Contents

To the Instructor xi

To the Student xiii

Introduction and Outline of the Book xvii

Acknowledgments xxi

Part 1. Rebuilding the Calculus Building

Chapter 1. The Real Number System Revisited 3

1.1. The Algebraic Axioms 5

1.2. The Order Axioms 7

1.3. Absolute Value, Distance, and Neighborhoods 9

1.4. Natural Numbers and Mathematical Induction 12

1.5. The Axiom of Completeness and Its Uses 21

1.6. The Complex Number System 33

Notes on Essence and Generalizability 39

Exercises 41

Chapter 2. Sequences and Series of Real Numbers 47

2.1. Real Sequences, Their Convergence, and Boundedness 48

2.2. Subsequences, Limit Superior and Limit Inferior 66

2.3. Cauchy Sequences 74

2.4. Sequences in Closed and Bounded Intervals 76

2.5. Series: Revisiting Some Convergence Tests 78

2.6. Rearrangements of Series 90

2.7. Power Series 92

Notes on Essence and Generalizability 96
Exercises 97

Chapter 3. Limit and Continuity of Real Functions 103
3.1. Limit Points and Some Other Classes of Points in $\mathbb{R}$ 104
3.2. A More General Definition of Limit 112
3.3. Limit at Infinity 126
3.4. One-Sided Limits 130
3.5. Continuity and Two Kinds of Discontinuity 136
3.6. Continuity on $[a, b]$: Results and Applications 142
3.7. Uniform Continuity 149
Notes on Essence and Generalizability 151
Exercises 152

Chapter 4. Derivative and Differentiation 159
4.1. The Why and What of the Concept of Derivative 160
4.2. The Basic Properties of Derivative 168
4.3. Local Extrema and Derivative 172
4.4. The Mean Value Theorem: More Applications of Derivative 175
4.5. Taylor Series: A First Glance 181
4.6. Taylor's Theorem and the Convergence of Taylor Series 185
Notes on Essence and Generalizability 189
Exercises 190

Chapter 5. The Riemann Integral 193
5.1. Motivation: The Area Problem 194
5.2. The Riemann Integral: Definition and Basic Results 197
5.3. Some Integrability Theorems 214
5.4. Antiderivatives and the Fundamental Theorem of Calculus 220
Notes on Essence and Generalizability 228
Exercises 228

Part 2. Abstraction and Generalization

Chapter 6. Basic Theory of Metric Spaces 235
6.1. A First Generalization: The Definition of Metric Space 239
6.2. Neighborhoods and Some Classes of Points 245
6.3. Open and Closed Sets 255
6.4. Metric Subspaces 262
6.5. Boundedness and Total Boundedness 265
Notes on Essence and Generalizability 269

Exercises 270

Chapter 7. Sequences in General Metric Spaces 275
7.1. Convergence and Divergence in Metric Spaces 275
7.2. Cauchy Sequences and Complete Metric Spaces 284
7.3. Compactness: Definition and Some Basic Results 287
7.4. Compactness: Some Equivalent Forms 290
7.5. Perfect Sets and Cantor's Set 294
Notes on Essence and Generalizability 296
Exercises 296

Chapter 8. Limit and Continuity of Functions in Metric Spaces 299
8.1. The Definition of Limit in General Metric Spaces 299
8.2. Continuity and Uniform Continuity 302
8.3. Continuity and Compactness 307
8.4. Connectedness and Its Relation to Continuity 310
8.5. Banach's Fixed Point Theorem 314
Notes on Essence and Generalizability 316
Exercises 317

Chapter 9. Sequences and Series of Functions 319
9.1. Sequences of Functions and Their Pointwise Convergence 319
9.2. Uniform Convergence 323
9.3. Weierstrass's Approximation Theorem 328
9.4. Series of Functions and Their Convergence 332
Notes on Essence and Generalizability 333
Exercises 334

Appendix 337
Real Sequences and Series 337
Limit and Continuity of Functions 339
The Concepts of Derivative and Differentiability 340
The Riemann Integral 340

Bibliography 343

Index 345