Preface

In American universities two distinct types of courses are often called Advanced Calculus: one, largely for engineers, emphasizes advanced computational techniques in calculus; the other, a more theoretical course, usually taken by majors in mathematics and physical sciences (and often called Elementary Analysis or Intermediate Analysis), concentrates on conceptual development and proofs. This Problems Based Course is of the latter type. It is not a place to look for post-calculus material on Fourier series, Laplace transforms, and the like. It is intended for students of mathematics, and others, who have completed (or nearly completed) a standard introductory calculus sequence and who wish to understand where all those rules and formulas come from.

Many advanced calculus texts contain more topics than this one. When students are encouraged to develop much of the subject matter for themselves, it is not possible to "cover" material at the same breathtaking pace that can be achieved by a truly determined lecturer. But, while no attempt has been made to make the book encyclopedic, I do think it nevertheless provides an integrated overview of calculus and, for those who continue, a solid foundation for a first-year graduate course in real analysis.

As the title of the present document is intended to suggest, *A Problems Based Course in Advanced Calculus*, is as much an extended problem set as a textbook. The proofs of most of the major results are either *exercises* or *problems*. The distinction here is that solutions to exercises are available online while solutions to problems are not given. I hope that this arrangement will provide flexibility for instructors who wish to use it as a text. For those who prefer a (modified) Moore-style development, where students work out and present most of the material, there is a quite large collection of problems for them to hone their skills on. For instructors who prefer a lecture format, it should be easy to base a coherent series of lectures on the presentation of solutions to thoughtfully chosen problems.

I have tried to make this document (in a rather highly qualified sense discussed below) self-contained. In it we investigate how the edifice of calculus can be grounded in carefully developed substrata of sets, logic, and numbers. Will it be a complete or totally rigorous development of the subject? Absolutely not. I am not aware of any serious enthusiasm among mathematicians I know for requiring rigorous courses in mathematical logic and axiomatic set theory as prerequisites for a first introduction to analysis. In the use of the tools from set theory and formal logic, there are many topics that because of their complexity and depth are cheated or not even mentioned. (For example, though used often, the *axiom of choice* is mentioned only once.) Even everyday topics such as arithmetic (see Appendix G) are not developed in any great detail.

Before embarking on the main ideas of calculus proper, one *ideally* should have a good background in all sorts of things: quantifiers, logical connectives, set operations, writing proofs, the arithmetic and order properties of the real numbers, mathematical induction, least upper bounds, functions, composition of functions, images and inverse images of sets under functions, finite and infinite sets, countable and uncountable sets. On the one hand all these are technically prerequisite to a careful discussion of the foundations of calculus. On the other hand any attempt to do all this business systematically at the beginning of a course will defer the discussion of anything concerning calculus proper to the middle of the academic year and may very well both bore and discourage students. Furthermore, in many schools there may be students who have already studied much of this material (in a Proofs course, for example). In the spirit of compromise and flexibility I have relegated this material to the appendices. Treat it any way you like. I teach in a large university where students show up for Advanced Calculus with a wide variety of backgrounds, so it is my practice to go over the appendices first, covering many of them in a quite rapid and casual way, my goal being to provide just enough detail so that everyone will know where to find the relevant material when they need it later in the course. After a rapid traversal of the appendices, I start Chapter 1.

For this text to be useful, a student should have previously studied introductory calculus, more for mathematical maturity than anything else. Familiarity with properties of elementary functions and techniques of differentiation and integration may be assumed and made use of in a few *examples*—but is never relied upon in the logical development of the material.

One motive for my writing this text is to make available in fairly simple form material that I think of as "calculus done right". For example, differential calculus as it appears in many texts is a morass of tedious epsilon-delta arguments and partial derivatives, the net effect of which is to almost totally obscure the beauty and elegance which results from a careful and patient elaboration of the concept of tangency. On the other hand texts in which things are done right (for example Loomis and Sternberg [**LS90**]) tend to be rather forbidding. I have tried to write a text which will be helpful to a determined student with an average background. (I seriously doubt that it will be of much use to those who are chronically lazy or totally unengaged.)

In my mind one aspect of doing calculus "correctly" is arranging things so that there is nothing to unlearn later. For example, in this text topological properties of the real line are discussed early on. Later, topological things (continuity, compactness, connectedness, and so on) are discussed in the context of metric spaces (because they unify the material conceptually and greatly simplify subsequent arguments). But the important thing is that *definitions* in the single variable case and the metric space case are the same. Students do not have to unlearn material as they go to more general settings. Similarly, the differential calculus is eventually developed in its natural habitat of normed linear spaces. But here again, the student who has mastered the one-dimensional case, which occurs earlier in the text, will encounter definitions and theorems and proofs that are virtually identical to the ones with which he/she is already familiar. There is nothing to unlearn.

In the process of writing this document, I have rethought the proofs of many standard theorems. Although some, perhaps most, results in advanced calculus have reached a final, optimal form, there are many others that, despite dozens of different approaches over the years, have proofs that are genuinely confusing to most students. To mention just one example, there is the theorem concerning change of order of differentiation whose conclusion is

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}\,.$$

This well-known result to this day still receives clumsy, impenetrable, and even incorrect proofs. I make an attempt, here and elsewhere in the text, to lead students to proofs that are conceptually clear, even when they may not be the shortest or most elegant. And throughout I try very hard to show that mathematics is about ideas and not about manipulation of symbols.

I extend my gratitude to the many students over the years who have endured various versions of this text and to colleagues and reviewers who have generously nudged me when they found me napping. I want especially to thank Dan Streeter, who provided me with much help in the technical aspects of getting this document produced. The text was prepared using AMS-LaTeX. For the diagrams I used the macro package XY-pic by Kristoffer H. Rose and Ross Moore supplemented by additional macros in the *diagxy* package by Michael Barr.

Finally it remains only to say that I will be delighted to receive, and will consider, any comments, suggestions, or error reports. My e-mail address is `erdmanj@comcast.net`.

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