gives us $|\sqrt{n} - x| > 1$.

Bounded and Unbounded Sequences. Now we turn to question **(2.b)**. What are the possible reasons for the divergence of a sequence? To answer this question, we start by finding a necessary condition for the convergence of sequences. Recall that a *necessary condition* for the truth of some statement p is another statement q which is implied by p.

To begin with, let $\{x_n\}$ be a convergent sequence with limit x. For $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for every $n \geq N$. Thus for all such n,

$$|x_n| - |x| \leq |x_n - x| < 1,$$

that is, $|x_n| < 1 + |x|$. If we let

$$M = \max\{|x_1|, \ldots, |x_{N-1}|, 1 + |x|\},$$

then it follows that $|x_n| \leq M$ for every $n \in \mathbb{N}$.

Since we defined sequences as functions from $\mathbb{N}$ into $\mathbb{R}$, the set

$$\{x_n : \ n \in \mathbb{N}\} \subset \mathbb{R}$$

that corresponds to a sequence $\{x_n\}$ may be called its *range*. With this terminology, our above discussion shows that the convergence of a sequence implies the boundedness of its range (Proposition 1.57). To state this result more simply, let us say that a sequence is *bounded* when its range satisfies the same property. Therefore, we have proved the following.

Theorem 2.20. *Every convergent sequence is bounded.*

When a sequence is not bounded, we say that it is *unbounded*. Notice that by the contrapositive law of mathematical logic, the following statement is equivalent to Theorem 2.20.

Every unbounded sequence is divergent.

In other words, unboundedness is another reason for divergence.

Various kinds of unbounded sequences.

It is clear that a sequence $\{x_n\}$ is unbounded if and only if it is either

(1) bounded from above and unbounded from below,
(2) unbounded from above and bounded from below, or
(3) unbounded from above and below.

Example 2.21. The following sequences are unbounded and, accordingly, divergent. In each case determine the reason for unboundedness.

(1) $\{\sqrt{n}\}$.
(2) $\{-n^2\}$.
(3) $\{(-1)^n n\}$.
(4) $\{a^n\}$, if a is any real number satisfying $|a| > 1$.

Solution.

(1) If $M > 0$ is arbitrary, we may find $N \in \mathbb{N}$ such that $N > M^2$. Then $\sqrt{n} > M$ for every $n \geq N$. Note that 1 is a lower bound for this sequence and the sequence is unbounded from above.
(2) For a given $L < 0$, $-L > 0$ and we may choose $N \in \mathbb{N}$ with $N > \sqrt{-L}$. Then $-n^2 < L$ for every $n \geq N$, and this shows that the sequence is unbounded from below. It should be noted that -1 is an upper bound for this sequence.
(3) The range of this sequence is $\{2k : k \in \mathbb{N}\} \cup \{-(2k-1) : k \in \mathbb{N}\}$. Since the first set is not bounded from above and the second one is not bounded from below, the sequence is unbounded from above and below.
(4) Since $|a| > 1$, we have two cases as follows.
   - $a > 1$. In this case a positive number c exists, namely $c = a - 1$, such that $a = 1 + c$. Then, by the Binomial Theorem we find that for every n,

$$a^n = (1+c)^n = 1 + nc + \frac{1}{2}n(n-1)c^2 + \cdots + c^n > nc.$$

   So if $M > 0$ is given, choosing $N \in \mathbb{N}$ so large that $N > M/c$, we find that $a^n > nc \geq Nc > M$ for every $n \geq N$. This proves that $\{a^n\}$ is unbounded from above and, hence, divergent in this case.

- $a < -1$. Here, the range is unbounded from above, as its subset

$$\{a^{2k} : \; k \in \mathbb{N}\}$$

is also. The range is also unbounded from below. To see this, it is enough to show that its subset $\{a^{2k-1} : k \in \mathbb{N}\}$ has the same property. But this follows easily because

$$\{a^{2k-1} : k \in \mathbb{N}\} = (1/a)\{a^{2k} : k \in \mathbb{N}\},$$

the right-hand set is unbounded from above, and $1/a < 0$. We therefore proved that the sequence $\{a^n\}$ is unbounded from above and below when $a < -1$.

Of course, a bounded sequence may be divergent. This fact reveals that unboundedness is not the only reason for divergence. We will pursue our study of the reasons of divergence in Section 2.2.

Example 2.22. Show that the sequence $\{\cos n\pi\}$ is bounded and nevertheless divergent.

Solution. It is known that for every n, $\cos n\pi = (-1)^n$ and hence $|\cos n\pi| = 1$, which shows that this sequence is bounded. Assume to the contrary that $\{(-1)^n\}$ converges to some $L \in \mathbb{R}$. For $\epsilon = 1$, we then find some $N \in \mathbb{N}$ such that for every $n \geq N$, $|(-1)^n - L| < 1$. Let $n_0 > N$ be even. Then $n_0 + 1$ is odd, and we obtain from the last inequality that

$$|(-1)^{n_0} - L| = |1 - L| < 1 \tag{2.2}$$

and

$$|(-1)^{n_0+1} - L| = |-1 - L| < 1. \tag{2.3}$$

But (2.2) yields $0 < L < 2$ and (2.3) gives $-2 < L < 0$. This contradiction shows that $\{\cos n\pi\}$ is divergent.

A note on Example 2.22.

Comparing Example 2.22 with Example 2.21(1), we observe two different divergence types. The terms of the sequence $\{\sqrt{n}\}$ are growing unlimitedly, and this prevents the sequence from being convergent. We describe this by saying that the sequence diverges to $+\infty$. On the other hand, the sequence $\{(-1)^n\}$ cannot converge to any real number, in spite of the fact that all of its terms are in the bounded set $\{-1, 1\}$.

The next result can be used to find the limit of many sequences.

Proposition 2.23. *Let $\{x_n\}$ be a bounded sequence. If $\{y_n\}$ is a sequence that converges to 0, then $\lim_{n\to\infty} x_n y_n = 0$.*

Proof. Let $M > 0$ be such that $|x_n| \leq M$ for every $n \in \mathbb{N}$. If $\epsilon > 0$ is given, find a natural number N such that $|y_n| < \epsilon/M$ for every $n \geq N$. Then, for all such n,

$$|x_n y_n - 0| = |x_n||y_n| < M \, \frac{\epsilon}{M} = \epsilon. \qquad \square$$

To interpret Proposition 2.23, let us call the sequence $\{x_n y_n\}$ the product of $\{x_n\}$ and $\{y_n\}$.

What does Proposition 2.23 say?

Proposition 2.23 says that the product of a sequence which converges to zero and a bounded sequence will also converge to zero.

Exercise 2.24. Let $\{x_n\}$ be a bounded sequence. If $\{y_n\}$ is a sequence that converges to some nonzero real number, is it necessarily true that $\{x_n y_n\}$ is convergent? Why?

Example 2.25. Find the value of the following limits.

(1) $\lim_{n\to\infty}(\sin n)/n$.
(2) $\lim_{n\to\infty}(-1)^n(\sqrt[n]{n}-1)$.

Solution.

(1) The sequence $\{\sin n\}$ is bounded and $\lim_{n\to\infty} 1/n = 0$. The desired limit is therefore equal to 0 by Proposition 2.23.
(2) We observed in the solution of Example 2.15(5) that $\lim_{n\to\infty}(\sqrt[n]{n}-1) = 0$. Now, it follows from the boundedness of $\{(-1)^n\}$ and Proposition 2.23 that the given limit is equal to 0.

Algebraic Operations and Convergence. Given arbitrary sequences $\{x_n\}$ and $\{y_n\}$, we can use the algebraic operations of addition, multiplication, and subtraction to define new sequences. These are $\{x_n + y_n\}$, $\{x_n y_n\}$, and $\{x_n - y_n\}$, respectively. We have already considered the product sequence $\{x_n y_n\}$ in Proposition 2.23. If $y_n \neq 0$ for every $n \in \mathbb{N}$, then we can also use division to obtain the sequence $\{x_n/y_n\}$. But, what is the relation between the convergence of $\{x_n\}$ and $\{y_n\}$ and that of the above-mentioned sequences? Clearly, we expect to see that the limit *respects* the algebraic operations, which is the content of our following result.

Theorem 2.26. *If $\{x_n\}$ and $\{y_n\}$ converge to x and y, respectively, then*

(1) $\lim_{n\to\infty}(x_n + y_n) = x + y$, $\lim_{n\to\infty}(x_n - y_n) = x - y$; *and*
(2) $\lim_{n\to\infty}(x_n y_n) = xy$;
(3) *if, in addition, $y_n \neq 0$ for every n and $y \neq 0$, then* $\lim_{n\to\infty} x_n/y_n = x/y$.

Proof. Let $\epsilon > 0$ be given.

(1) We only prove the first equality as the second one can be proved similarly. There exist N_1 and N_2 such that $|x_n - x| < \epsilon/2$ and $|y_n - y| < \epsilon/2$, for every $n \geq N_1$ and $n \geq N_2$, respectively. Let $N = \max\{N_1, N_2\}$. Then, by the triangle inequality, for every $n \geq N$,

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \epsilon/2 + \epsilon/2 = \epsilon.$$

(2) Since $\{x_n\}$ is convergent, it is bounded by Theorem 2.20. Let $M > 0$ be such that for every n, $|x_n| < M$. Then, by the triangle inequality,

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n||y_n - y| + |y||x_n - x| \\ &\leq M|y_n - y| + |y||x_n - x| \end{aligned}$$

for every n. Now letting $K = \max\{M, |y|\}$, we see that for every n,

$$|x_n y_n - xy| \leq K(|y_n - y| + |x_n - x|).$$

Choosing N_1 and N_2 so large that $n \geq N_1$ and $n \geq N_2$ imply $|x_n - x| < \epsilon/(2K)$ and $|y_n - y| < \epsilon/(2K)$, respectively, $|x_n y_n - xy| < \epsilon$ follows for all $n \geq \max\{N_1, N_2\}$.

(3) It is sufficient to show that with the assumptions we have, $\lim_{n\to\infty} 1/y_n = 1/y$, because the desired result then follows from this and item (2). To this end we note that

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y_n - y|}{|y_n||y|}. \tag{2.4}$$

Since $\lim_{n\to\infty} y_n = y \neq 0$, we may choose $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, $|y_n - y| < |y|/2$. But, $||y_n| - |y|| \leq |y_n - y|$, and hence it follows that

$$|y_n| > \frac{|y|}{2} \tag{2.5}$$

for every $n \geq N_1$. Combining (2.4) and (2.5) we see that for every $n \geq N_1$,

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| < \frac{2}{|y|^2}\ |y_n - y|.$$

So if we find $N_2 \in \mathbb{N}$ such that $|y_n - y| < \epsilon\ |y|^2/2$ for all $n \geq N_2$, then for every $n \geq \max\{N_1, N_2\}$, $|1/y_n - 1/y| < \epsilon$. □

What does Theorem 2.26 say?

The first equality in Theorem 2.26(1) says that the limit of the sum of two convergent sequences is the sum of their limits. The remaining equalities can be interpreted similarly.

Exercise 2.27. Show by means of an example that a sequence of nonzero real numbers may converge to 0. This shows that the assumption $y \neq 0$ is necessary in Theorem 2.26(3).

Since a sequence, all of whose terms are equal to some fixed number a, must converge to a itself, it follows that for an arbitrary sequence $\{x_n\}$ with limit x, $\lim_{n\to\infty}(a + x_n) = a + x$, $\lim_{n\to\infty}(a - x_n) = a - x$ and $\lim_{n\to\infty}(ax_n) = ax$. In particular, we find that $\lim_{n\to\infty}(-x_n) = -\lim_{n\to\infty} x_n = -x$.

Example 2.28. Find the value of the given limits.

(1) $\lim_{n\to\infty}(2n^2 - 1)/(3n^2 + 4n)$.
(2) $\lim_{n\to\infty}((0.1)^n + 2\sqrt[n]{n})/n^2$.

Solution.

(1) We first note that

$$\frac{2n^2-1}{3n^2+4n}=\frac{n^2(2-1/n^2)}{n^2(3+4/n)}=\frac{(2-1/n^2)}{(3+4/n)}.$$

Since $\lim_{n\to\infty} 1/n = 0$ by Example 2.15(1),

$$\lim_{n\to\infty} 2-\frac{1}{n^2}=2-\left(\lim_{n\to\infty}\frac{1}{n}\right)^2=2$$

and $\lim_{n\to\infty} 3+4/n = 3+4\lim_{n\to\infty} 1/n = 3 \neq 0$. Thus, Theorem 2.26(3) tells us that the given limit is 2/3.

(2) For every n, $((0.1)^n + 2\sqrt[n]{n})/n^2 = ((0.1)^n + 2\sqrt[n]{n})\left(1/n^2\right)$. Therefore, letting n tend to infinity in both sides of this equality and using (4) and (5) of Example 2.15 and Theorem 2.26(2), we find that

$$\lim_{n\to\infty}\frac{(0.1)^n+2\sqrt[n]{n}}{n^2}=(0+2)(0)=0.$$

Monotone Sequences. If $\{x_n\}$ is a bounded sequence, then its range

$$X=\{x_n:\ n\in\mathbb{N}\}$$

is a bounded subset of $\mathbb{R}$, by our definition. Let $x=\sup X$. By Theorem 2.4, x will be the limit of a sequence in X. Since $\{x_n\}$ itself is a sequence in X, we encounter a natural question.

When does a bounded sequence converge to the supremum of its range?

To answer this question, denote the assertion "$\{x_n\}$ converges to the supremum of its range" by q. We are therefore seeking a statement p concerning $\{x_n\}$ that implies q. By the contrapositive law, we should then be able to deduce the negation of p from that of q. So, let us begin with the negation of q; that is, let us assume that $\{x_n\}$ is a bounded sequence that does not converge to the supremum x of its range.

By Remark 2.18 there exists $\varepsilon>0$ such that for every $N\in\mathbb{N}$, $n\geq N$ can be found with $x-x_n=|x_n-x|\geq\varepsilon$. Since $x-\varepsilon<x$, we find $m\in\mathbb{N}$ such that

$$x_m > x-\varepsilon. \tag{2.6}$$

By our choice of ε, for $N=m+1$ we find $n\geq N$ such that

$$x-x_n\geq\varepsilon. \tag{2.7}$$

It then follows from (2.6) and (2.7) that

$$x_n\leq x-\varepsilon<x_m.$$

In summary, we found $m,n\in\mathbb{N}$ with $n>m$ and $x_n<x_m$. The negation of p for $\{x_n\}$ then reads as follows.

There exist $m,n\in\mathbb{N}$ such that $n>m$ and $x_n<x_m$.

The statement p itself therefore states the following.

For every $m,n\in\mathbb{N}$ with $n>m$, $x_n\geq x_m$.

When the above statement is true for a sequence $\{x_n\}$, we say that $\{x_n\}$ is *increasing*. Thus, we observed that *a bounded and increasing sequence necessarily converges to the supremum of its range*. We will present a more nicely arranged proof of this fact below. For now we give a formal definition of increasing sequences.

Definition 2.29. A sequence $\{x_n\}$ is *increasing* (resp., *decreasing*) if for every $n \in \mathbb{N}$, $x_n \leq x_{n+1}$ (resp., $x_n \geq x_{n+1}$). If $\leq$ (resp., $\geq$) is replaced by $<$ (resp., $>$), then we say that $\{x_n\}$ is *strictly increasing* (resp., *strictly decreasing*). A *monotone* sequence is one which is either increasing or decreasing.

Example 2.30. In each case determine whether the given sequence is monotone or not.

(1) $\{1 - 1/n^3\}$.
(2) $\{\sqrt{n}\}$.
(3) $\{(-1)^n \cos^2 n\}$.

Solution.

(1) It is clear that for every n, $(n+1)^3 > n^3$ and hence $-1/(n+1)^3 > -1/n^3$. Adding 1 to the both sides of this last inequality, we find that the sequence is strictly increasing.
(2) Since $\sqrt{n+1} > \sqrt{n}$ for every n, this sequence is also strictly increasing.
(3) The terms of this sequence are alternatively negative and positive. This shows that the sequence is neither increasing nor decreasing.

We now come to the formal statement of our observation above.

Theorem 2.31. *An increasing sequence which is bounded from above converges to the supremum of its range.*

Proof. Let $\{x_n\}$ be a bounded increasing sequence. The boundedness assumption ensures, in view of the axiom of completeness, that the range $X = \{x_n : n \in \mathbb{N}\}$ of the sequence has a supremum. Denote this by x. We prove that $\lim_{n\to\infty} x_n = x$. To see this, let $\varepsilon > 0$ be given. Since $x = \sup X$ and $x - \varepsilon < x$, $x - \varepsilon$ cannot be an upper bound for X. Thus, we can find $N \in \mathbb{N}$ such that $x_N > x - \varepsilon$. That $\{x_n\}$ is increasing shows that for every $n \geq N$,

$$x_n \geq x_N > x - \varepsilon. \tag{2.8}$$

On the other hand, the assumption $x = \sup X$ implies that for every $n \in \mathbb{N}$,

$$x_n \leq x < x + \varepsilon. \tag{2.9}$$

Now, we conclude from (2.8) and (2.9) that $|x_n - x| < \epsilon$ for every $n \geq N$. This completes the proof. □

A note on Theorem 2.31.

A bounded sequence may converge to the supremum of its range without being monotone. An example is the sequence

$$\left\{1 - \frac{(1 + (-1)^n)}{n}\right\},$$

which converges to 1, the supremum of its range, and is nevertheless not monotone. Thus, the property of being monotone is a *sufficient condition* for the convergence of a sequence to the supremum of its range. This property is by no means necessary.

Exercise 2.32. Prove that a decreasing sequence that is bounded from below converges to the infimum of its range.

As an immediate consequence of Theorem 2.31 and Exercise 2.32, we have the following corollary.

Corollary 2.33. *Every monotone and bounded sequence is convergent.*

Example 2.34. Show that the sequence

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

converges.

Solution. A simple calculation shows that for every n,

$$a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0.$$

Hence $\{a_n\}$ is strictly increasing. Also

$$a_n \leq n\left(\frac{1}{n+1}\right) < 1,$$

showing that $\{a_n\}$ is also bounded from above (and that it is bounded indeed, because $a_n \geq 0$). The sequence is therefore convergent by Theorem 2.31.

Example 2.35. For each $n \in \mathbb{N}$, let $x_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}$, where there are n square roots. For instance, $x_1 = \sqrt{2}$, $x_2 = \sqrt{2 + \sqrt{2}}$, and so on. Prove that $\{x_n\}$ is convergent.

Solution. It is enough to show that the sequence is bounded from above and increasing. To this end we use induction on n to show that

(1) 2 is an upper bound for $\{x_n\}$, and

(2) $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$.

To prove (1), note that $x_1 = \sqrt{2} < 2$, and if $x_n \leq 2$, then

$$x_{n+1} = \sqrt{2 + x_n} \leq \sqrt{2 + 2} = 2.$$

As for (2), we first see that $x_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = x_2$. Next, if $x_n \le x_{n+1}$, then

$$x_{n+1} = \sqrt{2 + x_n} \le \sqrt{2 + x_{n+1}} = x_{n+2},$$

which proves (2).

Finding the limit of the sequence $\{x_n\}$ in a mathematically rigorous way will be done in the next section (Example 2.56).

Exercise 2.36. Rework Example 2.35 with 2 replaced by a fixed real number $a > 0$ in the definition of $\{x_n\}$.

Divergence to Infinity. One important class of unbounded sequences is composed of those sequences that diverge to infinity. To have an idea of what is meant by divergence to infinity, consider the sequence $\{n\}$. Given any $M > 0$, Example 1.40(2) gives us some natural number N greater than M. Then for every $n \ge N$, n is also greater than M. In other words, the terms x_n of this sequence can go beyond any prescribed positive value when n is sufficiently large. Clearly, this implies that the sequence is unbounded and, hence, divergent. We describe this by saying that the sequence diverges to $+\infty$ or that it tends to $+\infty$. This is our motivation for the following definition, in which we also define divergence to $-\infty$.

Definition 2.37. Let $\{x_n\}$ be a sequence. We say that $\{x_n\}$ diverges to $+\infty$ (resp., $-\infty$) if the following statement is true.

Given any $M > 0$ (resp., $M < 0$), $N \in \mathbb{N}$ can be found such that $x_n > M$ (resp., $x_n < M$) for every $n \ge N$.

An obvious example for a sequence that diverges to $-\infty$ is $\{-n\}$.

Example 2.38. Prove that the sequence $\{(n^2+1)/(n-1)\}$ diverges to $+\infty$.

Solution. Denote the nth term of the sequence by a_n. If $M > 0$ is given, we should find $N \in \mathbb{N}$ such that $a_n > M$ for every $n \ge N$. Since this inequality is equivalent to $n^2 - nM > -(M+1)$ and $-(M+1) < 0$, we find that $a_n > M$ holds for every $n \in \mathbb{N}$ with $n^2 - nM > 0$, or equivalently with $n > M$. Thus, if we choose $N \in \mathbb{N}$ to be greater than M, then $a_n > M$ holds for every $n \ge N$.

Exercise 2.39. Verify that the sequence $\{(n^2+1)/(2-n)\}$ diverges to $-\infty$.

A note on unboundedness and divergence to infinity.

A sequence that diverges to $+\infty$ or $-\infty$ is necessarily unbounded. More precisely, the divergence of $\{x_n\}$ to $+\infty$ implies that $\{x_n\}$ is unbounded from above, and the divergence of $\{x_n\}$ to $-\infty$ tells us that $\{x_n\}$ is unbounded from below. Of course, an unbounded sequence is not necessarily divergent to infinity. An example is the sequence $\{(-1)^n n\}$ which is unbounded from above and below, and is nevertheless not divergent to infinity.

The following theorem complements Theorem 2.31.

Theorem 2.40. *An increasing sequence that is unbounded from above diverges to $+\infty$.*

Proof. Let $\{x_n\}$ be an unbounded and increasing sequence. Let $M > 0$ be arbitrary. Since $\{x_n\}$ is not bounded from above, M is not an upper bound for its range. Thus, there exists some $N \in \mathbb{N}$ such that $x_N > M$. Since $\{x_n\}$ is increasing, $x_n \geq x_N > M$ for every $n \geq N$. This shows that $\lim_{n\to\infty} x_n = +\infty$. □

> A note on increasing sequences.
>
> As a result of Theorem 2.31 and Theorem 2.40 we find that an increasing sequence either converges to the supremum of its range or diverges to $+\infty$.

Exercise 2.41. If $\{x_n\}$ diverges to $+\infty$, does it necessarily follow that $\{x_n\}$ is increasing?

Exercise 2.42. Prove that a decreasing sequence that is unbounded from below diverges to $-\infty$.

Convergence and Order. Now we turn to the relations that convergence and order may have. We begin by answering a natural question. If $\{x_n\}$ converges to x, what can be said about the sign of x and those of the terms x_n?

Lemma 2.43. *Let $\{x_n\}$ be a sequence with limit x. If $x < 0$, then $x_n < 0$ for all sufficiently large n.*

Proof. For $\epsilon = -x/2 > 0$, find $N \in \mathbb{N}$ such that for every $n \geq N$, $|x_n - x| < \epsilon$. Then, it follows from this inequality that for all such n, $x_n < x/2 < 0$. □

Note that the converse of Lemma 2.43 is not true. In fact, $\{-1/n\}$ is a sequence, all of whose terms are negative, that nevertheless converges to 0.

Exercise 2.44. If $\lim_{n\to\infty} x_n = x$ and $x > 0$, then prove that $x_n > 0$ for all sufficiently large n.

Lemma 2.43 and Exercise 2.44 explored the way the sign of the limit of a sequence affects those of its terms. The following theorem examines the converse of this.

Theorem 2.45. *If $x_n \geq 0$ for all sufficiently large n and $\lim_{n\to\infty} x_n = x$, then $x \geq 0$. In particular, if $x_n \geq y_n$ for all sufficiently large n, $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} y_n = y$, then $x \geq y$.*

Proof. If $x < 0$, we may use Lemma 2.43 to deduce that $x_n < 0$ for all sufficiently large n, contradicting our nonnegativity assumption. To prove the second assertion, note that the assumption implies $x_n - y_n \geq 0$ for all sufficiently large n, and therefore by the first part and Theorem 2.26, $x - y = \lim_{n\to\infty}(x_n - y_n) \geq 0$. □

> What does Theorem 2.45 say?
>
> The first part of Theorem 2.45 says that a sequence all of whose terms, except perhaps a finite number of them, are nonnegative cannot converge to a negative limit. The second part shows that the limit *respects* the order relation $<$.

Next we show that the limit can be interchanged with the square root. As we will emphasize in the proof, this useful result owes its legitimacy to Theorem 2.45.

Proposition 2.46. *If* $x_n \geq 0$ *for every* n *and* $\lim_{n\to\infty} x_n = x$*, then*

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}.$$

Proof. By Theorem 2.45, $x \geq 0$. Thus $\sqrt{x}$ exists as a real number. Let $\epsilon > 0$ be given. If $x = 0$, then it is enough to find N so large that $|\sqrt{x_n} - 0| = \sqrt{x_n} < \epsilon$ for every $n \geq N$. But $\sqrt{x_n} < \epsilon$ is equivalent to $x_n < \epsilon^2$. So, it is sufficient to find a natural number N such that $x_n < \epsilon^2$ for every $n \geq N$. Since $\{x_n\}$ converges to x and $x_n = |x_n - 0|$, this can be done easily.

If $x > 0$, then $\sqrt{x} > 0$. Now $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0$ for every n, and we may write

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{1}{\sqrt{x}}\,|x_n - x|.$$

If we choose N so large that $|x_n - x| < \epsilon\sqrt{x}$ for every $n \geq N$, then $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ for all such n. ☐

To interpret Proposition 2.46 let us call $\{\sqrt{x_n}\}$ the square root of $\{x_n\}$.

What does Proposition 2.46 say?

Proposition 2.46 says that the limit of the square root of a convergent sequence of nonnegative numbers is the square root of its limit.

Example 2.47. Find the limit of the given sequences.

(1) $\lim_{n\to\infty} \sqrt{(n+1)/(n+3)}$.

(2) $\lim_{n\to\infty} \sqrt{n^2+n} - n$.

Solution. (1) Since for every n,

$$\frac{n+1}{n+3} = \frac{1+1/n}{1+3/n},$$

it follows by letting $n \to \infty$ and using Theorem 2.26 that

$$\lim_{n\to\infty} \frac{n+1}{n+3} = 1.$$

It now follows from Proposition 2.46 that the given limit is also equal to 1.

(2) The limit cannot be computed in this way. So we write

$$\begin{aligned} \sqrt{n^2+n} - n &= \frac{(\sqrt{n^2+n} - n)\,(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} \\ &= \frac{n}{\sqrt{n^2+n} + n} = \frac{1}{\sqrt{1+1/n} + 1}. \end{aligned}$$

Since by Proposition 2.46

$$\lim_{n\to\infty} \sqrt{1+1/n} = \sqrt{\lim_{n\to\infty}(1+1/n)} = \sqrt{1} = 1,$$

letting n tend to infinity in both sides of the equality

$$\sqrt{n^2+n}-n=\frac{1}{\sqrt{1+1/n}+1},$$

we find by (1) and (3) of Theorem 2.26, that the desired value is 1/2.

Exercise 2.48. Show that the sequence $\{\sqrt{n^2+1}-n\}$ converges, and find the value of its limit.

The following is a limit-analogue of the real number property that $y \le x \le y$ implies $x = y$.

Theorem 2.49 (The Squeeze Theorem). *If $y_n \le x_n \le z_n$ for all sufficiently large n and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x$, then $\lim_{n\to\infty} x_n = x$.*

Proof. Suppose N_1 is so large that $y_n \le x_n \le z_n$ for every $n \ge N_1$. Let $\epsilon > 0$ be given. There exists $N_2 \in \mathbb{N}$ such that $|z_n - x| < \epsilon$ for every $n \ge N_2$. Similarly, one finds $N_3 \in \mathbb{N}$ such that $|y_n - x| < \epsilon$ for each $n \ge N_3$. Let $N = \max\{N_1, N_2, N_3\}$. Then for every $n \ge N$,

$$-\epsilon < y_n - x \le x_n - x \le z_n - x < \epsilon.$$

This gives $|x_n - x| < \epsilon$ for all $n \ge N$, and accordingly completes the proof. □

What does the Squeeze Theorem say?

The Squeeze Theorem says that if a sequence is *squeezed* between two sequences that converge to a common limit, then this sequence will also converge to the same limit.

Example 2.50. Find the value of the given limits.

(1) $\lim_{n\to\infty} 1/2^n$.
(2) $\lim_{n\to\infty}(1+\cos^2 n!)/(3n^2+n!)$.
(3) $\lim_{n\to\infty} n!/n^n$.
(4) $\lim_{n\to\infty} \sqrt[n]{1+1/2+\cdots+1/n}$.
(5) $\lim_{n\to\infty} \left(1/n^2 + 1/(n+1)^2 + \cdots + 1/(2n)^2\right)$.

Solution. (1) As we observed in Example 1.27, $2^n > n$ for every $n \in \mathbb{N}$. Hence for each n, $1/2^n < 1/n$ and we deduce from $\lim_{n\to\infty} 1/n = 0$ and the Squeeze Theorem that $\lim_{n\to\infty} 1/2^n = 0$.

(2) Since for every $n \in \mathbb{N}$,

$$0 < 1/(3n^2+n!) < 1/n! \le 1/n \tag{2.10}$$

and $\lim_{n\to\infty} 1/n = 0$, the Squeeze Theorem gives $\lim_{n\to\infty} 1/(3n^2+n!) = 0$. On the other hand, the sequence $\{1+\cos^2 n!\}$ is bounded, as $1 \le 1+\cos^2 n! \le 2$ for every n. Thus, the given limit is equal to 0 by Proposition 2.23. Note that from (2.10) and the Squeeze Theorem, we also deduce that $\lim_{n\to\infty} 1/n! = 0$.

(3) For each n,

$$0 < \frac{n!}{n^n} = \frac{1}{n} \cdots \frac{n}{n} \leq \frac{1}{n}.$$

Thus we may deduce as in (1) that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

(4) It follows from the inequalities

$$1 = \sqrt[n]{1} \leq \sqrt[n]{1 + \frac{1}{2} + \cdots + \frac{1}{n}} \leq \sqrt[n]{n},$$

which are valid for every $n \in \mathbb{N}$, Example 2.15(5) and the Squeeze Theorem that

$$\lim_{n\to\infty} \sqrt[n]{1 + 1/2 + \cdots + 1/n} = 1.$$

(5) For arbitrary $n \in \mathbb{N}$, let a_n denote the nth term of the given sequence. Then,

$$a_n > \frac{1}{(2n)^2} + \cdots + \frac{1}{(2n)^2} = \frac{n+1}{(2n)^2} > \frac{n}{4n^2} = \frac{1}{4n}.$$

On the other hand,

$$a_n < \frac{1}{n^2} + \cdots + \frac{1}{n^2} = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2}.$$

Consequently for every n,

$$\frac{1}{4n} < a_n < \frac{1}{n} + \frac{1}{n^2}.$$

It now follows from the Squeeze Theorem that $\lim_{n\to\infty} a_n = 0$.

Exercise 2.51. If $y_n \leq x_n \leq z_n$ for all sufficiently large n and $\{y_n\}$ and $\{z_n\}$ are convergent with distinct limits, does it necessarily follow that $\{x_n\}$ is convergent?

2.2. Subsequences, Limit Superior and Limit Inferior

In this section we continue our study of the reasons of divergence to provide a more complete answer to question **(2.b)**. To motivate our main concepts, let us begin with considering the sequence $x_n = (-1)^n(1 + 1/n)$. In this sequence those terms which have an even index, namely the terms $x_{2k} = 1 + 1/(2k)$, form a sequence $\{1 + 1/(2k)\}$. Since the terms of this new sequence are chosen from those of the original one $\{x_n\}$, we say that $\{x_{2k}\}$ is a *subsequence* of $\{x_n\}$. Here, the terms of our subsequence are chosen from the x_n's using a given strictly increasing sequence of natural numbers, namely $\{2k\}$. If we work with the sequence of odd natural numbers $\{2k-1\}$, which is again a strictly increasing sequence of natural numbers, we get another subsequence of $\{x_n\}$. This is the sequence $\{-(1 + 1/(2k - 1))\}$. Based on what we learned in the previous section, it is easy to see that

$$\lim_{k\to\infty} x_{2k} = \lim_{k\to\infty} 1 + \frac{1}{2k} = 1$$

and

$$\lim_{k\to\infty} x_{2k-1} = \lim_{k\to\infty} -\left(1 + \frac{1}{2k-1}\right) = -1.$$

Chapter 7

Sequences in General Metric Spaces

As we mentioned in Chapter 2, the notion of sequence has meaning in any nonempty set X. But if we want to talk about the convergence or divergence of sequences in X, we need to have a distance function. Our aim in this chapter is to generalize the theory of real sequences to a theory for sequences in general metric spaces. We will see that some of the aspects of real sequence theory are generalizable, while some others are not. An instance of the latter aspects is the theory of real series, because the series are defined using addition, and this is not meaningful in arbitrary metric spaces. This is why the word *series* is not contained in the title of this chapter.

7.1. Convergence and Divergence in Metric Spaces

Based on our experience with real sequences, we can immediately generalize the concept of convergence to the context of general metric spaces. This is a straightforward generalization as we only need to replace the Euclidean distance function d_e by a general one d.

Definition 7.1. Let $\{x_n\}$ be a sequence in a metric space (X, d), and let $x \in X$. We say that $\{x_n\}$ *converges* to x, and we write $\lim_{n\to\infty} x_n = x$ if the following statement is true.

(MC) For every $\varepsilon > 0$, *we can find* $N \in \mathbb{N}$ *such that* $d(x_n, x) < \varepsilon$ *for every* $n \geq N$.

(We used *MC* as an abbreviation for *metric space convergence*.)

When a sequence fails to converge to a point of the space, we say that it is *divergent*.

As we will see shortly and as it seems to be true by *(MC)*, the convergence or divergence of $\{x_n\}$ depends strongly on the distance function. For this reason, when *(MC)* is true, we sometimes say that $\{x_n\}$ converges to x with respect to d or that it converges to x in (X, d).

Exercise 7.2. Suppose $\{x_n\}$ is a sequence in a metric space (X, d). Verify that $\{x_n\}$ converges to some $x \in X$ if and only if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

What does *(MC)* say?

The statement *(MC)* has an interpretation similar to that of the convergence of real sequences. This can be stated as follows.

The terms x_n will be as close to x as we wish, provided that the index n is sufficiently large.

But this time, *closeness* is expressed in terms of the generic distance function d.

The following examples show the diversity of contexts in which convergence can be considered.

Example 7.3. Let X be any nonempty set equipped with the discrete metric. If $\{x_n\}$ is a sequence in X and $x \in X$, prove that $\{x_n\}$ converges to x if and only if there exists $N \in \mathbb{N}$ such that $x_n = x$ for every $n \geq N$.

Solution. It is clear that when $x_n = x$ for all sufficiently large n, then $\{x_n\}$ converges to x. This is indeed true in any metric space. To prove the converse, note that when $\{x_n\}$ converges to x, for $\varepsilon = 1/2$ we can find $N \in \mathbb{N}$ such that for every $n \geq N$, $x_n \in N_\varepsilon(x) = \{x\}$. This completes the proof.

What does Example 7.3 say?

Example 7.3 says that a sequence in a discrete metric space is convergent if and only if it is constant for all sufficiently large indices. As a result of this example we see that the sequence $\{1/n\}$, which converges to 0 in the Euclidean space $\mathbb{R}$, is divergent when we equip $\mathbb{R}$ with the discrete metric. This shows that the convergence or divergence of a sequence depends on the metric we consider on the underlying set.

Our next example shows that the convergence situation may change if we fix the metric and change the underlying set.

Example 7.4. The sequence $\{1/n\}$ converges to 0 in the Euclidean space $\mathbb{R}$. If we consider this as a sequence in the subspace $Y = (0, 1]$, then the sequence is divergent in Y, simply because 0 is not an element of Y.

In our next two examples, we examine the convergence of sequences in the Euclidean space $\mathbb{R}^n$ and in spaces $C([a, b])$.

Example 7.5. Let $\{\overrightarrow{x_m}\}$ be a sequence in $\mathbb{R}^n$, where $\overrightarrow{x_m} = (x_{m1}, \ldots, x_{mn})$ for each m. If $\overrightarrow{x} = (x_1, \ldots, x_n)$, then prove that $\lim_{m\to\infty} \overrightarrow{x_m} = \overrightarrow{x}$ with respect to the Euclidean metric of $\mathbb{R}^n$ if and only if $\lim_{m\to\infty} x_{mi} = x_i$, for each $i \in \{1, \ldots, n\}$, with respect to the Euclidean metric of $\mathbb{R}$.

Solution. Let $\varepsilon > 0$ be given. If $\lim_{m\to\infty} \overrightarrow{x_m} = \overrightarrow{x}$, then there exists $N \in \mathbb{N}$ such that for every $m \geq N$,

$$d_e^n(\overrightarrow{x_m}, \overrightarrow{x}) = \sqrt{(x_{m1} - x_1)^2 + \cdots + (x_{mn} - x_n)^2} < \varepsilon.$$

So, for every $i \in \{1, \ldots, n\}$ and every $m \geq N$,

$$d_e(x_{mi}, x_i) = |x_{mi} - x_i| = \sqrt{(x_{mi} - x_i)^2} \leq d_e^n(\overrightarrow{x_m}, \overrightarrow{x}) < \varepsilon.$$

This proves that for each $i \in \{1, \ldots, n\}$, $\lim_{m\to\infty} x_{mi} = x_i$.

To prove the converse, assume that for every $i \in \{1, \ldots, n\}$, $\lim_{m\to\infty} x_{mi} = x_i$. Then for each i there exists $N_i \in \mathbb{N}$ such that $m \geq N_i$ implies

$$d_e(x_{mi}, x_i) = |x_{mi} - x_i| < \frac{\varepsilon}{n}.$$

Then Example 1.63 shows that for every $m \geq N := \max\{N_1, \ldots, N_n\}$,

$$d_e^n(\overrightarrow{x_m}, \overrightarrow{x}) \leq \sum_{i=1}^{n} |x_{mi} - x_i| < \sum_{i=1}^{n} \frac{\varepsilon}{n} = \varepsilon.$$

This shows that $\lim_{m\to\infty} \overrightarrow{x_m} = \overrightarrow{x}$.

What does Example 7.5 say?

Example 7.5 says that convergence in the space $(\mathbb{R}^n, d_e^n)$ is in fact *coordinatewise*.

The above example can be used to find the limit of sequences of n-tuples. For example,

$$\lim_{n\to\infty} \left(\frac{1}{n}, \frac{2n-1}{n+3}\right) = (0, 2)$$

and

$$\lim_{n\to\infty} \left(\sqrt[n]{n}, \frac{n-3}{n^2+1}\right) = (1, 0).$$

It also shows that the sequence $\{(n, 1 - \frac{1}{n})\}$ diverges in the Euclidean space $\mathbb{R}^2$, because $\{n\}$ is divergent in the Euclidean space $\mathbb{R}$.

Exercise 7.6. Let $\{\overrightarrow{x_m}\}$ and $\overrightarrow{x}$ be as in Example 7.5. Prove that $\{\overrightarrow{x_m}\}$ converges to $\overrightarrow{x}$ with respect to d_s^n or d_m^n if and only if $\lim_{m\to\infty} x_{mi} = x_i$ for each $i \in \{1, \ldots, n\}$ with respect to the metric d_e.

Example 7.7. Let $\{f_n\}$ be a sequence in $C([a, b])$ and $f \in C([a, b])$. Prove that $\{f_n\}$ converges to f with respect to d_u if and only if the following statement is true.

(UCO) For every $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that for every $n \geq N$ and every $x \in [a, b]$, $|f_n(x) - f(x)| < \varepsilon$.

(We used *UCO* as an abbreviation for *uniform convergence*.)

In particular, if $\{f_n\}$ converges to f with respect to d_u, then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every $x \in [a,b]$.

Solution. Note that $\{f_n\}$ converges to f with respect to the metric d_u if and only if

for every $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that for every* $n \geq N$*,*

$$d_u(f_n, f) = \sup\{|f_n(x) - f(x)| : x \in [a,b]\} < \varepsilon.$$

That the above statement is equivalent to *(UCO)* can be easily verified using the properties of the supremum. It is also clear that *(UCO)* implies the convergence of $\{f_n(x)\}$ to $f(x)$ in the space $(\mathbb{R}, d_e)$, for every $x \in [a,b]$.

Remark 7.8. If we want to formally describe that $\{f_n(x)\}$ converges to $f(x)$ for every $x \in [a,b]$, then we may write the following.

For every $\varepsilon > 0$ *and every* $x \in [a,b]$ *there exists* $N_x \in \mathbb{N}$ *such that for every* $n \geq N_x$*,* $|f_n(x) - f(x)| < \varepsilon$*.*

The subscript x in N_x shows that this natural number is not only related to ε, but it also depends on x. The statement *(UCO)* therefore presents a more general notion of convergence for the sequence $\{f_n\}$, to which we will refer, in Chapter 9, as *uniform convergence*. The subscript u in d_u refers to *uniform*, and the metric d_u will be frequently called the *uniform metric*. Note that in *(UCO)*, there is some $N \in \mathbb{N}$ that works for all $x \in [a,b]$ in the formal definition of $\lim_{n\to\infty} f_n(x) = f(x)$. When this last equality holds for all $x \in [a,b]$, we say that $\{f_n\}$ converges to f *pointwise* on $[a,b]$.

Exercise 7.9. For $n \in \mathbb{N}$ and $x \in [0,1]$ define $f_n(x) = x^{n^2}$. Prove that $\{f_n\}$ converges to the function

$$f(x) = \begin{cases} 0 & 0 \leq x < 1, \\ \\ 1 & x = 1 \end{cases}$$

pointwise on $[0,1]$. Does $\{f_n\}$ converge to f with respect to the uniform metric of $C([0,1])$?

From now on, we will try to generalize some of the basic facts that we proved for real sequences to the context of metric spaces. The following is the first instance of such generalizations.

Proposition 7.10. *In any metric space a convergent sequence has a unique limit.*

The proof is similar to that of Proposition 2.12, where we proved that the limit of real sequences is unique. The details are therefore left to the reader as exercise.

Monotonicity cannot be generalized to the metric space context.

Of course, many results proved for real sequences have no meaning for general metric spaces. An instance is the fact that every monotone and bounded sequence of real numbers converges. This is because monotone sequences were defined using the order $<$ on $\mathbb{R}$, but as we know, such an order may not exist in an arbitrary metric space. For this reason, another important result which is not generalizable is Theorem 2.59 which asserts that every sequence of real numbers has a monotone subsequence. So, the notion of monotonicity and its allied theorems cannot be considered in the metric space setting.

On the other hand, some arguments which do not seem to be generalizable can be extended using some tricks. For example, boundedness of real sequences was also defined in terms of the order $<$, but in the general context of metric spaces, we may define bounded sequences via the notion of boundedness we defined in the previous chapter.

Definition 7.11. If $\{x_n\}$ is a sequence in a metric space X, then we say that $\{x_n\}$ is *bounded* if the set $\{x_n : n \in \mathbb{N}\}$, known as the *range* of $\{x_n\}$, is a bounded subset of X.

Now, we can prove the following straightforward generalization of Theorem 2.20, where we proved that convergent sequences of real numbers are bounded.

Theorem 7.12. *In any metric space convergent sequences are bounded.*

Example 7.13. In any discrete metric space, every sequence is bounded. This is because, as we saw in Example 6.71, all subsets of a discrete space are bounded.

Example 7.14. Prove that in the metric space $(C([a, b]), d_u)$, a sequence $\{f_n\}$ is bounded if and only if the following statement is true.

(UBO) There exists $M > 0$ such that for every n and every $x \in [a, b]$, $|f_n(x)| \leq M$.

(We use *UBO* as an abbreviation for *uniform boundedness.*)

Solution. If $\{f_n\}$ is bounded, there exist $f \in C([a, b])$ and $\varepsilon > 0$ such that for every n, $d_u(f_n, f) < \varepsilon$. If M_1 is an upper bound for the values of $|f|$ on $[a, b]$, whose existence is a consequence of the continuity of this function, then for every $x \in [a, b]$ and every n,

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < \varepsilon + M_1.$$

If we let $M = \varepsilon + M_1$, this completes the first part of the proof.

Conversely, if we assume that *(UBO)* is true, then for every n and every $x \in [a, b]$,

$$|f_n(x) - f_1(x)| \leq |f_n(x)| + |f_1(x)| \leq 2M,$$

so that the range of $\{f_n\}$ is contained in $N_K(f_1)$ when $K > 2M$ is arbitrary. Thus, $\{f_n\}$ is bounded.

Note that *(UBO)* shows that M is an upper bound for the values of $|f_n|$ on $[a, b]$, for each n. For this reason, when *(UBO)* is the case, we say that $\{f_n\}$ is a *uniformly bounded* sequence of real-valued functions.

Example 7.15. Which of the given sequences is bounded in the metric space $(C([0,1]), d_u)$?

(1) $f_n(x) = \sin nx$.

(2) $g_n(x) = n(1+x)$.

Solution. (1) For every $n \in \mathbb{N}$ and every $x \in [0,1]$,

$$|f_n(x)| = |\sin nx| \leq 1.$$

So, Example 7.14 shows that $\{f_n\}$ is bounded.

(2) If $M > 0$ is given, choose n_0 so large that

$$g_{n_0}(1) = (1+1)n_0 > M.$$

It follows that the statement

for every $n \in \mathbb{N}$ and every $x \in [0,1]$, $|g_n(x)| \leq M$

is not true. Since M was arbitrary, we find in view of Example 7.14 that $\{g_n\}$ is not a bounded sequence in $(C([0,1]), d_u)$.

Subsequences. The notion of subsequence has meaning for sequences in arbitrary sets. If $\{x_n\}$ is a sequence in some set X, and $\{n_k\}$ is a strictly increasing sequence of natural numbers, then $\{x_{n_k}\}$ is called a subsequence of $\{x_n\}$. Although the theory of subsequences for general metric spaces has some differences with that in the Euclidean space $\mathbb{R}$, the following generalization of Theorem 2.55 can be proved similarly.

Theorem 7.16. *Let $\{x_n\}$ be a sequence in a metric space (X, d), and let $x \in X$. Then, the following conditions are equivalent.*

(1) *The sequence $\{x_n\}$ converges to x.*

(2) *Every subsequence of $\{x_n\}$ converges to x.*

(3) *The subsequences of even- and odd-indexed terms of $\{x_n\}$ converge to x.*

The first difference between subsequence theory in the Euclidean space $\mathbb{R}$ and that in general metric spaces is that the concepts of limit superior and limit inferior have no meaning in the latter context. This is because the concepts are defined using suprema and infima, and these are defined using the order relation $<$ on $\mathbb{R}$, which is not present in arbitrary metric spaces. Nevertheless, the concept of subsequential limit can be defined in our current setting.

Definition 7.17. Let $\{x_n\}$ be a sequence in a metric space (X, d). An element x of X is called a *subsequential limit* of $\{x_n\}$ if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$. The set of all subsequential limits of $\{x_n\}$ will be denoted by $\overline{\{x_n\}}$.

If $\{x_n\}$ is a sequence in the Euclidean space $\mathbb{R}$, the set of subsequential limits defined in the above definition may differ from the set $E(\{x_n\})$ considered in Section 2.2. For example, for the sequence

$$x_n = \begin{cases} \frac{1}{n} & n \text{ is even,} \\ \\ n & n \text{ is odd,} \end{cases}$$

the set $E(\{x_n\})$ is $\{0, +\infty\}$, while $\overline{\{x_n\}} = \{0\}$. The symbols $-\infty$ and $+\infty$ were adjoined to $\mathbb{R}$ in Chapter 1 to improve the theory of suprema and infima. Hence we cannot expect to see them in our discussion of subsequences in general metric spaces.

It follows from Theorem 7.16 that a sequence $\{x_n\}$ converges to x if and only if $\overline{\{x_n\}} = \{x\}$.

Example 7.18. If we consider the sequence $\{n\}$ in the Euclidean space $\mathbb{R}$, then it is clear that $\overline{\{n\}} = \emptyset$. Thus, the set of all subsequential limits of a real sequence, in the sense of Definition 7.17, may be empty. Compare this with what we observed in Section 2.2: the set $E(\{x_n\})$ is nonempty for every sequence $\{x_n\}$ of real numbers.

The second difference between the theory of subsequences in $\mathbb{R}$ and in the general context is related to the existence of convergent subsequences. As you may remember, a central result in Chapter 2 was the *Bolzano–Weierstrass theorem* which asserts that any bounded sequence of real numbers has a convergent subsequence. Although the same assertion can be phrased in the metric space setting, the following example shows that this theorem cannot be generalized to the abstract theory of metric spaces, even if we work in $\mathbb{R}$ with a different metric.

Example 7.19. If we consider the discrete metric on $\mathbb{R}$, $\{n\}$ is a bounded sequence in this metric space which has no convergent subsequences by Example 7.3.

Example 7.20. For each $n \in \mathbb{N}$, define a function f_n on $[0,1]$ by $f_n(x) = \sin(x/n)$ if n is even and by $f_n(x) = \cos(x/n)$ when n is odd. Find the set $\overline{\{f_n\}}$ of subsequential limits of $\{f_n\}$ in the space $(C([0,1]), d_u)$.

Solution. The subsequences $\{f_{2k}\}$ and $\{f_{2k-1}\}$ converge to the constant functions $f \equiv 0$ and $g \equiv 1$, respectively. To see why, note that for every $k \in \mathbb{N}$ and every $x \in [0,1]$,

$$|f_{2k}(x) - 0| = \left|\sin \frac{x}{2k}\right| \leq \left|\frac{x}{2k}\right| \leq \frac{1}{2k}.$$

So, if $\varepsilon > 0$ is given and we choose $N \in \mathbb{N}$ so large that $1/N < 2\varepsilon$, then for every $k \geq N$ and every $x \in [0,1]$,

$$|f_{2k}(x) - 0| \leq \frac{1}{2k} \leq \frac{1}{2N} < \varepsilon.$$

This proves that for all $k \geq N$,

$$d_u(f_{2k}, f) < \varepsilon,$$

and therefore that $\{f_{2k}\}$ converges to f with respect to the uniform metric. Similar reasoning shows that $\{f_{2k-1}\}$ converges to g with respect to d_u. The set $\overline{\{f_n\}}$ of subsequential limits of $\{f_n\}$ is therefore $\{f, g\}$.

Limit Points, Closure, and Closedness in Terms of Sequences. One application of sequences in metric space theory is that we may use them to determine the closedness of sets. The following is the main result in this connection.

Proposition 7.21. *If A is a subset of a metric space X and $x \in A$, then*

(1) $x \in A'$ *if and only if* x *is the limit of a sequence in* $A \backslash \{x\}$, *and*

(2) $x \in \overline{A}$ *if and only if* x *is the limit of a sequence of elements of* A.

Proof. We only prove (2) because (1) is a straightforward generalization of Proposition 3.6. To prove (2), note that when $x \in \overline{A}$, we have two cases:

- $x \in A$, in which case the sequence defined by $x_n = x$ for every $n \in \mathbb{N}$ is a sequence in A that converges to x;
- $x \in A'$, in which (1) gives us a sequence $\{x_n\}$ in A such that $x_n \neq x$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$.

As for the converse, let x be such that $\lim_{n\to\infty} x_n = x$ for a sequence $\{x_n\}$ in A. If $x = x_n$ for some $n \in \mathbb{N}$, then $x \in A$. Otherwise, (1) shows that x is a limit point of A. Thus, in either case, x belongs to $\overline{A}$. ☐

Item (2) of the above proposition says that x is a cluster point of A if and only if it is the limit of a sequence in A.

An important corollary of Proposition 7.21 is the following interesting result.

Corollary 7.22. *If A is a subset of $\mathbb{R}$ (equipped with d_e) which is bounded from above and we let $x = \sup A$, then $x \in \overline{A}$.*

Proof. This follows easily from Proposition 7.21(2) because x is the limit of a sequence of the elements of A by Theorem 2.4. ☐

What does Corollary 7.22 say?

Corollary 7.22 says that the supremum of a set A of real numbers is a cluster point of the set. Based on our understanding of closure and cluster points, this result confirms the previously mentioned fact that the supremum of a set A of real numbers is *adhered* to the set.

As another consequence of Proposition 7.21, we can prove the following *sequential* characterization of closed subsets of metric spaces.

Corollary 7.23. *For a subset A of a metric space X, the following conditions are equivalent.*

(1) *The set A is closed.*

(2) *If $\{x_n\}$ is a sequence in A that converges to some x, then $x \in A$.*

Proof. This follows from the above proposition and the fact that A is closed if and only if A equals $\overline{A}$. ☐

What does Corollary 7.23 say?

Corollary 7.23 says that a subset A of a metric space is closed if and only if it contains the limit of each of the sequences that lie entirely in A. Recall that in Proposition 2.80 a similar assertion was proved for intervals of the form $[a, b]$ as subsets of the Euclidean space $\mathbb{R}$. Thus, as we mentioned there, such intervals are closed subsets of $(\mathbb{R}, d_e)$.

Exercise 7.24. Prove that $x \in A'$ if and only if x is the limit of a sequence with distinct terms in A.

Example 7.25. In the Euclidean space $\mathbb{R}$, $(0, 1)$ is not closed, as we knew. One way to see this is to note that the sequence $\{\frac{1}{2n}\}$ is a sequence in $(0, 1)$ that converges to 0 in $\mathbb{R}$, but $0 \notin (0, 1)$

Example 7.26. In the Euclidean space $\mathbb{R}^2$, determine whether

$$A = \left\{ \left(\frac{1}{n}, \frac{n}{2n - 3} \right) : n \in \mathbb{N} \right\}$$

is closed or not.

Solution. The set A is not closed because

$$\lim_{n\to\infty} \left(\frac{1}{n}, \frac{n}{2n - 3} \right) = \left(0, \frac{1}{2} \right),$$

and $(0, 1/2) \in \overline{A} \backslash A$. It is clear that $\overline{A} = A \cup \{(0, 1/2)\}$.

The following proposition generalizes the above example to the context of metric spaces.

Proposition 7.27. *If $\{x_n\}$ is a sequence in a metric space (X, d) and we let $A = \{x_n : n \in \mathbb{N}\}$, then $\overline{A} = A \cup \overline{\{x_n\}}$. If, in particular, $\{x_n\}$ converges to some x, then $\overline{A} = A \cup \{x\}$.*

Proof. If $y \in \overline{\{x_n\}}$, y is the limit of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Since $\{x_{n_k}\}$ is a sequence in A, Proposition 7.21 tells us that $y \in \overline{A}$. Thus, $A \cup \overline{\{x_n\}} \subseteq \overline{A}$.

On the other hand, if $z \in \overline{A}$ is arbitrary, then Proposition 7.21(2) gives us a sequence $\{z_k\}$ in A that converges to z. Consider two cases for the range Z of $\{z_k\}$.

- Z is finite. Then there exists $N \in \mathbb{N}$ such that for every $k \geq N$, $z_k = z$. Thus $z \in A$ in this case.
- Z is infinite. In this case we may choose a subsequence of $\{z_k\}$ with distinct terms, which is therefore such a subsequence of $\{x_n\}$ that converges to z. So, $z \in \overline{\{x_n\}}$.

Thus, in either case, $z \in A \cup \overline{\{x_n\}}$. This shows that $\overline{A} \subseteq A \cup \overline{\{x_n\}}$. □

What does Proposition 7.27 say?

Proposition 7.27 says that when we consider a sequence $\{x_n\}$ as a set A, the cluster points of A are the elements of A and the subsequential limits of $\{x_n\}$.

Example 7.28. In the metric space $(C([0,1]), d_u)$ find the closure of the set

$$A = \{g_k : k \in \mathbb{N}\} \cup \{h_k : k \in \mathbb{N}\},$$

where $g_k(x) = \sin \frac{x}{2k}$ and $h_k(x) = \cos \frac{x}{2k-1}$ for every $k \in \mathbb{N}$ and every $x \in [0,1]$.

Solution. We saw in Example 7.20 that A is the range of the sequence $\{f_n\}$ defined by $f_n(x) = \sin(x/n)$ if n is even, and $f_n(x) = \cos(x/n)$ when n is odd. We also observed in that example that $\overline{\{f_n\}} = \{f, g\}$, where $f \equiv 0$ and $g \equiv 1$ on $[0,1]$. So $\overline{A} = A \cup \{f, g\}$.

7.2. Cauchy Sequences and Complete Metric Spaces

Just as easily as we defined the convergence of sequences, the notion of Cauchy sequence can be generalized to the context of metric spaces.

Definition 7.29. Let $\{x_n\}$ be a sequence in a metric space (X, d). We say that $\{x_n\}$ is a *Cauchy sequence* if the following statement is true.

For every $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

As we saw in Chapter 2, Cauchy sequences in $\mathbb{R}$ and $[a, b]$ are convergent. This leads us to guess that the same conclusion is true in any metric space. But, it can be easily seen that $\{\frac{1}{n}\}$ is a Cauchy sequence in $(0,1]$ which is divergent there. Thus $(0,1]$, considered as a subspace of the Euclidean space $\mathbb{R}$, is a metric space in which Cauchy sequences are not necessarily convergent. This shows that Theorem 2.78, stating that every Cauchy sequence of real numbers converges in $\mathbb{R}$, cannot be generalized to the metric space setting. This is quite surprising, as our definition of the Cauchy sequence was (in some sense) a copy of that of Cauchy sequences in $\mathbb{R}$.

Here, the divergence of $\{\frac{1}{n}\}$ is a *flaw* of the space $(0,1]$, not of the sequence itself. If we replace $(0,1]$ by an appropriate larger subspace of the Euclidean space $\mathbb{R}$ that contains 0, then $\{\frac{1}{n}\}$ converges in this space. We describe this flaw by saying that the space $(0,1]$ is not as *complete* as we wish. This is our motivation for the following definition.

Definition 7.30. A metric space (X, d) is called *complete* if every Cauchy sequence in X converges.

With this definition, the sets $\mathbb{R}$ and $[a, b]$ with their Euclidean metric are complete.

Example 7.31. Prove that any discrete metric space is complete.