Preface

While most calculus books are so similar that they seem to have been tested in the same wind tunnel, there is a lot more variety between books on real analysis, both with respect to content and level. If we start with levels, it is easy to distinguish at least three. The most elementary one is made up of books whose main purpose is to redo single-variable calculus in a more rigorous way – classical examples are Frank Morgan's *Real Analysis*, Colin Clark's *The Theoretical Side of Calculus*, and Stephen Abbott's *Understanding Analysis*. On the intermediate level we have undergraduate texts like Walter Rudin's *Principles of Mathematical Analysis*, Tom Körner's *A Companion to Analysis*, and Kenneth R. Davidson and Allan P. Donsig's *Real Analysis and Applications*, just to mention a few. In these texts, metric or normed spaces usually play a central part. On the third level we find graduate texts like H. L. Royden's classic *Real Analysis* (now in a new edition by Patrick Fitzpatrick), Gerald B. Folland's *Real Analysis: Modern Techniques and Their Applications*, and John N. McDonald and Neil A. Weiss' *A Course in Real Analysis* – books where measure theory is usually the point of departure. Above these again we have research level texts on different aspects of real analysis.

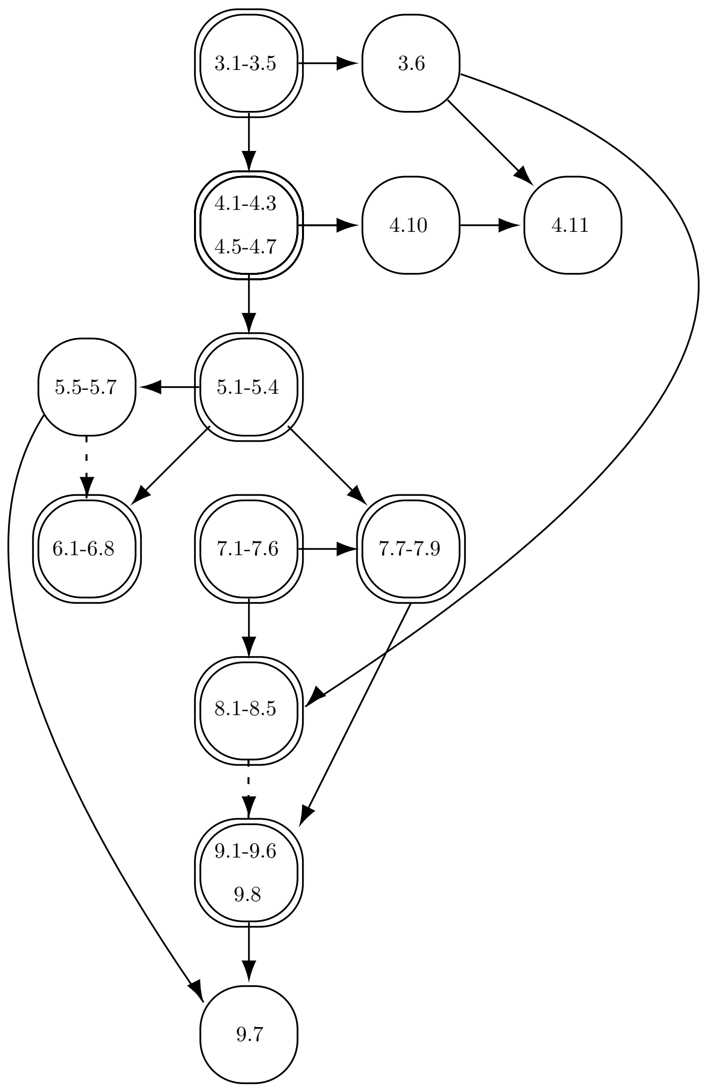
The present book is intended to be on the second level – it is written for students with a good background in (advanced) calculus and linear algebra but not more (although many students would undoubtedly benefit from a course on proofs and mathematical thinking). Books on this level are to a varying degree forward-looking or backward-looking, where backward-looking means reflecting on material in previous courses from a more advanced point of view, and forward-looking means providing the tools necessary for the next courses. The distinction is neatly summed up in the subtitle of Körner's book: *A Second First or a First Second Course in Analysis*. While Körner aims to balance the two aspects, this book in unabashedly forward-looking – it is definitely intended as a first second course in analysis. For that reason I have dropped some of the staple ingredients of courses on this level in favor of more advanced material; for example, I don't redo Riemann integration but go directly to Lebesgue integrals, and I do differentiation in normed spaces rather than refining differentiation in euclidean spaces. Although the exposition is still aimed at students on the second level, these choices bring in material that are usually taught on the third level, and I have tried to compensate by putting a lot of emphasis on examples and motivation, and by writing out arguments in greater detail than what is usually done in books on the third level. I have also included an introductory chapter on the foundation of calculus for students who have not had much previous exposure to the theoretical side of the subject.

The central concepts of the book are completeness, compactness, convergence, and continuity, and students get to see them from many different perspectives – first in the context of metric spaces, then in normed spaces, and finally in measure theory and Fourier analysis. As the book is forward-looking, my primary aim has been to provide students with the platform they need to understand applications, read more advanced texts, and follow more specialized courses. Although the book is definitely not a full-fledged course in functional analysis or measure theory, it does provide students with many of the tools they need in more advanced courses, such as Banach's Fixed Point Theorem, the Arzelà-Ascoli Theorem, the Stone-Weierstrass Theorem, Baire's Category Theorem, the Open Mapping Theorem, the Inverse and Implicit Function Theorems, Lebegsue's Dominated Convergence Theorem, the Riesz-Fischer Theorem on the completeness of L^p, Carathéodory's Extension Theorem, Fubini's Theorem, the L^2-convergence of Fourier series, Fejér's Theorem, and Dini's Test for pointwise convergence.

The main danger with a forward-looking course of this kind is that it becomes all method and no content – that the only message to students is: "Believe me, you will need this when you grow up!" This is definitely a danger also with the present text, but I have tried to include a selection of examples and applications (mainly to differential equations and Fourier analysis) that I hope will convince students that all the theory is worthwhile.

Various versions of the text have been used for a fourth-semester course at the University of Oslo, but I should warn you that I have never been able to cover all the material in the same semester – some years the main emphasis has been on measure theory (Chapters 7 and 8) and other years on normed spaces (Chapters 5 and 6). The chart below shows the major dependencies between the main Chapters 3-9, but before we turn to it, it may be wise to say a few words about the introductory chapters 1 and 2. Chapter 1 is a short introduction to sets, functions, and relations from an abstract point of view. As most of our students don't have this background, I usually cover it during the first week of classes. The second chapter is meant as a service to students who lack a conceptual grasp of calculus, either because they have taken a more computational-oriented calculus sequence, or because they haven't really understood the theoretical parts of their courses. I have never lectured on this chapter as our students are supposed to have the background needed to go directly to Chapter 3 on metric spaces, but my feeling is that many have found it useful for review and consolidation. One small point: I always have to pick up the material on lim inf and lim sup in Section 2.2 as it is not covered by our calculus courses.

Let us now turn to the chart showing the main logical dependencies between the various parts of the book. It is not as complicated as it may seem at first glance. The doubly ringed parts form the theoretical backbone of the book. This doesn't mean that the other parts are uninteresting (in fact, you will find deep and important theorems such as the Stone-Weierstrass Theorem and the Baire Category Theorem in these parts), but they are less important for the continuity.



The two dotted arrows indicate less important dependencies – Chapter 6 only depends on Sections 5.5-5.7 through the Bounded Inverse Theorem in Section 5.7, and Chapter 9 only depends on Chapter 8 through Theorem 8.5.6 which states that the continuous functions are dense in $L^p([a,b],\mu)$. In my opinion, both these results can be postulated. Note that some sections, such as 3.7, 4.4, 4.8-4.9, and 6.9-6.11, don't appear in the chart at all. This just means that no later sections depend directly on them, and that I don't consider them part of the core of the book.

At the end of each chapter there is a brief section with a historical summary and suggestions for further reading. I have on purpose made the reading lists short as I feel that long lists are more intimidating than helpful at this level. You will probably find many of your favorite books missing (so are some of mine!), but I had to pick the ones I like and find appropriate for the level.

Acknowledgments. The main acknowledgments should probably go to all the authors I have read and all the lecturers I have listened to, but I have a feeling that the more important their influence is, the less I am aware of it – some of the core material "is just there", and I have no recollection of learning it for the first time. During the writing of the book, I have looked up innumerable texts, some on real analysis and some on more specialized topics, but I hope I have managed to make all the material my own. An author often has to choose between different approaches, and in most cases I have chosen what to me seems intuitive and natural rather than sleek and elegant.

There are probably some things an author should not admit, but let me do it anyway. I never had any plans for a book on real analysis until the textbook for the course I was teaching in the Spring of 2011 failed to show up. I started writing notes in the hope that the books would be there in a week or two, but when the semester was over, the books still hadn't arrived, and I had almost 200 pages of class notes. Over the intervening years, I and others have taught from ever-expanding versions of the notes, some years with an emphasis on measure theory, other years with an emphasis on functional analysis and differentiability.

I would like to thank everybody who has made constructive suggestions or pointed out mistakes and weaknesses during this period, in particular Snorre H. Christiansen, Geir Ellingsrud, Klara Hveberg, Erik Løw, Nils Henrik Risebro, Nikolai Bjørnestøl Hansen, Bernt Ivar Nødland, Simon Foldvik, Marius Jonsson (who also helped with the figure of vibrating strings in Chapter 9), Daniel Aubert, Lisa Eriksen, and Imran Ali. I would also like to extend my thanks to anonymous but very constructive referees who have helped improve the text in a number of ways, and to my enthusiastic editor Ina Mette and the helpful staff of the AMS.

If you find a misprint or a more serious mistake, please send a note to

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