# Introduction – Mainly to the Students

This is a book on real analysis, and real analysis is a continuation of calculus. Many of the words you find in this book, such as "continuity", "convergence", "derivative", and "integral", are familiar from calculus, but they will now appear in new contexts and with new interpretations.

The biggest change from calculus to real analysis is a shift in emphasis from calculations to arguments. If you thumb through the book, you will find surprisingly few long calculations and fancy graphs, and instead a lot of technical words and unfamiliar symbols. This is what advanced mathematics books usually look like – calculations never lose their importance, but they become less dominant. However, this doesn't mean that the book reads like a novel; as always, you have to read mathematics with pencil and paper at hand.

Your calculus courses have probably been divided into single-variable and multivariable calculus. Single-variable calculus is about functions of one variable $x$ while multivariable calculus is about functions of several variables $x_1, x_2, \ldots, x_n$. Another way of looking at it is to say that multivariable calculus is still about functions of one variable, but that this variable is now a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Real analysis covers single- and multivariable calculus in one sweep and at the same time opens the door to even more general situations – functions of infinitely many variables! This is not as daunting as it may sound: Just as functions of several variables can be thought of as functions of a single, vector-valued variable, functions of infinitely many variables can often be thought of as functions of a single, functioned-valued variable (intuitively, a function is infinite dimensional as its graph consists of infinitely many points). Hence you should be prepared to deal with functions of the form $F(y)$ where $y$ is a function.

As real analysis deals with functions of one, several, and infinitely many variables at the same time, it is necessarily rather abstract. It turns out that in order to study such notions as convergence and continuity, we don't really need to specify what kinds of objects we are dealing with (numbers, vectors, functions, etc.) – all we need to know is that there is a reasonable way to measure the distance between them. This leads to the theory of *metric spaces* that will be the foundation for most of what we shall be doing in this book. If we also want to differentiate functions, we need the notion of a *normed space*, i.e., a metric space that is also a vector space (recall linear algebra). For integration, we shall invent another kind of space, called a *measure space*, which is tailored to measuring the size of sets. These spaces will again give rise to new kinds of normed spaces.

What I have just written probably doesn't make too much sense to you at this stage. What is a space, after all? Well, in mathematics a space is just a set (i.e., a collection of objects) with some additional structure that allows us to operate with the objects. In linear algebra, you have met vector spaces which are just collections of objects that can be added and multiplied by numbers in the same way that ordinary vectors can. The metric spaces that we shall study in this book are just collections of objects equipped with a function that measures the distance between them in a reasonable manner. In the same way, the measure spaces we shall study toward the end of the book consist of a set and a function that measure the size of (some of the) subsets of that set.

Spaces are abstractly defined by rules (often called axioms); anything that satisfies the rules is a space of the appropriate kind, and anything that does *not* satisfy the rules is not a space of this kind. These abstract definitions give real analysis a different flavor from calculus – in calculus it often suffices to have an intuitive understanding of a concept; in real analysis you need to read the definitions carefully as they are all you have to go by. As the theory develops, we get more information about the spaces we study. This information is usually formulated as propositions or theorems, and you need to read these propositions and theorems carefully to see when they apply and what they mean.

Students often complain that there are too few examples in books on advanced mathematics. That is true in one sense and false in another. It's true in the sense that if you count the labelled examples in this book, there are far fewer of them than you are used to from calculus. However, there are lots of examples under a different label – and that is the label "proof". Advanced mathematics is about arguments and proofs, and every proof is an example you can learn from. The aim of your mathematics education is to make you able of producing your own mathematical arguments, and the only practical way to learn how to make proofs is to read and understand proofs. Also, I should add, knowing mathematics is much more about knowing ways to argue than about knowing theorems and propositions.

So how does one read proofs? There are probably many ways, but the important thing is to try to understand the idea behind the proof and how that idea can be turned into a logically valid argument. A trick that helped me as a student was to read the proof one day, understand it as well as I could, and then return to it a day or two later to see if I could do it on my own without looking at the book. As I don't have a photographic memory, this technique forced me to concentrate on the ideas of the proof. If I had understood the main idea (which can usually be summed up in a sentence or a drawing once you have understood it), I could usually reconstruct the rest of the proof without any problem. If I had not understood the main idea, I would be hopelessly lost.

Let us take a closer look at the contents of the book. The first two chapters contain preliminary material that is not really about real analysis as such. The first chapter gives a quick introduction to proofs, sets, and functions. If you have taken a course in mathematical reasoning or the foundations of mathematics, there is probably little new here, otherwise you should read it carefully. The second chapter reviews the theoretical foundation of calculus. How much you have to read here depends on the calculus sequence you have taken. If it was fairly theoretical, this chapter may just be review; if it was mainly oriented toward calculations, it's probably a good idea to work carefully through most of this chapter. I'm sure your instructor will advise you on what to do.

The real contents of the book start with Chapter 3 on metric spaces. This is the theoretical foundation for the rest of the book, and it is important that you understand the basic ideas and become familiar with the concepts. Pay close attention to the arguments – they will reappear with small variations throughout the text. Chapter 4 is a continuation of Chapter 3 and focuses on spaces where the elements are continuous functions. This chapter is less abstract than Chapter 3 as it deals with objects that you are already familiar with (continuous functions, sequences, power series, differential equations), but some of the arguments are perhaps tougher as we have more structure to work with and try to tackle problems that are closer to "real life".

In Chapter 5 we turn to normed spaces which are an amalgamation of metric spaces and the vector spaces you know from linear algebra. The big difference between this chapter and linear algebra is that we are now primarily interested in infinite dimensional spaces. The last two sections are quite theoretical, otherwise this is a rather friendly chapter. In Chapter 6 we use tools from Chapter 5 to study derivatives of functions between normed spaces in a way that generalizes many of the concepts you know from calculus (the Chain Rule, directional derivatives, partial derivatives, higher order derivatives, Taylor's formula). We also prove two important theorems on inverse and implicit functions that you may not have seen before.

Chapter 7 deals with integration and is a new start in two ways – both because most of the chapter is independent of the previous chapters, and also because it presents an approach to integration that is totally different from what you have seen in calculus. This new approach is based on the notion of measure, which is a very general way of assigning size to sets. Toward the end of the chapter, you will see how these measures lead to a new class of normed spaces with attractive properties. Chapter 8 is a continuation of Chapter 7. Here you will learn how to construct measures and see some important applications.

The final chapter is on Fourier analysis. It shows you an aspect of real analysis that has to some degree been neglected in the previous chapters – the power of concrete calculations. It also brings together techniques from most of the other chapters in the book and illustrates in a striking manner a phenomenon that appears again and again throughout the text: The convergence of a sequence or series of functions is a tricky business!

At the end of each chapter there is a short section with notes and references. Here you will find a brief historical summary and some suggestions for further reading. If you want to be a serious student of mathematics, I really recommend that you take a look at its history. Mathematics – and particularly the abstracts parts – is so much easier to appreciate when you know where it comes from. In fact, learning mathematics without knowing something of its history is a bit like watching a horror movie with the sound turned off: You see that people get scared and take their precautions, but you don't understand why. This is particularly true of real analysis where much of the theory developed out of a need to deal with (what at the time felt like) counter-intuitive examples.

I hope you will enjoy the book. I know it's quite tough and requires hard work, but I have done my best to explain things as clearly as I can. Good Luck!