Preface

The idea for this textbook was conceived as a direct result of my experience teaching the “introduction to proofs” course at Allegheny College. When I first started teaching this course, there were only a handful of appropriate textbooks on the market. My experience teaching from various textbooks clarified in my mind what I wanted to accomplish in such a course and how to accomplish it.

One possible title (and the one used at Allegheny College) for an “introduction to proofs” course is “Foundations of Mathematics”, which can conjure up at least two possibilities for the focus of the course. The word “foundations” could be interpreted in the sense of the logician: working axiomatically and in the language of set theory, or working within a formal proof system. On the other hand, the course can be viewed as a student’s first exposure to proofs, sets, functions, etc., *as mathematicians use them*, giving students a practical collection of tools that will enable them to be successful in later mathematics courses, such as abstract algebra and real analysis. As interesting as the first interpretation is, in my opinion the second interpretation is the right one for a first exposure to these ideas and for the average mathematics major.

It is important that students begin writing proofs as early as possible in the course, hopefully by the end of the second week of classes. To achieve this, I present only enough logic for students to be able to work with the propositional connectives and the quantifiers. Although the initial treatment of this material is streamlined, the importance of this material is emphasized throughout the book. Students are frequently reminded, especially in the early chapters, of the importance of the logical structure of a mathematical statement as a framework for finding a proof of that statement. In particular, the importance of the logical structure of a mathematical definition as a framework for proving that an object has (or does not have) that property is a constant theme throughout the textbook.

Focusing on logical structure is an important first step in addressing the question, “How do I start?” that students who are learning to write proofs often ask.

To help students learn that searching for a proof is a *process*, I have adopted Velleman's approach [**15**] of using a "Given-Goal diagram" to organize what is known versus what is to be proved. Together, these methods teach students that the logical structure of the goal dictates the structure of a proof. Given-Goal diagrams can be modified by unravelling the logical structure of the statement to be proved, thereby organizing the search for a proof.

For many theorems and sample solutions to problems, this book presents several paragraphs of "scratchwork" before presenting a correct formal proof. The goal is to walk students through a complete analysis of a problem: understanding the logical structure of the statement, creating one or more Given-Goal diagrams, deploying various techniques to build a bridge from the given to the goal, and finally writing a complete concise proof. I hope that this will help demystify the process of searching for a proof.

After the introductory material on logic, my goal is to introduce students to various proof techniques. The focus is on proving simple statements about integers, rational numbers, and real numbers. It is important that students know what the "ground rules" are at the beginning: what may they assume and what requires proof? To achieve this, I provide students with the "basic properties" of integers and real numbers; all other statements require proof. My goal here is not to develop the entire theory of the integers (or the real numbers) from the axioms. Rather, my goal is to be clear about the assumptions we make. Occasionally, I feel that assuming additional axioms (such as assuming the existence part of the Fundamental Theorem of Arithmetic when proving the existence of infinitely many prime numbers) will expedite discussion and illustrate concepts. In such cases, I clearly point out when we're assuming more than we ought and emphasize that we will pay our debts later.

Quantifiers and the proper use of variables are given very careful treatment. The focus, as before, is that the logical structure of the statement determines the shape of the proof, as well as guides the search for the proof. My approach to quantifiers and variables is informed by the rigorous framework provided by first-order logic (see, for example, [**10**]). I pay particular attention to the concept of *existential instantiation*[1] as a means to stress the proper use of variables, as well as the difference between a quantified statement and a nonquantified statement; namely, if we know $(\exists x)P(x)$, then we may fix a particular element d in the universe such that $P(d)$ is true, as long as we use a new variable that doesn't already have a particular meaning in the proof. In my experience, students need time to get used to working with a single quantifier before moving on to more complicated statements. Consequently, I postpone proofs of statements *beginning* with mixed quantifiers, such as the ε-δ proofs that give students so much trouble, until after the students have more experience proving statements beginning with only one quantifier. Given-Goal diagrams are particularly effective here as a means of teaching students to unravel the logical complexity of a statement as a means of organizing and searching for the proof of that statement.

[1] I mention this terminology exactly once in the text!

One of the most difficult decisions one has to make when writing a textbook of this sort is the order in which to present the concepts of relation and function. In particular, one has to decide whether to define a function as an abstract set of ordered pairs with a particular property or as a triple consisting of two sets and a “correspondence”. If a function is a triple, then one must further decide how function equality is defined. (Must the codomains agree, or not? Both definitions occur in the literature.) The cost of defining a function as a particular kind of set of ordered pairs is that other concepts, such as function composition, become more abstract and harder to work with. In my experience, this formal approach is confusing and unnecessarily abstract for students who are just beginning to learn about mathematical proofs. Functions are rarely treated as sets of ordered pairs in other undergraduate mathematics courses (unless a student takes a course in formal set theory), so doing so in this course would present an abstract view of the concept that won’t be useful to students later when they need to use functions in other courses. I have chosen to define the concept of function first, as a triple consisting of two sets and a “correspondence”. I include agreement of the codomains when defining function equality, to avoid the situation that can otherwise occur, in which two functions can be equal with one being onto its codomain and the other not.

In the early part of the textbook, the focus is on giving students practice with sets and the various proof techniques. Chapter 1 provides the necessary background on the logic of the propositional connectives and the quantifiers, as well as an introduction to the concept of proof. In Chapter 2, I present a variety of direct and indirect proof techniques. Chapter 3 is devoted to induction.

The goal of the later chapters is to provide students with the foundational material on sets, functions, equivalence relations, number theory, finite and infinite sets, and introductory analysis they will need in order to succeed in their later proof-based courses, particularly linear algebra, number theory, abstract algebra, and real analysis. Chapters 4 and 5 deal with sets and functions, respectively. The focus of Chapter 6 is introductory number theory: the Division and Euclidean Algorithms and elementary facts about congruences, which are also important in abstract algebra. The material in Section 6.4 and Section 6.5 on congruences and congruence classes can be delayed until the more general discussion of relations, equivalence relations, and equivalence classes in Chapter 7, which provides a further introduction to ideas important in abstract algebra. In Chapter 8, I discuss finite and infinite sets, with a focus on the material needed in real analysis, namely, the difference between countability and uncountability. Finally, Chapter 9 presents the axiomatic foundations of analysis, including the Completeness Axiom, the existence of $\sqrt{2}$, and the Archimedean Property.

One of the more difficult aspects of learning to write a proof is learning to effectively communicate that proof to others (the instructor or other students). Particularly at the beginning, I emphasize the difference between the search for a proof (including any scratchwork) and the final written proof. I have included some guidelines for writing mathematics in the appendix. These guidelines were originally inspired by the “Writing checklist” that Dr. Annalisa Crannell (Franklin & Marshall College) so generously shared with me many years ago.

I have deliberately not included "answers to selected problems" at the back of the textbook. The process of learning to find a proof, and learning to recognize when what one has written is a correct proof, is an active one, which the passive reading of solutions circumvents. The text provides plenty of examples and a great deal of commentary. Students will learn more by grappling with the problems, perhaps in consultation with the instructor or other students, without a solution easily accessible. When appropriate, I include hints for some of the more difficult problems. There are many exercises at the end of sections. Text cross-references to exercises are in the form, for example, Exercise 1.1.2a. This is a reference to exercise 2, part (a), in the section Exercises 1.1, which is at the end of Section 1.1.

Proofs are ended with the usual end-of-proof character □. I have used the symbol ◊ to mark the end of examples.

To Students

Chances are you are studying the material in this book because you are enrolled in an "introduction to proofs" course at your college or university. You will be learning how to use mathematical language and notation and how to communicate mathematical ideas clearly and precisely. And you will be learning the foundations of mathematical reasoning, the mathematician's standard of truth. Logical reasoning skills and the ability to use mathematical language and notation properly are also essential for other scientific disciplines, such as physics and computer science.

In my experience, this type of course is usually a completely new experience for students. It is normal to feel a bit disoriented at first. It is important to persevere. It is especially important to study *actively*, by reading the textbook equipped with pencil and paper, by writing lots of proofs, and by discussing the mathematics with your instructor and fellow students. You should never expect to simply write down a proof immediately after reading the statement to be proved. As illustrated in this textbook, finding a proof is a *process* that must take you from an analysis of the statement to be proved, through the scratchwork of Given-Goal diagrams and false starts, to a final polished and correct proof.

It is important that you learn any new mathematical definition or notation right away; you cannot hope to succeed if you don't know what the words and notation mean! This may be a new experience for you, particularly since there is not a lot of "leeway" in mathematical definitions. One needs to know the precise meaning of the words, rather than the "general idea". Finally, you may find it strange to be writing so much in a math course. Keep in mind that our job is to *communicate* what we know, and how we know it, to others. Learning to write mathematics well requires a lot of practice and can be difficult at the beginning. One way to improve your writing is to read your mathematical statements out loud. Since notation simply abbreviates a mathematical statement and since our statements have a grammar, speaking out loud *exactly* what you've written (no more and no less) can help you improve your mathematical writing.

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I have coded the graphics in this book using pstricks and TikZ. In particular, the code for the Venn diagrams on page 75 can be found in "Example: Set operations illustrated with Venn diagrams", authored by Uwe Ziegenhagen, published on TEXample.net on 3/18/2010. I have replaced the use of color in that example with grayscale, moved the labels, and added the rectangular box for the universe $\mathcal{U}$. The use and adaptation of this code is permitted by the Creative Commons License Deed found at creativecommons.org/licenses/by/2.5/.

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My desire to write this textbook was motivated by my desire to equip students with the skills they need in order to be successful in upper-level mathematics courses. Teaching Allegheny's introduction to proofs course helped me better understand why some students find proof writing so difficult, and it helped me improve as a teacher. I am grateful to my own teachers, as well. In particular, I would like to thank Iraj Kalantari at Western Illinois University for sharing with me his enthusiasm and expertise as a mathematician, teacher, and expositor.

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