A Problems Based Course in Advanced Calculus

John M. Erdman

Department of Mathematics, Portland State University, PO Box 751, Portland, OR 97207

Current address: 3412 NE Hancock St, Portland, OR 97212

E-mail address: `erdmanj@comcast.net`

Appendix Q

SOLUTIONS TO EXERCISES

Q.1. Exercises in chapter 01

Q.1.1. (Solution to 1.1.3) Find those numbers x such that $d(x, -2) \leq 5$. In other words, solve the inequality

$$|x + 2| = |x - (-2)| \leq 5.$$

This may be rewritten as

$$-5 \leq x + 2 \leq 5,$$

which is the same as

$$-7 \leq x \leq 3.$$

Thus the points in the closed interval $[-7, 3]$ are those that lie within 5 units of -2.

Q.1.2. (Solution to 1.1.12) If $x \in J_\delta(a)$, then $|x - a| < \delta \leq \epsilon$. Thus x lies within ϵ units of a; that is, $x \in J_\epsilon(a)$.

Q.1.3. (Solution to 1.2.6) Factor the left side of the inequality $x^2 - x - 6 \geq 0$. This yields $(x + 2)(x - 3) \geq 0$. This inequality holds for those x satisfying $x \leq -2$ and for those satisfying $x \geq 3$. Thus $A = (-\infty, -2] \cup [3, \infty)$. No neighborhood of -2 or of 3 lies in A. Thus $A^\circ = (-\infty, -2) \cup (3, \infty)$ and $A^\circ \neq A$.

Q.1.4. (Solution to 1.2.11) Since $A \cap B \subseteq A$ we have $(A \cap B)^\circ \subseteq A^\circ$ by 1.2.9. Similarly, $(A \cap B)^\circ \subseteq B^\circ$. Thus $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. To obtain the reverse inclusion take $x \in A^\circ \cap B^\circ$. Then there exist $\epsilon_1, \epsilon_2 > 0$ such that $J_{\epsilon_1}(x) \subseteq A$ and $J_{\epsilon_2}(x) \subseteq B$. Then $J_\epsilon(x) \subseteq A \cap B$ where $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. This shows that $x \in (A \cap B)^\circ$.

Q.1.5. (Solution to 1.2.12) Since $A \subseteq \bigcup \mathfrak{A}$ for every $A \in \mathfrak{A}$, we can conclude from proposition 1.2.9 that $A^\circ \subseteq \left(\bigcup \mathfrak{A}\right)^\circ$ for every $A \in \mathfrak{A}$. Thus $\bigcup\{A^\circ : A \in \mathfrak{A}\} \subseteq \left(\bigcup \mathfrak{A}\right)^\circ$. (See F.1.8.)

Q.2. Exercises in chapter 02

Q.2.1. (Solution to 2.1.7) (a) Let $\mathfrak{S}$ be a family of open subsets of $\mathbb{R}$. By proposition 2.1.4 each nonempty member of $\mathfrak{S}$ is a union of bounded open intervals. But then $\bigcup\mathfrak{S}$ is itself a union of bounded open intervals. So $\bigcup\mathfrak{S}$ is open.

(b) We show that if S_1 and S_2 are open subsets of $\mathbb{R}$, then $S_1 \cap S_2$ is open. From this it follows easily by mathematical induction that if $S_1, \dots, S_n$ are all open in $\mathbb{R}$, then $S_1 \cap \dots \cap S_n$ is open. Let us suppose then that S_1 and S_2 are open subsets of $\mathbb{R}$. If $S_1 \cap S_2 = \emptyset$, there is nothing to prove; so we assume that $S_1 \cap S_2 \neq \emptyset$. Let x be an arbitrary point in $S_1 \cap S_2$. Since $x \in S_1 = {S_1}^\circ$, there exists $\epsilon_1 > 0$ such that $J_{\epsilon_1}(x) \subseteq S_1$. Similarly, there exists $\epsilon_2 > 0$ such that $J_{\epsilon_2}(x) \subseteq S_2$. Let ϵ be the smaller of ϵ_1 and ϵ_2. Then clearly $J_\epsilon(x) \subseteq S_1 \cap S_2$, which shows that x is an interior point of $S_1 \cap S_2$. Since every point of the set $S_1 \cap S_2$ is an interior point of the set, $S_1 \cap S_2$ is open.

Q.2.2. (Solution to 2.2.5) Since no point in $\mathbb{R}$ has an ϵ-neighborhood consisting entirely of rational numbers, $A^\circ = \emptyset$. If $x \geq 0$, then every ϵ-neighborhood of x contains infinitely many positive rational numbers. Thus each such x belongs to A'. If $x < 0$, it is possible to find an ϵ-neighborhood of x that contains no positive rational number. Thus $A' = [0, \infty)$ and $\overline{A} = A \cup A' = [0, \infty)$.

Q.2.3. (Solution to 2.2.9) (a) First we show that $\overline{A^c} \subseteq A^{\circ c}$. If $x \in \overline{A^c}$, then either $x \in A^c$ or x is an accumulation point of A^c. If $x \in A^c$, then x is certainly not in the interior of A; that is, $x \in A^{\circ c}$. On the other hand, if x is an accumulation point of A^c, then every ϵ-neighborhood of x contains points of A^c. This means that no ϵ-neighborhood of x lies entirely in A. So, in this case too, $x \in A^{\circ c}$.

For the reverse inclusion suppose $x \in A^{\circ c}$. Since x is not in the interior of A, no ϵ-neighborhood of x lies entirely in A. Thus either x itself fails to be in A, in which case x belongs to A^c and therefore to $\overline{A^c}$, or else every ϵ-neighborhood of x contains a point of A^c different from x. In this latter case also, x belongs to the closure of A^c.

Remark. It is interesting to observe that the proof of (a) can be accomplished by a single string of "iff" statements. That is, each step of the argument uses a reversible implication. One needs to be careful, however, with the negation of quantifiers (see section D.4 of appendix D). It will be convenient to let $J^*_\epsilon(x)$ denote the ϵ-neighborhood of x with x deleted; that is, $J^*_\epsilon(x) = (x - \epsilon, x) \cup (x, x + \epsilon)$. The proof goes like this:

$$
\begin{aligned}
x \in A^{\circ c} \text{ iff } & \sim (x \in A^\circ) \\
\text{iff } & \sim ((\exists \epsilon > 0)\, J_\epsilon(x) \subseteq A) \\
\text{iff } & (\forall \epsilon > 0)\, J_\epsilon(x) \nsubseteq A \\
\text{iff } & (\forall \epsilon > 0)\, J_\epsilon(x) \cap A^c \neq \emptyset \\
\text{iff } & (\forall \epsilon > 0)\, (x \in A^c \text{ or } J^*_\epsilon(x) \cap A^c \neq \emptyset) \\
\text{iff } & (x \in A^c \text{ or } (\forall \epsilon > 0)\, J^*_\epsilon(x) \cap A^c \neq \emptyset) \\
\text{iff } & (x \in A^c \text{ or } x \in (A^c)') \\
\text{iff } & x \in \overline{A^c}.
\end{aligned}
$$

Proofs of this sort are not universally loved. Some people admire their precision and efficiency. Others feel that reading such a proof has all the charm of reading computer code.

(b) The easiest proof of (b) is produced by substituting A^c for A in part (a). Then $\overline{A} = \overline{A^{cc}} = A^{coc}$. Take complements to get $\overline{A}^c = A^{co}$.

Q.3. Exercises in chapter 03

Q.3.1. (Solution to 3.2.4) Let $a \in \mathbb{R}$. Given $\epsilon > 0$, choose $\delta = \epsilon/5$. If $|x - a| < \delta$, then $|f(x) - f(a)| = 5|x - a| < 5\delta = \epsilon$.

Q.3.2. (Solution to 3.2.5)Given $\epsilon > 0$, choose $\delta = \min\{1, \epsilon/7\}$, the smaller of the numbers 1 and $\epsilon/7$. If $|x - a| = |x + 1| < \delta$, then $|x| = |x + 1 - 1| \leq |x + 1| + 1 < \delta + 1 \leq 2$. Therefore,

$$
\begin{aligned}
&\text{(i)} & |f(x) - f(a)| &= |x^3 - (-1)^3| \\
&\text{(ii)} & &= |x^3 + 1| \\
&\text{(iii)} & &= |x + 1|\,|x^2 - x + 1| \\
&\text{(iv)} & &\leq |x + 1|(x^2 + |x| + 1) \\
&\text{(v)} & &\leq |x + 1|\,(4 + 2 + 1) \\
&\text{(vi)} & &= 7|x + 1| \\
&\text{(vii)} & &< 7\delta \\
&\text{(viii)} & &\leq \epsilon.
\end{aligned}
$$

Remark. How did we know to choose $\delta = \min\{1, \epsilon/7\}$? As scratch work we do steps (i)–(iv), a purely algebraic process, and obtain

$$|f(x) - f(a)| \leq |x + 1|(x^2 + |x| + 1)\,.$$

Now how do we guarantee that the quantity in parentheses doesn't get "too large". The answer is to require that x be "close to" $a = -1$. What do we mean by close? Almost *anything* will work. Here it was decided, arbitrarily, that x should be no more than 1 unit from -1. In other words, we wish δ to be no larger than 1. Then $|x| \leq 2$ and consequently $x^2 + |x| + 1 \leq 7$; so we arrive at step (vi)

$$|f(x) - f(a)| \leq 7|x + 1|\,.$$

Since we assume that $|x - (-1)| = |x + 1| < \delta$, we have (vii)

$$|f(x) - f(a)| < 7\delta\,.$$

What we *want* is $|f(x) - f(a)| < \epsilon$. This can be achieved by choosing δ to be no greater than $\epsilon/7$. Notice that we have required two things of δ:

$$\delta \leq 1 \quad \text{and} \quad \delta \leq \epsilon/7\,.$$

The easiest way to arrange that δ be no larger than each of two numbers is to make it the smaller of the two. Thus our choice is $\delta = \min\{1, \epsilon/7\}$.

A good exercise is to repeat the preceding argument, but at (iv) require x to be within 2 units of -1 (rather than 1 unit as above). This will change some things in the proof, but should not create any difficulty.

Q.3.3. (Solution to 3.2.6) Let $a \in \mathbb{R}$. Given $\epsilon > 0$, choose $\delta = \min\{1, (4|a|+2)^{-1}\epsilon\}$. If $|x - a| < \delta$, then

$$|x| \leq |x - a| + |a| < 1 + |a|.$$

Therefore

$$\begin{aligned} |f(x) - f(a)| &= |(2x^2 - 5) - (2a^2 - 5)| \\ &= 2|x^2 - a^2| \\ &= 2|x - a|\,|x + a| \\ &\leq 2|x - a|\,(|x| + |a|) \\ &\leq 2|x - a|\,(1 + |a| + |a|) \\ &\leq (4|a| + 2)|x - a| \\ &< (4|a| + 2)\delta \\ &\leq \epsilon. \end{aligned}$$

Q.3.4. (Solution to 3.2.12) Suppose f is continuous. Let V be an open subset of $\mathbb{R}$. To show that $f^{\leftarrow}(V)$ is open it suffices to prove that each point of $f^{\leftarrow}(V)$ is an interior point of that set. (Notice that if $f^{\leftarrow}(V)$ is empty, then there is nothing to prove. The null set is open.) If $a \in f^{\leftarrow}(V)$, then V is a neighborhood of $f(a)$. Since f is continuous at a, the set $f^{\leftarrow}(V)$ contains a neighborhood of a, from which we infer that a is an interior point of $f^{\leftarrow}(V)$.

Conversely, suppose that $f^{\leftarrow}(V) \overset{\circ}{\subseteq} \mathbb{R}$ whenever $V \overset{\circ}{\subseteq} \mathbb{R}$. To see that f is continuous at an arbitrary point a in $\mathbb{R}$, notice that if V is a neighborhood of $f(a)$, then $a \in f^{\leftarrow}(V) \overset{\circ}{\subseteq} \mathbb{R}$. That is, $f^{\leftarrow}(V)$ is a neighborhood of a. So f is continuous at a.

Q.4. Exercises in chapter 04

Q.4.1. (Solution to 4.1.11) Let $\epsilon > 0$. Notice that $x_n \in J_\epsilon(0)$ if and only if $|x_n| \in J_\epsilon(0)$. Thus (x_n) is eventually in $J_\epsilon(0)$ if and only if $(|x_n|)$ is eventually in $J_\epsilon(0)$.

Q.4.2. (Solution to 4.3.7) Let $x_n \to a$ where (x_n) is a sequence of real numbers. Then by problem 4.1.7(a) there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n - a| < 1$. Then by J.4.7(c)

$$\big||x_n| - |a|\big| \leq |x_n - a| < 1$$

for all $n \geq n_0$. Thus, in particular,

$$|x_n| < |a| + 1$$

for all $n \geq n_0$. If $M = \max\{|x_1|, |x_2|, \dots, |x_{n_0-1}|, |a| + 1\}$, then $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Q.4.3. (Solution to 4.4.3) Let (a_n) be a sequence in $\mathbb{R}$. As suggested in the hint, consider two cases. First, suppose that there is a subsequence (a_{n_k}) consisting of peak terms. Then for each k,

$$a_{n_k} \geq a_{n_{k+1}}\,.$$

That is, the subsequence (a_{n_k}) is decreasing.

Now consider the second possibility: there exists a term a_p beyond which there are no peak terms. Let $n_1 = p + 1$. Since a_{n_1} is not a peak term, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$. Since a_{n_2} is not a peak term, there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. Proceeding in this way we choose an increasing (in fact, strictly increasing) subsequence (a_{n_k}) of the sequence (a_n).

In both of the preceding cases we have found a monotone subsequence of the original sequence.

Q.4.4. (Solution to 4.4.11) Suppose that $b \in \overline{A}$. There are two possibilities; $b \in A$ or $b \in A'$. If $b \in A$, then the constant sequence $(b, b, b, \dots)$ is a sequence in A that converges to b. On the other hand, if $b \in A'$, then for every $n \in \mathbb{N}$ there is a point $a_n \in J_{1/n}(b)$ such that $a_n \in A$ and $a_n \neq b$. Then (a_n) is a sequence in A that converges to b.

Conversely, suppose there exists a sequence (a_n) in A such that $a_n \to b$. Either $a_n = b$ for some n (in which case $b \in A$) or else a_n is different from b for all n. In the latter case every neighborhood of b contains points of A—namely, the a_n's for n sufficiently large—other than b. Thus in either case $b \in \overline{A}$. □

Q.4.5. (Solution to 4.4.17) If $\ell = \lim_{n\to\infty} x_n$ exists, then taking limits as $n \to \infty$ of both sides of the expression $4x_{n+1} = {x_n}^3$ yields $4\ell = \ell^3$. That is,

$$\ell^3 - 4\ell = \ell(\ell - 2)(\ell + 2) = 0.$$

Thus if ℓ exists, it must be -2, 0, or 2. Next notice that

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{4}{x_n}^3 - x_n \\ &= \frac{1}{4}x_n(x_n - 2)(x_n + 2). \end{aligned}$$

Therefore

(94) $$x_{n+1} > x_n \quad \text{if} \quad x_n \in (-2, 0) \cup (2, \infty)$$

and

(95) $$x_{n+1} < x_n \quad \text{if} \quad x_n \in (-\infty, -2) \cup (0, 2).$$

Now consider the seven cases mentioned in the hint. Three of these are trivial: if $x_1 = -2$, 0, or 2, then the resulting sequence is constant (therefore certainly convergent).

Next suppose $x_1 < -2$. Then $x_n < -2$ for every n. [The verification is an easy induction: If $x_n < -2$, then ${x_n}^3 < -8$; so $x_{n+1} = \frac{1}{4}{x_n}^3 < -2$.] From this and (95) we see that $x_{n+1} < x_n$ for every n. That is, the sequence (x_n) decreases. Since the only possible limits are -2, 0, and 2, the sequence cannot converge. (It must, in fact, be unbounded.)

The case $x_1 > 2$ is similar. We see easily that $x_n > 2$ for all n and therefore [by (94)] the sequence (x_n) is increasing. Thus it diverges (and is unbounded).

If $-2 < x_1 < 0$, then $-2 < x_n < 0$ for every n. [Again an easy inductive proof: If $-2 < x_n < 0$, then $-8 < {x_n}^3 < 0$; so $-2 < \frac{1}{4}{x_n}^3 = x_{n+1} < 0$.] From (94) we conclude that (x_n) is increasing. Being bounded above it must converge [see proposition 4.3.3] to some real number ℓ. The only available candidate is $\ell = 0$.

Similarly, if $0 < x_1 < 2$, then $0 < x_n < 2$ for all n and (x_n) is decreasing. Again the limit is $\ell = 0$.

We have shown that the sequence (x_n) converges if and only if $x_1 \in [-2, 2]$. If $x_1 \in (-2, 2)$, then $\lim x_n = 0$; if $x_1 = -2$, then $\lim x_n = -2$; and if $x_1 = 2$, then $\lim x_n = 2$.

Q.5. Exercises in chapter 05

Q.5.1. (Solution to 5.1.2) Suppose there exists a nonempty set U that is properly contained in A and that is both open and closed in A. Then, clearly, the sets U and U^c (both open in A) disconnect A. Conversely, suppose that A is disconnected by sets U and V (both open in A). Then the set U is not the null set, is not equal to A (because V, its complement with respect to A, is nonempty), is open in A, and is closed in A (because V is open in A).

Q.5.2. (Solution to 5.2.1) Let A be a subset of $\mathbb{R}$ and $f\colon A \to \mathbb{R}$ be continuous. Suppose that $\operatorname{ran} f$ is disconnected. Then there exist disjoint nonempty sets U and V both open in $\operatorname{ran} f$ whose union is $\operatorname{ran} f$. By 3.3.11 the sets $f^{\leftarrow}(U)$ and $f^{\leftarrow}(V)$ are open subsets of A. Clearly these two sets are nonempty, they are disjoint, and their union is A. This shows that A is disconnected.

Q.5.3. (Solution to 5.2.3) The continuous image of an interval is an interval (by theorem 5.2.2). Thus the range of f (that is, $f^{\rightarrow}(J)$) is an interval. So a point belongs to the range of f if it lies between two other points that belong to the range of f. That is, if $a, b \in \operatorname{ran} f$ and z lies between a and b, then there exists a point x in the domain of f such that $z = f(x)$.

Q.5.4. (Solution to 5.2.4) Let

$$f(x) = x^{27} + 5x^{13} + x - x^3 - x^5 - \frac{2}{\sqrt{1+3x^2}}$$

for all $x \in \mathbb{R}$. The function f is defined on an interval (the real line), and it is easy, using the results of chapter 3, to show that f is continuous. Notice that $f(0) = -2$ and that $f(1) = 4$. Since -2 and 4 are in the range of f and 0 lies between -2 and 4, we conclude from the *intermediate value theorem* that 0 is in the range of f. In fact, there exists an x such that $0 < x < 1$ and $f(x) = 0$. Such a number x is a solution to (10). Notice that we have not only shown the existence of a solution to (10) but also located it between consecutive integers.

Q.5.5. (Solution to 5.2.5) Let $f\colon [a, b] \to [a, b]$ be continuous. If $f(a) = a$ or $f(b) = b$, then the result is obvious; so we suppose that $f(a) > a$ and $f(b) < b$. Define $g(x) = x - f(x)$. The function g is continuous on the interval $[a, b]$. (Verify the last assertion.) Notice that $g(a) = a - f(a) < 0$ and that $g(b) = b - f(b) > 0$. Since $g(a) < 0 < g(b)$, we may conclude from the *intermediate value theorem* that $0 \in \operatorname{ran} g$. That is, there exists z in (a, b) such that $g(z) = z - f(z) = 0$. Thus z is a fixed point of f.

Q.6. Exercises in chapter 06

Q.6.1. (Solution to 6.2.3) Let A be a compact subset of $\mathbb{R}$. To show that A is closed, prove that A^c is open. Let y be a point of A^c. For each $x \in A$ choose numbers $r_x > 0$ and $s_x > 0$ such that the open intervals $J_{r_x}(x)$ and $J_{s_x}(y)$ are disjoint. The family $\{J_{r_x}(x) : x \in A\}$ is an open cover for A. Since A is compact there exist $x_1, \dots, x_n \in A$ such that $\{J_{r_{x_i}}(x_i) : 1 \le i \le n\}$ covers A. It is easy to see that if

$$t = \min\{s_{x_1}, \dots, s_{x_n}\}$$

then $J_t(y)$ is disjoint from $\bigcup_{i=1}^n J_{r_{x_i}}(x_i)$ and hence from A. This shows that $y \in A^{c\circ}$. Since y was arbitrary, A^c is open.

To prove that A is bounded, we need consider only the case where A is nonempty. Let a be any point in A. Then $\{J_n(a) : n \in \mathbb{N}\}$ covers A (it covers all of $\mathbb{R}$!). Since A is compact a finite subcollection of these open intervals will cover A. In this finite collection there is a largest open interval; it contains A.

Q.6.2. (Solution to 6.3.2) Let $\mathfrak{V}$ be a family of open subsets of $\mathbb{R}$ that covers $f^{\rightarrow}(A)$. The family

$$\mathfrak{U} := \{f^{\leftarrow}(V) : V \in \mathfrak{V}\}$$

is a family of open sets that covers A (see 3.2.12). Since A is compact we may choose sets $V_1, \dots, V_n \in \mathfrak{V}$ such that $\bigcup_{k=1}^n f^{\leftarrow}(V_k) \supseteq A$. We complete the proof by showing that $f^{\rightarrow}(A)$ is covered by the finite subfamily $\{V_1, \dots, V_n\}$ of $\mathfrak{V}$. If $y \in f^{\rightarrow}(A)$, then $y = f(x)$ for some $x \in A$. This element x belongs to at least one set $f^{\leftarrow}(V_k)$; so (by proposition M.1.22)

$$y = f(x) \in f^{\rightarrow}(f^{\leftarrow}(V_k)) \subseteq V_k\,.$$

Thus $f^{\rightarrow}(A) \subseteq \bigcup_{k=1}^n V_k$.

Q.6.3. (Solution to 6.3.3) We prove that f assumes a maximum on A. (To see that f has a minimum apply the present result to the function $-f$.) By theorem 6.3.2 the image of f is a compact subset of $\mathbb{R}$ and is therefore (see 6.2.3) closed and bounded. Since it is bounded the image of f has a least upper bound, say l. By example 2.2.7 the number l is in the closure of $\operatorname{ran} f$. Since the range of f is closed, $l \in \operatorname{ran} f$. Thus there exists a point a in A such that $f(a) = l \ge f(x)$ for all $x \in A$.

Q.7. Exercises in chapter 07

Q.7.1. (Solution to 7.1.3) Argue by contradiction. If $b \ne c$, then $\epsilon := |b - c| > 0$. Thus there exists $\delta_1 > 0$ such that $|f(x) - b| < \epsilon/2$ whenever $x \in A$ and $0 < |x - a| < \delta_1$ and there exists $\delta_2 > 0$ such that $|f(x) - c| < \epsilon/2$ whenever $x \in A$ and $0 < |x - a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Since $a \in A'$, the set $A \cap J_\delta(a)$ is nonempty. Choose a point x in this set. Then

$$\epsilon = |b - c| \le |b - f(x)| + |f(x) - c| < \epsilon/2 + \epsilon/2 = \epsilon\,.$$

This is a contradiction.

It is worth noticing that the preceding proof cannot be made to work if a is not required to be an accumulation point of A. To obtain a contradiction we must know that the condition $0 < |x - a| < \delta$ is satisfied for at least one x in the domain of f.

Q.7.2. (Solution to 7.2.3) Both halves of the proof make use of the fact that in A the open interval about a of radius δ is just $J_\delta(a) \cap A$ where $J_\delta(a)$ denotes the corresponding open interval in $\mathbb{R}$.

Suppose f is continuous at a. Given $\epsilon > 0$ choose $\delta > 0$ so that $x \in J_\delta(a) \cap A$ implies $f(x) \in J_\epsilon(f(a))$. If $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. That is, $\lim_{x\to a} f(x) = f(a)$.

Conversely, suppose that $\lim_{x\to a} f(x) = f(a)$. Given $\epsilon > 0$ choose $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ whenever $x \in A$ and $0 < |x - a| < \delta$. If $x = a$, then $|f(x) - f(a)| = 0 < \epsilon$. Thus $x \in J_\delta(a) \cap A$ implies $f(x) \in J_\epsilon(f(a))$. This shows that f is continuous at a.

Q.7.3. (Solution to 7.2.4) Let $g\colon h \mapsto f(a + h)$. Notice that $h \in \operatorname{dom} g$ if and only if $a + h \in \operatorname{dom} f$;

$$\operatorname{dom} f = a + \operatorname{dom} g\,. \tag{96}$$

That is, $\operatorname{dom} f = \{a + h\colon h \in \operatorname{dom} g\}$.

First we suppose that

$$l := \lim_{h\to 0} g(h) = \lim_{h\to 0} f(a + h) \text{ exists.} \tag{97}$$

We show that $\lim_{x\to a} f(x)$ exists and equals l. Given $\epsilon > 0$ there exists (by (97)) a number $\delta > 0$ such that

$$|g(h) - l| < \epsilon \text{ whenever } h \in \operatorname{dom} g \text{ and } 0 < |h| < \delta. \tag{98}$$

Now suppose that $x \in \operatorname{dom} f$ and $0 < |x - a| < \delta$. Then by (96)

$$x - a \in \operatorname{dom} g$$

and by (98)

$$|f(x) - l| = |g(x - a) - l| < \epsilon\,.$$

Thus given $\epsilon > 0$ we have found $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $x \in \operatorname{dom} f$ and $0 < |x - a| < \delta$. That is, $\lim_{x\to a} f(x) = l$.

The converse argument is similar. Suppose $l := \lim_{x\to a} f(x)$ exists. Given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $x \in \operatorname{dom} f$ and $0 < |x - a| < \delta$. If $h \in \operatorname{dom} g$ and $0 < |h| < \delta$, then $a + h \in \operatorname{dom} f$ and $0 < |(a + h) - a| < \delta$. Therefore $|g(h) - l| = |f(a + h) - l| < \epsilon$, which shows that $\lim_{h\to 0} g(h) = l$.

Q.8. Exercises in chapter 08

Q.8.1. (Solution to 8.1.9) Suppose that $T \in \mathfrak{L} \cap \mathfrak{o}$. Let $\epsilon > 0$. Since $T \in \mathfrak{o}$, there exists $\delta > 0$ so that $|Ty| \le \epsilon|y|$ whenever $|y| < \delta$. Since $T \in \mathfrak{L}$ there exists $m \in \mathbb{R}$ such that $Tx = mx$ for all x (see example 8.1.7). Now, suppose $0 < |y| < \delta$. Then

$$|m|\,|y| = |Ty| \le \epsilon y$$

so $|m| \le \epsilon$. Since ϵ was arbitrary, we conclude that $m = 0$. That is, T is the constant function 0.

Q.8.2. (Solution to 8.1.10) Let f, $g \in \mathfrak{O}$. Then there exist positive numbers M, N, δ, and η such that $|f(x)| \le M|x|$ whenever $|x| < \delta$ and $|g(x)| < N|x|$ whenever $|x| < \eta$. Then $|f(x) + g(x)| \le (M + N)|x|$ whenever $|x|$ is less than the smaller of δ and η. So $f + g \in \mathfrak{O}$.

If c is a constant, then $|cf(x)| = |c|\,|f(x)| \le |c|\,M\,|x|$ whenever $|x| < \delta$; so $cf \in \mathfrak{O}$.

Q.8.3. (Solution to 8.1.13) (The domain of $f \circ g$ is taken to be the set of all numbers x such that $g(x)$ belongs to the domain of f; that is, $\operatorname{dom}(f \circ g) = g^{\leftarrow}(\operatorname{dom} f)$.) Since $f \in \mathfrak{O}$ there exist $M, \delta > 0$ such that $|f(y)| \le M|y|$ whenever $|y| < \delta$. Let $\epsilon > 0$. Since $g \in \mathfrak{o}$ there exists $\eta > 0$ such that $|g(x)| \le \epsilon M^{-1}|x|$ whenever $|x| \le \eta$.

Now if $|x|$ is less than the smaller of η and $M\epsilon^{-1}\delta$, then $|g(x)| \le \epsilon M^{-1}|x| < \delta$, so that

$$|(f \circ g)(x)| \le M|g(x)| \le \epsilon|x|.$$

Thus $f \circ g \in \mathfrak{o}$.

Q.8.4. (Solution to 8.1.14) Since ϕ and f belong to $\mathfrak{O}$, there exist positive numbers M, N, δ, and η such that $|\phi(x)| \le M|x|$ whenever $|x| < \delta$ and $|f(x)| < N|x|$ whenever $|x| < \eta$. Suppose $\epsilon > 0$. If $|x|$ is less than the smallest of $\epsilon M^{-1}N^{-1}$, δ, and η, then

$$|(\phi f)(x)| = |\phi(x)|\,|f(x)| \le MNx^2 \le \epsilon|x|\,.$$

Q.8.5. (Solution to 8.2.3) Clearly $f(0) - g(0) = 0$. Showing that $\lim_{x\to 0} \dfrac{f(x) - g(x)}{x} = 0$ is a routine computation:

$$\begin{aligned}\lim_{x\to 0} \frac{f(x) - g(x)}{x} &= \lim_{x\to 0} \frac{1}{x}\left(x^2 - 4x - 1 - \frac{1}{3x^2 + 4x - 1}\right)\\ &= \lim_{x\to 0} \frac{3x^4 - 8x^3 - 20x^2}{x(3x^2 + 4x - 1)} = 0.\end{aligned}$$

Q.8.6. (Solution to 8.2.4) Reflexivity is obvious. Symmetry: if $f \simeq g$, then $f - g \in \mathfrak{o}$; so $g - f = (-1)(f - g) \in \mathfrak{o}$ (by proposition 8.1.11). Thus $g \simeq f$. Transitivity: if $f \simeq g$ and $g \simeq h$, then $g - f \in \mathfrak{o}$ and $h - g \in \mathfrak{o}$; so $h - f = (h - g) + (g - f) \in \mathfrak{o}$ (again by 8.1.11). Thus $f \simeq h$.

Q.8.7. (Solution to 8.2.5) Since $S \simeq f$ and $T \simeq f$, we conclude from proposition 8.2.4 that $S \simeq T$. Then $S - T \in \mathfrak{L} \cap \mathfrak{o}$ and thus $S - T = 0$ by proposition 8.1.9.

Q.8.8. (Solution to 8.3.4) For each h in the domain of $\Delta(f + g)_a$ we have

$$\begin{aligned}\Delta(f + g)_a(h) &= (f + g)(a + h) - (f + g)(a)\\ &= f(a + h) + g(a + h) - f(a) - g(a)\\ &= \Delta f_a(h) + \Delta g_a(h)\\ &= (\Delta f_a + \Delta g_a)(h).\end{aligned}$$

Q.8.9. (Solution to 8.3.6) For every h in the domain of $\Delta(g \circ f)_a$ we have

$$\begin{aligned}\Delta(g \circ f)_a(h) &= g(f(a + h)) - g(f(a))\\ &= g(f(a) + f(a + h) - f(a)) - g(f(a))\\ &= g(f(a) + \Delta f_a(h)) - g(f(a))\\ &= \big(\Delta g_{f(a)} \circ \Delta f_a\big)(h).\end{aligned}$$

Q.8.10. (Solution to 8.4.8) If f is differentiable at a, then

$$\Delta f_a = \big(\Delta f_a - df_a\big) + df_a \in \mathfrak{o} + \mathfrak{L} \subseteq \mathfrak{O} + \mathfrak{O} \subseteq \mathfrak{O}\,.$$

Q.8.11. (Solution to 8.4.9) If $f \in \mathcal{D}_a$, then $\Delta f_a \in \mathfrak{O} \subseteq \mathcal{C}_0$. Since Δf_a is continuous at 0, f is continuous at a.

Q.8.12. (Solution to 8.4.10) Since f is differentiable at a, its differential exists and $\Delta f_a \simeq df_a$. Then

$$\Delta(\alpha f)_a = \alpha\Delta f_a \simeq \alpha df_a$$

by propositions 8.3.3 and 8.2.6. Since αdf_a is a linear function that is tangent to $\Delta(\alpha f)_a$, we conclude that it must be the differential of αf at a (see proposition 8.4.2). That is, $\alpha df_a = d(\alpha f)_a$.

Q.8.13. (Solution to 8.4.12) It is easy to check that $\phi(a)df_a + f(a)d\phi_a$ is a linear function. From $\Delta f_a \simeq df_a$ we infer that $\phi(a)\Delta f_a \simeq \phi(a)df_a$ (by proposition 8.2.6), and from $\Delta\phi_a \simeq d\phi_a$ we infer that $f(a)\Delta\phi_a \simeq f(a)d\phi_a$ (also by 8.2.6). From propositions 8.4.8 and 8.1.14 we see that

$$\Delta\phi_a \cdot \Delta f_a \in \mathfrak{O} \cdot \mathfrak{O} \subseteq \mathfrak{o}\,;$$

that is, $\Delta\phi_a \cdot \Delta f_a \simeq 0$. Thus by propositions 8.3.5 and 8.2.6

$$\begin{aligned}\Delta(\phi f)_a &= \phi(a)\Delta f_a + f(a)\Delta\phi_a + \Delta\phi_a \cdot \Delta f_a \\ &\simeq \phi(a)df_a + f(a)d\phi_a + 0 \\ &= \phi(a)df_a + f(a)d\phi_a.\end{aligned}$$

Q.8.14. (Solution to 8.4.13) By hypothesis $\Delta f_a \simeq df_a$ and $\Delta g_{f(a)} \simeq dg_{f(a)}$. By proposition 8.4.8 $\Delta f_a \in \mathfrak{O}$. Then by proposition 8.2.8

$$\Delta g_{f(a)} \circ \Delta f_a \simeq dg_{f(a)} \circ \Delta f_a\,; \tag{99}$$

and by proposition 8.2.7

$$dg_{f(a)} \circ \Delta f_a \simeq dg_{f(a)} \circ df_a\,. \tag{100}$$

According to proposition 8.3.6

$$\Delta(g \circ f)_a = \Delta g_{f(a)} \circ \Delta f_a\,. \tag{101}$$

From (99), (100) , and (101), and proposition 8.2.4 we conclude that

$$\Delta(g \circ f)_a \simeq dg_{f(a)} \circ df_a\,.$$

Since $dg_{f(a)} \circ df_a$ is a linear function, the desired conclusion is an immediate consequence of proposition 8.4.2.

Q.9. Exercises in chapter 09

Q.9.1. (Solution to 9.1.2) If x, $y \in M$, then

$$0 = d(x,x) \le d(x,y) + d(y,x) = 2\,d(x,y)\,.$$

Q.9.2. (Solution to 9.3.2) It is clear that $\max\{a,b\} \le a + b$. Expanding the right side of $0 \le (a \pm b)^2$ we see that $2|ab| \le a^2 + b^2$. Thus

$$\begin{aligned}(a+b)^2 &\le a^2 + 2|ab| + b^2 \\ &\le 2(a^2+b^2)\,.\end{aligned}$$

Taking square roots we get $a + b \le \sqrt{2}\,\sqrt{a^2+b^2}$. Finally,

$$\begin{aligned}\sqrt{2}\,\sqrt{a^2+b^2} &\le \sqrt{2}\,\sqrt{2(\max\{a,b\})^2} \\ &= 2\max\{a,b\}\,.\end{aligned}$$

We have established the claim made in the hint. Now if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are points in $\mathbb{R}^2$, then

$$\begin{aligned}\max\{|x_1 - y_1|, |x_2 - y_2|\} &\leq |x_1 - y_1| + |x_2 - y_2| \\ &\leq \sqrt{2}\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &\leq 2\max\{|x_1 - y_1|, |x_2 - y_2|\}\,.\end{aligned}$$

In other words

$$d_u(x, y) \leq d_1(x, y) \leq \sqrt{2}\, d(x, y) \leq 2\, d_u(x, y)\,.$$

Q.10. Exercises in chapter 10

Q.10.1. (Solution to 10.1.3) The three open balls are $B_1(-1) = \{-1\}$, $B_1(0) = [0, 1)$, and $B_2(0) = \{-1\} \cup [0, 2)$.

Q.10.2. (Solution to 10.1.14) (a) $A^\circ = \emptyset$; $A' = \{0\}$; $\overline{A} = \partial A = \{0\} \cup A$.

(b) $A^\circ = \emptyset$; $A' = \overline{A} = \partial A = [0, \infty)$.

(c) $A^\circ = A$; $A' = \overline{A} = A \cup \{0\}$; $\partial A = \{0\}$.

Q.10.3. (Solution to 10.2.1) Let $t = r - d(a, c)$. (Note that $t > 0$.) If $x \in B_t(c)$, then $d(a, x) \leq d(a, c) + d(c, x) < (r - t) + t = r$; so $x \in B_r(a)$. (Easier solution: This is just a special case of proposition 9.2.19. Take $b = a$ and $s = r$ there.)

Q.10.4. (Solution to 10.2.2) (a) If $x \in A^\circ$, then there exists $r > 0$ such that $B_r(x) \subseteq A \subseteq B$. So $x \in B^\circ$.

(b) Since $A^\circ \subseteq A$ we may conclude from (a) that $A^{\circ\circ} \subseteq A^\circ$. For the reverse inclusion take $a \in A^\circ$. Then there exists $r > 0$ such that $B_r(a) \subseteq A$. By lemma 10.2.1, about each point b in the open ball $B_r(a)$ we can find an open ball $B_s(b)$ contained in $B_r(a)$ and hence in A. This shows that $B_r(a) \subseteq A^\circ$. Since some open ball about a lies inside A°, we conclude that $a \in A^{\circ\circ}$.

Q.10.5. (Solution to 10.2.4) (a) Since $A \subseteq \bigcup \mathfrak{A}$ for every $A \in \mathfrak{A}$, we can conclude from proposition 10.2.2(a) that $A^\circ \subseteq \left(\bigcup \mathfrak{A}\right)^\circ$ for every $A \in \mathfrak{A}$. Thus $\bigcup\{A^\circ \colon A \in \mathfrak{A}\} \subseteq \left(\bigcup \mathfrak{A}\right)^\circ$.

(b) Let $\mathbb{R}$ be the metric space and $\mathfrak{A} = \{A, B\}$ where $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$. Then $A^\circ \cup B^\circ = \emptyset$ while $(A \cup B)^\circ = \mathbb{R}$.

Q.11. Exercises in chapter 11

Q.11.1. (Solution to 11.1.6) Since A° is an open set contained in A (by 10.2.2(b)), it is contained in the union of all such sets. That is,

$$A^\circ \subseteq \bigcup\{U \colon U \subseteq A \text{ and } U \text{ is open}\}.$$

On the other hand, if U is an open subset of A, then (by 10.2.2(a)) $U = U^\circ \subseteq A^\circ$. Thus

$$\bigcup\{U \colon U \subseteq A \text{ and } U \text{ is open}\} \subseteq A^\circ\,.$$

Q.11.2. (Solution to 11.1.9) Let A be a subset of a metric space. Using problem 10.3.6 we see that

$$\begin{aligned} A \text{ is open } & \text{ iff } A = A^\circ \\ & \text{ iff } A^c = \left(A^\circ\right)^c = \overline{A^c} \\ & \text{ iff } A^c \text{ is closed.} \end{aligned}$$

Q.11.3. (Solution to 11.1.22) Suppose that D is dense in M. Argue by contradiction. If there is an open ball B that contains no point of D (that is, $B \subseteq D^c$), then

$$B = B^\circ \subseteq \left(D^c\right)^\circ = \left(\overline{D}\right)^c = M^c = \emptyset\,,$$

which is not possible.

Conversely, suppose that D is not dense in M. Then $\overline{D}$ is a proper closed subset of M, making $\left(\overline{D}\right)^c$ a nonempty open set. Choose an open ball $B \subseteq \left(\overline{D}\right)^c$. Since $\left(\overline{D}\right)^c \subseteq D^c$, the ball B contains no point of D.

Q.11.4. (Solution to 11.2.3) It is enough to show that $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. Thus we suppose that U is an open subset of (M, d_1) and prove that it is an open subset of (M, d_2). For $a \in M$, $r > 0$, and $k = 1, 2$ let $B_r^k(a)$ be the open ball about a of radius r in the space (M, d_k); that is,

$$B_r^k(a) = \{x \in M : d_k(x, a) < r\}\,.$$

To show that U is an open subset of (M, d_2) choose an arbitrary point x in U and find an open ball $B_r^2(x)$ about x contained in U. Since U is assumed to be open in (M, d_1) there exists $s > 0$ such that $B_s^1(x) \subseteq U$. The metrics d_1 and d_2 are equivalent, so, in particular, there is a constant $\alpha > 0$ such that $d_1(u, v) \le \alpha\, d_2(u, v)$ for all $u, v \in M$. Let $r = s\,\alpha^{-1}$. Then if $y \in B_r^2(x)$, we see that

$$d_1(y, x) \le \alpha\, d_2(y, x) < \alpha\, r = s\,.$$

Thus $B_r^2(x) \subseteq B_s^1(x) \subseteq U$.

Q.12. Exercises in chapter 12

Q.12.1. (Solution to 12.2.2) Suppose that A is closed in M. Let (a_n) be a sequence in A that converges to a point b in M. If b is in A^c then, since A^c is a neighborhood of b, the sequence (a_n) is eventually in A^c, which is not possible. Therefore $b \in A$.

Conversely, if A is not closed, there exists an accumulation point b of A that does not belong to A. Then for every $n \in \mathbb{N}$ we may choose a point a_n in $B_{1/n}(b) \cap A$. The sequence (a_n) lies in A and converges to b; but $b \notin A$.

Q.12.2. (Solution to 12.3.2) As in 12.3.1, let ρ_1 be the metric on M_1 and ρ_2 be the metric on M_2. Use the inequalities given in the hint to 9.3.2 with $a = \rho_1(x_1, y_1)$ and $b = \rho_2(x_2, y_2)$ to obtain

$$d_u(x, y) \le d_1(x, y) \le \sqrt{2}\, d(x, y) \le 2\, d_u(x, y)\,.$$

Q.13. Exercises in chapter 13

Q.13.1. (Solution to 13.1.6) Let C be the set of all functions defined on $[0, 1]$ such that $0 < g(x) < 2$ for all $x \in [0, 1]$. It is clear that $B_1(f) \subseteq C$. The reverse inclusion, however, is not correct. For example, let

$$g(x) = \begin{cases} 1, & \text{if } x = 0 \\ x, & \text{if } 0 < x \leq 1. \end{cases}$$

Then g belongs to C; but it does *not* belong to $B_1(f)$ since

$$d_u(f, g) = \sup\{|f(x) - g(x)| : 0 \leq x \leq 1\} = 1\,.$$

Q.13.2. (Solution to 13.1.10) Let f, g, $h \in \mathcal{B}(S)$. There exist positive constants M, N, and P such that $|f(x)| \leq M$, $|g(x)| \leq N$, and $|h(x)| \leq P$ for all x in S.

First show that d_u is real valued. (That is, show that d_u is never infinite.) This is easy:

$$\begin{aligned} d_u(f, g) &= \sup\{|f(x) - g(x)| \colon x \in S\} \\ &\leq \sup\{|f(x)| + |g(x)| \colon x \in S\} \\ &\leq M + N. \end{aligned}$$

Now verify conditions (1)–(3) of the definition of "metric" in 9.1.1. Condition (1) follows from the observation that

$$|f(x) - g(x)| = |g(x) - f(x)| \qquad \text{for all } x \in S.$$

To establish condition (2) notice that for every $x \in S$

$$\begin{aligned} |f(x) - h(x)| &\leq |f(x) - g(x)| + |g(x) - h(x)| \\ &\leq d_u(f, g) + d_u(g, h); \end{aligned}$$

and therefore

$$d_u(f, h) = \sup\{|f(x) - h(x)| \colon x \in S\} \leq d_u(f, g) + d_u(g, h)\,.$$

Finally, condition (3) holds since

$$\begin{aligned} d_u(f, g) = 0 &\text{ iff } |f(x) - g(x)| = 0 \qquad \text{for all } x \in S \\ &\text{ iff } f(x) = g(x) \qquad \text{for all } x \in S \\ &\text{ iff } f = g. \end{aligned}$$

Q.13.3. (Solution to 13.2.2) Let (f_n) be a sequence of functions in $\mathcal{F}(S, \mathbb{R})$ and suppose that $f_n \to g\,(\text{unif})$ in $\mathcal{F}(S, \mathbb{R})$. Then for every $x \in S$

$$|f_n(x) - g(x)| \leq \sup\{|f_n(y) - g(y)| \colon y \in S\} \to 0 \qquad \text{as } n \to \infty.$$

Thus $f_n \to g\,(\text{ptws})$.

On the other hand, if we define for each $n \in \mathbb{N}$ a function f_n on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1, & \text{if } x \geq n \\ 0, & \text{if } x < n, \end{cases}$$

then it is easy to see that the sequence (f_n) converges pointwise to the zero function $\mathbf{0}$; but since $d_u(f_n, \mathbf{0}) = 1$ for every n, the sequence does not converge uniformly to $\mathbf{0}$.

Q.13.4. (Solution to 13.2.4) (a) Since $f_n \to g$ (unif), there exists $m \in \mathbb{N}$ such that

$$|f_n(x) - g(x)| \le \sup\{|f_n(y) - g(y)| \colon y \in S\} < 1$$

whenever $n \ge m$ and $x \in S$. Thus, in particular,

$$|g(x)| \le |f_m(x) - g(x)| + |f_m(x)| < 1 + K$$

where K is a number satisfying $|f_m(x)| \le K$ for all $x \in S$.

(b) Let

$$f_n(x) = \begin{cases} x, & \text{if } |x| \le n \\ 0, & \text{if } |x| > n \end{cases}$$

and $g(x) = x$ for all x in $\mathbb{R}$. Then $f_n \to g$ (ptws), each f_n is bounded, but g is not.

Q.13.5. (Solution to 13.2.5) If $0 \le x < 1$, then

$$f_n(x) = x^n - x^{2n} \to 0 \qquad \text{as } n \to \infty\,.$$

This and the obvious fact that $f_n(1) = 0$ for every n tell us that

$$f_n \to \mathbf{0}\,(\text{ptws})\,.$$

Observe that $x^{2n} \le x^n$ whenever $0 \le x \le 1$ and $n \in \mathbb{N}$. Thus $f_n \ge 0$ for each n. Use techniques from beginning calculus to find the maximum value of each f_n. Differentiating we see that

$$f_n'(x) = n\,x^{n-1} - 2n\,x^{2n-1} = n\,x^{n-1}(1 - 2x^n)$$

for $n > 1$. Thus the function f_n has a critical point, in fact assumes a maximum, when $1 - 2x^n = 0$; that is, when $x = 2^{-1/n}$. But then

$$\sup\{|f_n(x) - 0| \colon 0 \le x \le 1\} = f_n(2^{-1/n}) = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}$$

for all $n > 1$. Since $\sup\{|f_n(x) - 0| \colon 0 \le x \le 1\} \nrightarrow 0$ as $n \to \infty$, the convergence is not uniform.

Is it possible that the sequence f_n converges uniformly to some function g other than the zero function? The answer is *no*. According to proposition 13.2.2 if $f_n \to g \ne 0$ (unif), then $f_n \to g$ (ptws). But this contradicts what we have already shown, namely, $f_n \to 0$ (ptws).

Note: Since each of the functions f_n belongs to $\mathcal{B}([0,1])$, it is permissible, and probably desirable, in the preceding proof to replace each occurrence of the rather cumbersome expression $\sup\{|f_n(x) - 0| \colon 0 \le x \le 1\}$ by $d_u(f_n, \mathbf{0})$.

Q.14. Exercises in chapter 14

Q.14.1. (Solution to 14.1.5) Suppose f is continuous. Let $U \overset{\circ}{\subseteq} M_2$. To show that $f^{\leftarrow}(U)$ is an open subset of M_1, it suffices to prove that each point of $f^{\leftarrow}(U)$ is an interior point of $f^{\leftarrow}(U)$. If $a \in f^{\leftarrow}(U)$, then $f(a) \in U$. Since f is continuous at a, the set U, which is a neighborhood of $f(a)$, must contain the image under f of a neighborhood V of a. But then

$$a \in V \subseteq f^{\leftarrow}(f^{\rightarrow}(V)) \subseteq f^{\leftarrow}(U)\,,$$

which shows that a lies in the interior of $f^{\leftarrow}(U)$.

Conversely, suppose that $f^{\leftarrow}(U) \overset{\circ}{\subseteq} M_1$ whenever $U \overset{\circ}{\subseteq} M_2$. To see that f is continuous at an arbitrary point a in M_1, notice that if V is a neighborhood of $f(a)$, then $a \in f^{\leftarrow}(V) \overset{\circ}{\subseteq} M_1$. Thus $f^{\leftarrow}(V)$ is a neighborhood of a whose image $f^{\rightarrow}(f^{\leftarrow}(V))$ is contained in V. Thus f is continuous at a.

Q.14.2. (Solution to 14.1.9) Show that M is continuous at an arbitrary point (a, b) in $\mathbb{R}^2$. Since the metric d_1 (defined in 9.2.10) is equivalent to the usual metric on $\mathbb{R}^2$, proposition 14.1.8 assures us that it is enough to establish continuity of the function M with respect to the metric d_1. Let $K = |a| + |b| + 1$. Given $\epsilon > 0$, choose $\delta = \min\{\epsilon/K, 1\}$. If (x, y) is a point in $\mathbb{R}^2$ such that $|x - a| + |y - b| = d_1((x, y), (a, b)) < \delta$, then

$$|x| \leq |a| + |x - a| < |a| + \delta \leq |a| + 1 \leq K.$$

Thus for all such points (x, y)

$$\begin{aligned} |M(x, y) - M(a, b)| &= |xy - xb + xb - ab| \\ &\leq |x|\,|y - b| + |x - a|\,|b| \\ &\leq K|y - b| + K|x - a| \\ &< K\delta \\ &\leq \epsilon. \end{aligned}$$

Q.14.3. (Solution to 14.1.26) Suppose that f is continuous at a. Let $x_n \to a$ and B_2 be a neighborhood of $f(a)$. There exists a neighborhood B_1 of a such that $f^{\rightarrow}(B_1) \subseteq B2$. Choose $n_0 \in \mathbb{N}$ so that $x_n \in B_1$ whenever $n \geq n_0$. Then $f(x_n) \in f^{\rightarrow}(B_1) \subseteq B_2$ whenever $n \geq n_0$. That is, the sequence $\big(f(x_n)\big)$ is eventually in the neighborhood B_2. Since B_2 was arbitrary, $f(x_n) \to f(a)$.

Conversely, suppose that f is not continuous at a. Then there exists $\epsilon > 0$ such that the image under f of $B_\delta(a)$ contains points that do not belong to $B_\epsilon(f(a))$, no matter how small $\delta > 0$ is chosen. Thus for every $n \in \mathbb{N}$ there exists $x_n \in B_{1/n}(a)$ such that $f(x_n) \notin B_\epsilon(f(a))$. Then clearly $x_n \to a$ but $f(x_n) \not\to f(a)$.

Q.14.4. (Solution to 14.2.1) Let (a, b) be an arbitrary point in $M_1 \times M_2$. If $(x_n, y_n) \to (a, b)$ in $M_1 \times M_2$, then proposition 12.3.4 tells us that $\pi_1(x_n, y_n) = x_n \to a = \pi_1(a, b)$. This shows that π_1 is continuous at (a, b). Similarly, π_2 is continuous at (a, b).

Q.14.5. (Solution to 14.2.3) Suppose f is continuous. Then the components $f^1 = \pi_1 \circ f$ and $f^2 = \pi_2 \circ f$, being composites of continuous functions, are continuous. Conversely, suppose that f^1 and f^2 are continuous. Let a be an arbitrary point in N. If $x_n \to a$, then (by proposition 14.1.26) $f^1(x_n) \to f^1(a)$ and $f^2(x_n) \to f^2(a)$. From proposition 12.3.4 we conclude that

$$f(x_n) = (f^1(x_n), f^2(x_n)) \to (f^1(a), f^2(a)) = f(a)\,;$$

so f is continuous at a.

Q.14.6. (Solution to 14.2.4) (a) The function fg is the composite of continuous functions (that is, $fg = M \circ (f, g)$); so it is continuous by corollary 14.1.4.

(b) This is just a special case of (a) where f is the constant function whose value is α.

Q.14.7. (Solution to 14.2.12) For each $a \in M_1$ let $j_a\colon M_2 \to M_1 \times M_2$ be defined by $j_a(y) = (a, y)$. Then j_a is continuous. (Proof: If $y_n \to c$, then $j_a(y_n) = (a, y_n) \to (a, c) = j_a(c)$.) Since $f(a, \cdot\,) = f \circ j_a$, it too is continuous. The continuity of each $f(\,\cdot\, b)$ is established in a similar manner.

Q.14.8. (Solution to 14.2.15) Show that g is continuous at an arbitrary point a in M. Let $\epsilon > 0$. Since $f_n \to g$ (unif) there exists $n \in \mathbb{N}$ such that

$$|f_m(x) - g(x)| \leq \sup\{|f_n(y) - g(y)|\colon y \in M\} < \epsilon/3$$

whenever $m \geq n$ and $x \in M$. Since f_n is continuous at a, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(a)| < \epsilon/3$$

whenever $d(x, a) < \delta$. Thus for $x \in B_\delta(a)$

$$\begin{aligned} |g(x) - g(a)| &\leq |g(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - g(a)| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon\,. \end{aligned}$$

This shows that g is continuous at a.

Q.14.9. (Solution to 14.3.3) Argue by contradiction. If $b \neq c$, then $\epsilon = d(b, c) > 0$. Thus there exists $\delta_1 > 0$ such that $d(f(x), b) < \epsilon/2$ whenever $x \in A$ and $0 < d(x, a) < \delta_1$, and there exists $\delta_2 > 0$ such that $d(f(x), c) < \epsilon/2$ whenever $x \in A$ and $0 < d(x, a) < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Since $a \in A'$, the set $A \cap B_\delta(a)$ is nonempty. Choose a point x in this set. Then

$$\epsilon = d(b, c) \leq d(b, f(x)) + d(f(x), c) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon\,.$$

This is a contradiction.

It is worth noticing that the preceding proof cannot be made to work if a is not required to be an accumulation point of A. To obtain a contradiction we must know that the condition $0 < d(x, a) < \delta$ is satisfied for at least one x in the domain of f.

Q.14.10. (Solution to 14.3.8) Let $g(x) = \lim_{y\to b} f(x, y)$ for all $x \in \mathbb{R}$. It is enough to show that $\lim_{x\to a}\bigl(\lim_{y\to b} f(x, y)\bigr) = \lim_{x\to a} g(x) = l$. Given $\epsilon > 0$, choose $\delta > 0$ so that $|f(x, y) - l| < \epsilon/2$ whenever $0 < d\bigl((x, y), (a, b)\bigr) < \delta$. (This is possible because $l = \lim_{(x,y)\to(a,b)} f(x, y)$.) Suppose that $0 < |x - a| < \delta/\sqrt{2}$. Then (by the definition of g) there exists $\eta_x > 0$ such that $|g(x) - f(x, y)| < \epsilon/2$ whenever $0 < |y - b| < \eta_x$. Now choose any y such that $0 < |y - b| < \min\{\delta/\sqrt{2}, \eta_x\}$. Then (still supposing that $0 < |x - a| < \delta/\sqrt{2}$) we see that

$$\begin{aligned} 0 &< d\bigl((x, y), (a, b)\bigr) \\ &= \bigl((x - a)^2 + (y - b)^2\bigr)^{1/2} \\ &< \bigl((\delta^2/2 + \delta^2/2)\bigr)^{1/2} \\ &= \delta \end{aligned}$$

so

$$|f(x, y) - l| < \epsilon/2$$

and therefore

$$\begin{aligned}|g(x) - l| &\le |g(x) - f(x,y)| + |f(x,y) - l| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.\end{aligned}$$

That is, $\lim_{x\to a} g(x) = l$.

Q.14.11. (Solution to 14.3.9) By the remarks preceding this exercise, we need only find two distinct values that the function f assumes in every neighborhood of the origin. This is easy. Every neighborhood of the origin contains points $(x, 0)$ distinct from $(0, 0)$ on the x-axis. At every such point $f(x, y) = f(x, 0) = 0$. Also, every neighborhood of the origin contains points (x, x) distinct from $(0, 0)$ that lie on the line $y = x$. At each such point $f(x, y) = f(x, x) = 1/17$. Thus f has no limit at the origin since in every neighborhood of $(0, 0)$ it assumes both the values 0 and $1/17$. (Notice, incidentally, that both iterated limits, $\lim_{x\to 0}\bigl(\lim_{y\to 0} f(x, y)\bigr)$ and $\lim_{y\to 0}\bigl(\lim_{x\to 0} f(x, y)\bigr)$ exist and equal 0.)

Q.15. Exercises in chapter 15

Q.15.1. (Solution to 15.1.2) Suppose that A is compact. Let $\mathfrak{V}$ be a family of open subsets of M that covers A. Then

$$\mathfrak{U} := \{V \cap A \colon V \in \mathfrak{V}\}$$

is a family of open subsets of A that covers A. Since A is compact we may choose a subfamily $\{U_1, \dots, U_n\} \subseteq \mathfrak{U}$ that covers A. For $1 \le k \le n$ choose $V_k \in \mathfrak{V}$ so that $U_k = V_k \cap A$. Then $\{V_1, \dots, V_n\}$ is a finite subfamily of $\mathfrak{V}$ that covers A.

Conversely, Suppose every cover of A by open subsets of M has a finite subcover. Let $\mathfrak{U}$ be a family of open subsets of A that covers A. According to proposition 11.2.1 there is for each U in $\mathfrak{U}$ a set V_U open in M such that $U = V_U \cap A$. Let $\mathfrak{V}$ be $\{V_U \colon U \in \mathfrak{U}\}$. This is a cover for A by open subsets of M. By hypothesis there is a finite subfamily $\{V_1, \dots, V_n\}$ of $\mathfrak{V}$ that covers A. For $1 \le k \le n$ let $U_k = V_k \cap A$. Then $\{U_1, \dots, U_n\}$ is a finite subfamily of $\mathfrak{U}$ that covers A. Thus A is compact.

Q.16. Exercises in chapter 16

Q.16.1. (Solution to 16.1.9) If a metric space M is not totally bounded then there exists a positive number ϵ such that for every finite subset F of M there is a point a in M such that $F \cap B_\epsilon(a) = \emptyset$. Starting with an arbitrary point x_1 in M, construct a sequence (x_k) inductively as follows: having chosen $x_1, \dots, x_n$ no two of which are closer together than ϵ, choose $x_{n+1} \in M$ so that

$$B_\epsilon(x_{n+1}) \cap \{x_1, \dots, x_n\} = \emptyset\,.$$

Since no two terms of the resulting sequence are closer together than ϵ, it has no convergent subsequence.

Q.16.2. (Solution to 16.2.1) (a) $\Rightarrow$ (b): Suppose that there exists an infinite subset A of M that has no accumulation point in M. Then $\overline{A} = A$, so that A is closed. Each point $a \in A$ has a neighborhood B_a that contains no point of A other than a. Thus $\{B_a \colon a \in A\}$ is a cover for A by open subsets of M no finite subfamily of which covers A. This shows that A is not compact. We conclude from proposition 15.1.3 that M is not compact.

(b) $\Rightarrow$ (c): Suppose that (2) holds. Let a be a sequence in M. If the range of a is finite, then a has a constant (therefore convergent) subsequence. Thus we assume that the image of a is infinite. By hypothesis $\operatorname{ran} a$ has an accumulation point $m \in M$. We define $n \colon \mathbb{N} \to \mathbb{N}$ inductively: let $n(1) = 1$; if $n(1), \dots, n(k)$ have been defined, let $n(k+1)$ be an integer greater than $n(k)$ such that $a_{n_{k+1}} \in B_{1/(k+1)}(m)$. (This is possible since $B_{1/(k+1)}(m)$ contains infinitely many distinct points of the range of a.) It is clear that $a \circ n$ is a subsequence of a and that $a_{n_k} \to m$ as $k \to \infty$.

(c) $\Rightarrow$ (a): Let $\mathfrak{U}$ be an open cover for M. By corollary 16.1.15 the space M is separable. Let A be a countable dense subset of M and $\mathfrak{B}$ be the family of all open balls $B(a; r)$ such that

(i) $a \in A$;

(ii) $r \in \mathbb{Q}$ and;

(iii) $B_r(a)$ is contained in at least one member of $\mathfrak{U}$.

Then for each $B \in \mathfrak{B}$ choose a set U_B in $\mathfrak{U}$ that contains B. Let

$$\mathfrak{V} = \{U_B \in \mathfrak{U} : B \in \mathfrak{B}\}.$$

It is clear that $\mathfrak{V}$ is countable; we show that V covers M.

Let $x \in M$. There exist U_0 in $\mathfrak{U}$ and $r > 0$ such that $B_r(x) \subseteq U_0$. Since A is dense in M, proposition 11.1.22 allows us to select a point a in $A \cap B_{\frac{1}{3}r}(x)$. Next let s be any rational number such that $\frac{1}{3}r < s < \frac{2}{3}r$. Then $x \in B_s(a) \subseteq B_r(x) \subseteq U_0$. [Proof: if $y \in B_s(a)$, then

$$d(y, x) \le d(y, a) + d(a, x) < s + \frac{1}{3}r < r\,;$$

so $y \in B_r(x)$.] This shows that $B_s(a)$ belongs to $\mathfrak{B}$ and that $x \in U_{B_s(a)} \in \mathfrak{V}$. Thus $\mathfrak{V}$ covers M. Now enumerate the members of $\mathfrak{V}$ as a sequence $(V_1, V_2, V_3, \dots)$ and let $W_n = \cup_{k=1}^{n} V_k$ for each $n \in \mathbb{N}$. To complete the proof it suffices to find an index n such that $W_n = M$. Assume there is no such n. Then for every k we may choose a point x_k in ${W_k}^c$. The sequence (x_k) has, by hypothesis, a convergent subsequence (x_{n_k}). Let b be the limit of this sequence. Then for some m in $\mathbb{N}$ we have $b \in V_m \subseteq W_m$. Thus W_m is an open set that contains b but only finitely many of the points x_{n_k}. (W_m contains at most the points $x_1, \dots, x_{m-1}$.) Since $x_{n_k} \to b$, this is not possible.

Q.16.3. (Solution to 16.4.1) A compact subset of *any* metric space is closed and bounded (by problem 15.1.5). It is the converse we are concerned with here.

Let A be a closed and bounded subset of $\mathbb{R}^n$. Since it is bounded, there exist closed bounded intervals $J_1, \dots, J_n$ in $\mathbb{R}$ such that

$$A \subseteq J \equiv J_1 \times \dots \times J_n\,.$$

Each J_k is compact by example 6.3.5. Their product J is compact by corollary 16.3.2. Since J is a compact subset of $\mathbb{R}^n$ under the product metric, it is a compact subset of $\mathbb{R}^n$ under its usual Euclidean metric (see proposition 11.2.3

and the remarks preceding it). Since A is a closed subset of J, it is compact by proposition 15.1.3.

Q.17. Exercises in chapter 17

Q.17.1. (Solution to 17.1.6) Suppose there exists a nonempty set U that is properly contained in M and that is both open and closed. Then, clearly, the open sets U and U^c disconnect M. Conversely, suppose that the space M is disconnected by sets U and V. Then the set U is not the null set, is not equal to M (because its complement V is nonempty), is open, and is closed (because V is open).

Q.17.2. (Solution to 17.1.8) If N is disconnected, it can be written as the union of two disjoint nonempty sets U and V that are open in N. (These sets need not, of course, be open in M.) We show that U and V are mutually separated. It suffices to prove that $U \cap \overline{V}$ is empty, that is, that $U \subseteq \overline{V}^c$. To this end suppose that $u \in U$. Since U is open in N, there exists $\delta > 0$ such that

$$N \cap B_\delta(u) = \{x \in N \colon d(x,u) < \delta\} \subseteq U \subseteq V^c .$$

Clearly $B_\delta(u)$ is the union of two sets: $N \cap B_\delta(u)$ and $N^c \cap B_\delta(u)$. We have just shown that the first of these is contained in V^c. The second contains no points of N and therefore no points of V. Thus $B_\delta(u) \subseteq V^c$. This shows that u does not belong to the closure (in M) of the set V; so $u \in \overline{V}^c$. Since u was an arbitrary point of U, we conclude that $U \subseteq \overline{V}^c$.

Conversely, suppose that $N = U \cup V$ where U and V are nonempty sets mutually separated in M. To show that the sets U and V disconnect N, we need only show that they are open in N, since they are obviously disjoint.

We prove that U is open in N. Let $u \in U$ and notice that since $U \cap \overline{V}$ is empty, u cannot belong to $\overline{V}$. Thus there exists $\delta > 0$ such that $B_\delta(u)$ is disjoint from V. Then certainly $N \cap B_\delta(u)$ is disjoint from V. Thus $N \cap B_\delta(u)$ is contained in U. Conclusion: U is open in N.

Q.17.3. (Solution to 17.1.11) Let $G = \{(x,y) \colon y = \sin x\}$. The function $x \mapsto (x, \sin x)$ is a continuous surjection from $\mathbb{R}$ (which is connected by proposition 5.1.9) onto $G \subseteq \mathbb{R}^2$. Thus G is connected by theorem 17.1.10.

Q.17.4. (Solution to 17.1.13) Let the metric space M be the union of a family $\mathfrak{C}$ of connected subsets of M and suppose that $\bigcap \mathfrak{C} \neq \emptyset$. Argue by contradiction. Suppose that M is disconnected by disjoint nonempty open sets U and V. Choose an element p in $\bigcap \mathfrak{C}$. Without loss of generality suppose that $p \in U$. Choose $v \in V$. There is at least one set C in $\mathfrak{C}$ such that $v \in C$. We reach a contradiction by showing that the sets $U \cap C$ and $V \cap C$ disconnect C. These sets are nonempty [p belongs to $C \cap U$ and v to $C \cap V$] and open in C. They are disjoint because U and V are, and their union is C, since

$$(U \cap C) \cup (V \cap C) = (U \cup V) \cap C = M \cap C = C .$$

Q.17.5. (Solution to 17.1.15) Between an arbitrary point x in the unit square and the origin there is a straight line segment, denote it by $[0,x]$. Line segments are connected because they are continuous images of (in fact, homeomorphic to) intervals in $\mathbb{R}$. The union of all the segments $[0,x]$ where x is in the unit square is the square itself. The intersection of all these segments is the origin. Thus by proposition 17.1.13 the square is connected.

Q.17.6. (Solution to 17.2.7) The set $B = \{(x, \sin x^{-1}) \colon 0 < x \leq 1\}$ is a connected subset of $\mathbb{R}^2$ since it is the continuous image of the connected set $(0, 1]$ (see theorem 17.1.10). Then by proposition 17.1.9 the set $M := \overline{B}$ is also connected. Notice that $M = A \cup B$ where $A = \{(0, y) \colon |y| \leq 1\}$.

To show that M is not arcwise connected, argue by contradiction. Assume that there exists a continuous function $f \colon [0, 1] \to M$ such that $f(0) \in A$ and $f(1) \in B$. We arrive at a contradiction by showing that the component function $f^2 = \pi_2 \circ f$ is not continuous at the point $t_0 = \sup f^{\leftarrow}(A)$.

To this end notice first that, since A is closed in M and f is continuous, the set $f^{\leftarrow}(A)$ is closed in $[0, 1]$. By example 2.2.7 the point t_0 belongs to $f^{\leftarrow}(A)$. Without loss of generality we may suppose that $f^2(t_0) \leq 0$. We need only show that for every $\delta > 0$ there exists a number $t \in [0, 1]$ such that $|t - t_0| < \delta$ and $|f^2(t) - f^2(t_0)| \geq 1$.

Let $\delta > 0$. Choose a point t_1 in $(t_0, t_0 + \delta) \cap [0, 1]$. By proposition 5.1.9 the interval $[t_0, t_1]$ is connected, so its image $\left(f^1\right)^{\rightarrow}[t_0, t_1]$ under the continuous function $f^1 = \pi_1 \circ f$ is also a connected subset of $[0, 1]$ (by theorem 17.1.10) and therefore itself an interval. Let $c = f^1(t_1)$. From $t_1 > t_0$ infer that $t_1 \in f^{\leftarrow}(B)$ and that therefore $c > 0$. Since $t_0 \in f^{\leftarrow}(A)$ it is clear that $f^1(t_0) = 0$. Thus the interval $[0, c]$ is not a single point and it is contained in $\left(f^1\right)^{\rightarrow}[t_0, t_1]$. Choose $n \in \mathbb{N}$ sufficiently large that

$$x = \frac{2}{(4n + 1)\pi} < c\,.$$

Since x belongs to $\left(f^1\right)^{\rightarrow}[t_0, t_1]$, there exists $t \in [t_0, t_1]$ such that $x = f^1(t)$. And since $x > 0$ the point $f(t)$ belongs to B. This implies that

$$f(t) = \left(f^1(t), f^2(t)\right) = (x, \sin x^{-1}) = (x, \sin(4n + 1)\tfrac{\pi}{2}) = (x, 1)\,.$$

But then (since $f^2(t_0) \leq 0$)

$$|f^2(t) - f^2(t_0)| = |1 - f^2(t_0)| \geq 1\,.$$

Q.17.7. (Solution to 17.2.8) Let A be a connected open subset of $\mathbb{R}^n$. If A is empty the result is obvious, so suppose that it is not. Choose $a \in A$. Let U be the set of all points x in A for which there exists a continuous function $f \colon [0, 1] \to A$ such that $f(0) = a$ and $f(1) = x$. The set U is nonempty [it contains a]. Let $V = A \setminus U$. We wish to show that V is empty. Since A is connected it suffices to show that both U and V are open.

To show that U is open let $u \in U$ and let $f \colon [0, 1] \to A$ be a continuous function such that $f(0) = a$ and $f(1) = u$. Since A is an open subset of $\mathbb{R}^n$ there exists $\delta > 0$ such that $B_\delta(u) \subseteq A$. Every point b in $B_\delta(u)$ can be joined to u by the parametrized line segment $\ell \colon [0, 1] \to A$ defined by

$$\ell(t) = \left((1 - t)u_1 + tb_1, \dots, (1 - t)u_n + tb_n\right).$$

It is easy to see that since b can be joined to u and u to a, the point b can be joined to a. [Proof: If $f \colon [0, 1] \to A$ and $\ell \colon [0, 1] \to A$ are continuous functions satisfying

$f(0) = a$, $f(1) = u$, $\ell(0) = u$, and $\ell(1) = b$, then the function $g\colon [0,1] \to A$ defined by

$$g(t) = \begin{cases} f(2t), & \text{for } 0 \le t \le \frac{1}{2} \\ \ell(2t-1), & \text{for } \frac{1}{2} < t \le 1 \end{cases}$$

is continuous, $g(0) = a$, and $g(1) = b$.] This shows that $B_\delta(u) \subseteq U$.

To see that V is open let $v \in V$ and choose $\epsilon > 0$ so that $B_\epsilon(v) \subseteq A$. If some point y in $B_\epsilon(v)$ could be joined to a by an arc in A, then v could be so joined to a (*via* y). Since this is not possible, we have that $B_\epsilon(v) \subseteq V$.

Q.18. Exercises in chapter 18

Q.18.1. (Solution to 18.1.4) Suppose (x_n) is a convergent sequence in a metric space and a is its limit. Given $\epsilon > 0$ choose $n_0 \in \mathbb{N}$ so that $d(x_n, a) < \frac{1}{2}\epsilon$ whenever $n \ge n_0$. Then $d(x_m, x_n) \le d(x_m, a) + d(a, x_n) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ whenever $m, n \ge n_0$. This shows that (x_n) is Cauchy.

Q.18.2. (Solution to 18.1.5) Suppose that (x_{n_k}) is a convergent subsequence of a Cauchy sequence (x_n) and that $x_{n_k} \to a$. Given $\epsilon > 0$ choose n_0 such that $d(x_m, x_n) < \frac{1}{2}\epsilon$ whenever $m, n \ge n_0$. Next choose $k \in \mathbb{N}$ such that $n_k \ge n_0$ and $d(x_{n_k}, a) < \frac{1}{2}\epsilon$. Then for all $m \ge n_0$

$$d(x_m, a) \le d(x_m, x_{n_k}) + d(x_{n_k}, a) < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon\,.$$

Q.18.3. (Solution to 18.1.6) A sequence in a metric space M, being a function, is said to be bounded if its range is a bounded subset of M. If (x_n) is a Cauchy sequence in M, then there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < 1$ whenever $m, n \ge n_0$. For $1 \le k \le n_0 - 1$, let $d_k = d(x_k, x_{n_0})$; and let $r = \max\{d_1, \dots, d_{n_0-1}, 1\}$. Then for every $k \in \mathbb{N}$ it is clear that x_k belongs to $C_r(x_{n_0})$ (the closed ball about x_{n_0} of radius r). Thus the range of the sequence (x_n) is bounded.

Q.18.4. (Solution to 18.2.9) Let (M, d) and (N, ρ) be complete metric spaces. Let d_1 be the usual product metric on $M \times N$ (see 12.3.3). If $((x_n, y_n))_{n=1}^{\infty}$ is a Cauchy sequence in $M \times N$, then (x_n) is a Cauchy sequence in M since

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_n) + \rho(y_m, y_n) \\ &= d_1((x_m, y_m), (x_n, y_n)) \to 0 \end{aligned}$$

as $m, n \to \infty$.

Similarly, (y_n) is a Cauchy sequence in N. Since M and N are complete, there are points a and b in M and N respectively such that $x_n \to a$ and $y_n \to b$. By proposition 12.3.4, $(x_n, y_n) \to (a, b)$. Thus $M \times N$ is complete.

Q.18.5. (Solution to 18.2.10) It suffices to show that if (M, d) is complete, then (M, ρ) is. There exist $\alpha, \beta > 0$ such that $d(x, y) \le \alpha\,\rho(x, y)$ and $\rho(x, y) \le \beta\,d(x, y)$ for all $x, y \in M$. Let (x_n) be a Cauchy sequence in (M, ρ). Then since

$$d(x_m, x_n) \le \alpha\,\rho(x_m, x_n) \to 0 \qquad \text{as } m, n \to \infty\,,$$

the sequence (x_n) is Cauchy in (M, d). By hypothesis (M, d) is complete; so there is a point a in M such that $x_n \to a$ in (M, d). But then $x_n \to a$ in (M, ρ) since

$$\rho(x_n, a) \le \beta\,d(x_n, a) \to 0 \qquad \text{as } n \to \infty\,.$$

This shows that (M,ρ) is complete.

Q.18.6. (Solution to 18.2.12) Let (f_n) be a Cauchy sequence in $\mathcal{B}(S,\mathbb{R})$. Since for every $x\in S$

$$|f_m(x)-f_n(x)|\le d_u(f_m,f_n)\to 0 \qquad \text{as } m,\, n\to\infty\,,$$

it is clear that $\big(f_n(x)\big)$ is a Cauchy sequence in $\mathbb{R}$ for each $x\in S$. Since $\mathbb{R}$ is complete, there exists, for each $x\in S$, a real number $g(x)$ such that $f_n(x)\to g(x)$ as $n\to\infty$. Consider the function g defined by

$$g\colon S\to\mathbb{R}\colon x\mapsto g(x)\,.$$

We show that g is bounded and that $f_n\to g\,(\text{unif})$. Given $\epsilon>0$ choose $n_0\in\mathbb{N}$ so that $d_u(f_m,f_n)<\epsilon$ whenever $m,\, n\ge n_0$. Then for each such m and n

$$|f_m(x)-f_n(x)|<\epsilon \quad \text{whenever } x\in S\,.$$

Take the limit as $m\to\infty$ and obtain

$$|g(x)-f_n(x)|\le\epsilon$$

for every $n\ge n_0$ and $x\in S$. This shows that $g-f_n$ is bounded and that $d_u(g,f_n)\le\epsilon$. Therefore the function

$$g=(g-f_n)+f_n$$

is bounded and $d_u(g,f_n)\to 0$ as $n\to\infty$.

Q.19. Exercises in chapter 19

Q.19.1. (Solution to 19.1.2) **19.1.2** Let $f: M\to N$ be a contraction and let $a\in M$. We show f is continuous at a. Given $\epsilon>0$, choose $\delta=\epsilon$. If $d(x,a)<\delta$ then $d(f(x),f(a))\le cd(x,a)\le d(x,a)<\delta=\epsilon$, where c is a contraction constant for f.

Q.19.2. (Solution to 19.1.3) If (x,y) and (u,v) are points in $\mathbb{R}^2$, then

$$\begin{aligned}
d\big(f(x,y),f(u,v)\big) &= \tfrac{1}{3}\left[(u-x)^2+(y-v)^2+(x-y-u+v)^2\right]^{1/2}\\
&= \tfrac{1}{3}\left[2(x-u)^2+2(y-v)^2-2(x-u)(y-v)\right]^{1/2}\\
&\le \tfrac{1}{3}\left[2(x-u)^2+2(y-v)^2+2|x-u|\,|y-v|\right]^{1/2}\\
&\le \tfrac{1}{3}\left[2(x-u)^2+2(y-v)^2+(x-u)^2+(y-v)^2\right]^{1/2}\\
&= \tfrac{\sqrt{3}}{3}\left[(x-u)^2+(y-v)^2\right]^{1/2}\\
&= \tfrac{1}{\sqrt{3}}d\big((x,y),(u,v)\big)\,.
\end{aligned}$$

Q.19.3. (Solution to 19.1.5) Let M be a complete metric space and $f\colon M\to M$ be contractive. Since f is contractive, there exists $c\in(0,1)$ such that

$$d(f(x),f(y))\le c\,d(x,y) \tag{102}$$

for all $x,\, y\in M$. First we establish the existence of a fixed point. Define inductively a sequence $\big(x_n\big)_{n=0}^{\infty}$ of points in M as follows: Let x_0 be an arbitrary point in M.

Having chosen $x_0, \dots, x_n$ let $x_{n+1} = f(x_n)$. We show that (x_n) is Cauchy. Notice that for each $k \in \mathbb{N}$

$$d(x_k, x_{k+1}) \le c^k d(x_0, x_1)\,. \tag{103}$$

[Inductive proof: If $k = 1$, then $d(x_1, x_2) = d\,(f(x_0), f(x_1)) \le c\,d(x_0, x_1)$. Suppose that (103) holds for $k = n$. Then $d\big(x_{n+1}, x_{n+2}\big) = d\big(f(x_n), f(x_{n+1})\big) \le c\,d\big(x_n, x_{n+1}\big) \le c \cdot c^n\, d\big(x_0, x_1\big) = c^{n+1} d\big(x_0, x_1\big)$.) Thus whenever $m < n$

$$\begin{aligned} d\big(x_m, x_n\big) &\le \sum_{k=m}^{n-1} d\big(x_k, x_{k+1}\big) \\ &\le \sum_{k=m}^{n-1} c^k d\big(x_0, x_1\big) \\ &\le d\big(x_0, x_1\big) \sum_{k=m}^{\infty} c^k \\ &= d\big(x_0, x_1\big) \frac{c^m}{1-c}\,. \end{aligned}$$

Since $c^m \to 0$ as $m \to \infty$, we see that

$$d(x_m, x_n) \to 0 \qquad \text{as } m,\, n \to \infty\,.$$

That is, the sequence (x_n) is Cauchy. Since M is complete there exists a point p in M such that $x_n \to p$. By 19.1.2 the function f is continuous, so

$$f(p) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = p\,.$$

Finally, to show that there is at most one point fixed by f, argue by contradiction. Assume that $f(p) = p$, $f(q) = q$, and $p \ne q$. Then

$$\begin{aligned} d(p, q) &= d\big(f(p), f(q)\big) \\ &\le c\,d(p, q) \\ &< d(p, q) \end{aligned}$$

which certainly cannot be true.

Q.19.4. (Solution to 19.1.9) (a) In inequality (20) of example 19.1.6 we found that $c = 0.4$ is a contraction constant for the mapping T. Thus, according to 19.1.7,

$$\begin{aligned} d_1(x_4, p) &\le d_1(x_0, x_1) \frac{c^4}{1-c} \\ &= (0.7 + 1.1) \frac{(0.4)^4}{1 - 0.4} \\ &= 0.0768. \end{aligned}$$

(b) The d_1 distance between x_4 and p is

$$\begin{aligned} d_1(x_4, p) &= |1.0071 - 1.0000| + |0.9987 - 1.0000| \\ &= 0.0084\,. \end{aligned}$$

(c) We wish to choose n sufficiently large that $d_1(x_n, p) \leq 10^{-4}$. According to corollary 19.1.7 it suffices to find n such that

$$d_1(x_0, x_1)\frac{c^n}{1-c} \leq 10^{-4}\,.$$

This is equivalent to requiring

$$(0.4)^n \leq \frac{10^{-4}(0.6)}{1.8} = \tfrac{1}{3}10^{-4}\,.$$

For this, $n = 12$ suffices.

Q.19.5. (Solution to 19.2.1) Define T on $\mathcal{C}([0,1],\mathbb{R})$ as in the hint. The space $\mathcal{C}([0,1],\mathbb{R})$ is a complete metric space by example 18.2.13. To see that T is contractive, notice that for $f, g \in \mathcal{C}([0,1],\mathbb{R})$ and $0 \leq x \leq 1$

$$\begin{aligned}
|Tf(x) - Tg(x)| &= \left|\int_0^x t^2 f(t)\,dt - \int_0^x t^2 g(t)\,dt\right| \\
&= \left|\int_0^x t^2 (f(t) - g(t))\,dt\right| \\
&\leq \int_0^x t^2 |f(t) - g(t)|\,dt \\
&\leq d_u(f,g) \int_0^x t^2\,dt \\
&= \tfrac{1}{3}x^3\,d_u(f,g)\,.
\end{aligned}$$

Thus

$$\begin{aligned}
d_u(Tf, Tg) &= \sup\{|Tf(x) - Tg(x)| : 0 \leq x \leq 1\} \\
&\leq \tfrac{1}{3}\,d_u(f,g)\,.
\end{aligned}$$

This shows that T is contractive.

Theorem 19.1.5 tells us that the mapping T has a unique fixed point in $\mathcal{C}([0,1],\mathbb{R})$. That is, there is a unique continuous function on $[0,1]$ that satisfies (21).

To find this function we start, for convenience, with the function $g_0 = 0$ and let $g_{n+1} = Tg_n$ for all $n \geq 0$. Compute g_1, g_2, g_3, and g_4.

$$\begin{aligned}
g_1(x) &= Tg_0(x) = \tfrac{1}{3}x^3, \\
g_2(x) &= Tg_1(x) \\
&= \tfrac{1}{3}x^3 + \int_0^x t^2\left(\tfrac{1}{3}t^3\right)\,dt \\
&= \tfrac{1}{3}x^3 + \tfrac{1}{3\cdot 6}x^6\,, \\
g_3(x) &= Tg_2(x) \\
&= \tfrac{1}{3}x^3 + \int_0^x t^2\left(\tfrac{1}{3}t^3 + \tfrac{1}{3\cdot 6}x^6\right)\,dt \\
&= \tfrac{1}{3}x^3 + \tfrac{1}{3\cdot 6}x^6 + \tfrac{1}{3\cdot 6\cdot 9}x^9\,,
\end{aligned}$$

It should now be clear that for every $n \in \mathbb{N}$

$$g_n(x) = \sum_{k=1}^{n} \frac{1}{3^k\,k!} x^{3k} = \sum_{k=1}^{n} \frac{1}{k!} \left(\frac{x^3}{3}\right)^k$$

and that the uniform limit of the sequence (g_n) is the function f represented by the power series

$$\sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{x^3}{3}\right)^k .$$

Recall from elementary calculus that the power series expansion for e^y (also written $\exp(y)$) is

$$\sum_{k=0}^{\infty} \frac{1}{k!} y^k$$

for all y in $\mathbb{R}$; that is,

$$\sum_{k=1}^{\infty} \frac{1}{k!} y^k = e^y - 1 .$$

Thus

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{x^3}{3}\right)^k \\ &= \exp\left(\tfrac{1}{3}x^3\right) - 1 . \end{aligned}$$

Finally we check that this function satisfies (21) for all x in $\mathbb{R}$.

$$\begin{aligned} \tfrac{1}{3}x^3 + \int_0^x t^2 f(t)\,dt &= \tfrac{1}{3}x^3 + \int_0^x t^2 \left(\exp\left(\tfrac{1}{3}t^3\right) - 1\right)\,dt \\ &= \tfrac{1}{3}x^3 + \left(\exp\left(\tfrac{1}{3}t^3\right) - \tfrac{1}{3}t^3\right)\Big|_0^x \\ &= \exp\left(\tfrac{1}{3}x^3\right) - 1 \\ &= f(x). \end{aligned}$$

Q.20. Exercises in chapter 20

Q.20.1. (Solution to 20.1.2) Suppose that $\mathbf{0}$ and $\mathbf{0}'$ are vectors in V such that $x + \mathbf{0} = x$ and $x + \mathbf{0}' = x$ for all $x \in V$. Then

$$\mathbf{0}' = \mathbf{0}' + \mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0} .$$

Q.20.2. (Solution to 20.1.4) The proof takes one line:

$$x = x + \mathbf{0} = x + (x + (-x)) = (x + x) + (-x) = x + (-x) = \mathbf{0} .$$

Q.20.3. (Solution to 20.1.5) Establish (a), (b), and (c) of the hint.

(a) Since $\alpha\mathbf{0} + \alpha\mathbf{0} = \alpha(\mathbf{0} + \mathbf{0}) = \alpha\mathbf{0}$, we conclude from 20.1.4 that $\alpha\mathbf{0} = \mathbf{0}$.

(b) Use the same technique as in (a): since $0x + 0x = (0 + 0)x = 0x$, we deduce from 20.1.4 that $0x = \mathbf{0}$.

(c) We suppose that $\alpha \neq 0$ and that $\alpha x = \mathbf{0}$. We prove that $x = \mathbf{0}$. Since the real number α is not zero, its reciprocal a^{-1} exists. Then

$$x = 1 \cdot x = \big(\alpha^{-1}\alpha\big)x = \alpha^{-1}(\alpha x) = \alpha^{-1}\mathbf{0} = \mathbf{0}\,.$$

(The last equality uses part (a).)

Q.20.4. (Solution to 20.1.6) Notice that $(-x) + x = x + (-x) = \mathbf{0}$. According to 20.1.3, the vector x must be the (unique) additive inverse of $-x$. That is, $x = -(-x)$.

Q.20.5. (Solution to 20.1.13) The set W is closed under addition and scalar multiplication; so vector space axioms (1) and (4) through (8) hold in W because they do in V. Choose an arbitrary vector x in W. Then using (c) we see that the zero vector $\mathbf{0}$ of V belongs to W because it is just the result of multiplying x by the scalar 0 (see exercise 20.1.5). To show that (3) holds we need only verify that if $x \in W$, then its additive inverse $-x$ in V also belongs to W. But this is clear from problem 20.1.7 since the vector $-x$ is obtained by multiplying x by the scalar -1.

Q.20.6. (Solution to 20.1.19) Use proposition 20.1.13.

(a) The zero vector belongs to every member of $\mathfrak{S}$ and thus to $\bigcap\mathfrak{S}$. Therefore $\bigcap\mathfrak{S} \neq \emptyset$.

(b) Let x, $y \in \bigcap\mathfrak{S}$. Then x, $y \in S$ for every $S \in \mathfrak{S}$. Since each member of $\mathfrak{S}$ is a subspace, $x + y$ belongs to S for every $S \in \mathfrak{S}$. Thus $x + y \in \bigcap\mathfrak{S}$.

(c) Let $x \in \bigcap\mathfrak{S}$ and $\alpha \in \mathbb{R}$. Then $x \in S$ for every $S \in \mathfrak{S}$. Since each member of $\mathfrak{S}$ is closed under scalar multiplication, αx belongs to S for every $S \in \mathfrak{S}$. Thus $\alpha x \in \bigcap\mathfrak{S}$.

Q.20.7. (Solution to 20.2.2) We wish to find scalars α, β, γ, and δ such that

$$\alpha(1,0,0) + \beta(1,0,1) + \gamma(1,1,1) + \delta(1,1,0) = (0,0,0)\,.$$

This equation is equivalent to

$$(\alpha+\beta+\gamma+\delta, \gamma+\delta, \beta+\gamma) = (0,0,0)\,.$$

Thus we wish to find a (not necessarily unique) solution to the system of equations:

$$\begin{aligned} \alpha+\beta+\gamma+\delta &= 0 \\ +\gamma+\delta &= 0 \\ \beta+\gamma+\quad &= 0 \end{aligned}$$

One solution is $\alpha = \gamma = 1$, $\beta = \delta = -1$.

Q.20.8. (Solution to 20.3.2) We must find scalars α, β, $\gamma \geq 0$ such that $\alpha+\beta+\gamma = 1$ and

$$\alpha(1,0) + \beta(0,1) + \gamma(3,0) = (2,1/4)\,.$$

This last vector equation is equivalent to the system of scalar equations

$$\begin{cases} \alpha + 3\gamma = 2 \\ \qquad \beta = \frac{1}{4}\,. \end{cases}$$

From $\alpha + \frac{1}{4} + \gamma = 1$ and $\alpha + 3\gamma = 2$, we conclude that $\alpha = \frac{1}{8}$ and $\gamma = \frac{5}{8}$.

Q.20.9. (Solution to 20.3.10) In order to say that the intersection of the family of all convex sets that contain A is the "smallest convex set containing A", we must know that this intersection is indeed a convex set. This is an immediate consequence of the fact that the intersection of any family of convex sets is convex. (Proof. Let $\mathfrak{A}$ be a family of convex subsets of a vector space and let $x, y \in \bigcap \mathfrak{A}$. Then $x, y \in A$ for every $A \in \mathfrak{A}$. Since each A in $\mathfrak{A}$ is convex, the segment $[x, y]$ belongs to A for every A. Thus $[x, y] \subseteq \bigcap \mathfrak{A}$.)

Q.21. Exercises in chapter 21

Q.21.1. (Solution to 21.1.2) If x, $y \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} T(x+y) &= T(x_1+y_1, x_2+y_2, x_3+y_3) \\ &= (x_1+y_1+x_3+y_3, x_1+y_1-2x_2-2y_2) \\ &= (x_1+x_3, x_1-2x_2) + (y_1+y_3, y_1-2y_2) \\ &= Tx + Ty \end{aligned}$$

and

$$\begin{aligned} T(\alpha x) &= T(\alpha x_1, \alpha x_2, \alpha x_3) \\ &= (\alpha x_1 + \alpha x_3, \alpha x_1 - 2\alpha x_2) \\ &= \alpha(x_1 + x_3, x_1 - 2x_2) \\ &= \alpha T x. \end{aligned}$$

Q.21.2. (Solution to 21.1.4) Write $(2, 1, 5)$ as $2e^1 + e^2 + 5e^3$. Use the linearity of T to see that

$$\begin{aligned} T(2,1,5) &= T(2e^1 + e^2 + 5e^3) \\ &= 2Te^1 + Te^2 + 5Te^3 \\ &= 2(1,0,1) + (0,2,-1) + 5(-4,-1,3) \\ &= (2,0,2) + (0,2,-1) + (-20,-5,15) \\ &= (-18,-3,16). \end{aligned}$$

Q.21.3. (Solution to 21.1.6) (a) Let x be any vector in V; then (by proposition 20.1.5) $0\,x = \mathbf{0}$. Thus $T(\mathbf{0}) = T(0x) = 0\,Tx = \mathbf{0}$.

(b) By proposition 20.1.7

$$\begin{aligned} T(x-y) &= T(x+(-y)) \\ &= T(x+(-1)y) \\ &= Tx + (-1)Ty \\ &= Tx - Ty. \end{aligned}$$

Q.21.4. (Solution to 21.1.13) First we determine where T takes an arbitrary vector (x, y, z) in its domain.

$$\begin{aligned} T(x, y, z) &= T(xe^1 + ye^2 + ze^3) \\ &= xTe^1 + yTe^2 + zTe^3 \\ &= x(1, -2, 3) + y(0, 0, 0) + z(-2, 4, -6) \\ &= (x - 2z, -2x + 4z, 3x - 6z). \end{aligned}$$

A vector (x, y, z) belongs to the kernel of T if and only if $T(x, y, z) = (0, 0, 0)$; that is, if and only if $x - 2z = 0$. (Notice that the two remaining equations, $-2x + 4z = 0$ and $3x - 6z = 0$, have exactly the same solutions.) Thus the kernel of T is the set of points (x, y, z) in $\mathbb{R}^3$ such that $x = 2z$. This is a plane in $\mathbb{R}^3$ that contains the y-axis. A vector (u, v, w) belongs to the range of T if an only if there exists a vector $(x, y, z) \in \mathbb{R}^3$ such that $(u, v, w) = T(x, y, z)$. This happens if an only if

$$(u, v, w) = (x - 2z, -2x + 4z, 3x - 6z)\,;$$

that is, if an only if

$$\begin{aligned} u &= x - 2z \\ v &= -2x + 4z = -2u \\ w &= 3x - 6z = 3u. \end{aligned}$$

Consequently, only points of the form $(u, -2u, 3u) = u(1, -2, 3)$ belong to $\operatorname{ran} T$. Thus the range of T is the straight line in $\mathbb{R}^3$ through the origin that contains the point $(1, -2, 3)$.

Q.21.5. (Solution to 21.1.17) According to proposition 20.1.13 we must show that $\operatorname{ran} T$ is nonempty and that it is closed under addition and scalar multiplication. That it is nonempty is clear from proposition 21.1.6(a): $0 = T0 \in \operatorname{ran} T$. Suppose that $u, v \in \operatorname{ran} T$. Then there exist $x, y \in V$ such that $u = Tx$ and $v = Ty$. Thus

$$u + v = Tx + Ty = T(x + y)\,;$$

so $u + v$ belongs to $\operatorname{ran} T$. This shows that $\operatorname{ran} T$ is closed under addition. Finally, to show that it is closed under scalar multiplication let $u \in \operatorname{ran} T$ and $\alpha \in \mathbb{R}$. There exists $x \in V$ such that $u = Tx$; so

$$\alpha u = \alpha T x = T(\alpha x)$$

which shows that αu belongs to $\operatorname{ran} T$.

Q.21.6. (Solution to 21.2.1) Recall from example 20.1.11 that under pointwise operations of addition and scalar multiplication $\mathcal{F}(V, W)$ is a vector space. To prove that $\mathfrak{L}(V, W)$ is a vector space it suffices to show that it is a vector subspace of $\mathcal{F}(V, W)$. This may be accomplished by invoking proposition 20.1.13, according to which we need only verify that $\mathfrak{L}(V, W)$ is nonempty and is closed under addition and scalar multiplication. Since the zero transformation (the one that takes every x in V to the zero vector in W) is certainly linear, $\mathfrak{L}(V, W)$ is not empty. To prove that it is closed under addition we verify that the sum of two linear transformations is itself linear. To this end let S and T be members of $\mathfrak{L}(V, W)$. Then for all x and y in V

$$\begin{aligned}(S+T)(x+y) &= S(x+y)+T(x+y)\\ &= Sx+Sy+Tx+Ty\\ &= (S+T)x+(S+T)y.\end{aligned} \tag{104}$$

(It is important to be cognizant of the reason for each of these steps. There is no "distributive law" ate work here. The first and last use the definition of addition as a pointwise operation, while the middle one uses the linearity of S and T.) Similarly, for all x in V and α in $\mathbb{R}$

$$\begin{aligned}(S+T)(\alpha x) &= S(\alpha x)+T(\alpha x)\\ &= \alpha Sx+\alpha Tx\\ &= \alpha(Sx+Tx)\\ &= \alpha(S+T)x.\end{aligned} \tag{105}$$

Equations (104) and (105) show that $S+T$ is linear and therefore belongs to $\mathfrak{L}(V, W)$.

We must also prove that $\mathfrak{L}(V, W)$ is closed under scalar multiplication. Let $T \in \mathfrak{L}(V, W)$ and $\alpha \in \mathbb{R}$, and show that the function αT is linear. For all $x, y \in V$

$$\begin{aligned}(\alpha T)(x+y) &= \alpha(T(x+y))\\ &= \alpha(Tx+Ty)\\ &= \alpha(Tx)+\alpha(Ty)\\ &= (\alpha T)x+(\alpha T)y.\end{aligned} \tag{106}$$

Finally, for all x in V and β in $\mathbb{R}$

$$\begin{aligned}(\alpha T)(\beta x) &= \alpha(T(\beta x))\\ &= \alpha(\beta(Tx))\\ &= (\alpha\beta)Tx\\ &= (\beta\alpha)Tx\\ &= \beta(\alpha(Tx))\\ &= \beta((\alpha T)x)\,.\end{aligned} \tag{107}$$

Equations (106) and (107) show that αT belongs to $\mathfrak{L}(V, W)$.

Q.21.7. (Solution to 21.2.2) Since T is bijective there exists a function $T^{-1}\colon W \to V$ satisfying $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$. We must show that this function is linear. To this end let $u, v \in W$. Then

$$\begin{aligned}T(T^{-1}(u+v)) &= I_W(u+v)\\ &= u+v\\ &= I_W(u)+I_W(v)\\ &= T(T^{-1}(u)+T(T^{-1}(v)\\ &= T(T^{-1}u+T^{-1}v).\end{aligned}$$

Since T is injective the preceding computation implies that

$$T^{-1}(u+v) = T^{-1}u + T^{-1}v.$$

Similarly, from

$$TT^{-1}(\alpha x) = \alpha x = \alpha TT^{-1}x = T(\alpha T^{-1}x)$$

we infer that

$$T^{-1}(\alpha x) = \alpha T^{-1}x.$$

Q.21.8. (Solution to 21.3.1)

$$a+b = \begin{bmatrix} 5 & -3 & 3 & -2 \\ 2 & -2 & 1 & 4 \end{bmatrix} 3a \quad = \begin{bmatrix} 12 & 6 & 0 & -3 \\ -3 & -9 & 3 & 15 \end{bmatrix} a-2b = \begin{bmatrix} 2 & 12 & -6 & 1 \\ -7 & -5 & 1 & 7 \end{bmatrix}.$$

Q.21.9. (Solution to 21.3.3) $ab = \begin{bmatrix} 2(1)+3(2)+(-1)1 & 2(0)+3(-1)+(-1)(-2) \\ 0(1)+1(2)+4(1) & 0(0)+1(-1)+4(-2) \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 6 & -9 \end{bmatrix}.$

Q.21.10. (Solution to 21.3.8) $ax = (1,4,1)$.

Q.21.11. (Solution to 21.3.10(a)) To show that the vectors $a(x+y)$ and $ax+ay$ are equal show that $(a(x+y))_j$ (that is, the j^{th} component of $a(x+y)$) is equal to $(ax+ay)_j$ (the j^{th} component of $ax+ay$) for each j in $\mathbb{N}_m$. This is straight forward

$$\begin{aligned}
(a(x+y))_j &= \sum_{k=1}^{n} a^j_k (x+y)_k \\
&= \sum_{k=1}^{n} a^j_k (x_k + y_k) \\
&= \sum_{k=1}^{n} (a^j_k x_k + a^j_k y_k) \\
&= \sum_{k=1}^{n} a^j_k x_k + \sum_{k=1}^{n} a^j_k y_k \\
&= (ax)_j + (ay)_j \\
&= (ax+ay)_j.
\end{aligned}$$

Q.21.12. (Solution to 21.3.13)

$$xay = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = -6\,.$$

Q.21.13. (Solution to 21.3.17) Suppose a is an $n \times n$-matrix with inverses b and c. Then $ab = ba = I_n$ and $ac = ca = I_n$. Thus

$$b = bI_n = b(ac) = (ba)c = I_n c = c\,.$$

Q.21.14. (Solution to 21.3.18) Multiply a and b to obtain $ab = I_3$ and $ba = I_3$. By the uniqueness of inverses (proposition 21.3.17) b is the inverse of a.

Q.21.15. (Solution to 21.4.9) Expanding the determinant of a along the first row (fact 5) we obtain

$$\begin{aligned} \det a &= \sum_{k=1}^{3} a^1_k C^1_k \\ &= 1 \cdot (-1)^{1+1} \det \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} + 0 \cdot (-1)^{1+2} \det \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} + 2 \cdot (-1)^{1+3} \det \begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix} \\ &= 2 + 0 - 6 = -4. \end{aligned}$$

Since $\det a \neq 0$, the matrix a is invertible. Furthermore,

$$\begin{aligned} a^{-1} &= (\det a)^{-1} \begin{bmatrix} C^1_1 & C^1_2 & C^1_3 \\ C^2_1 & C^2_2 & C^2_3 \\ C^3_1 & C^3_2 & C^3_3 \end{bmatrix}^t \\ &= -\frac{1}{4} \begin{bmatrix} C^1_1 & C^2_1 & C^3_1 \\ C^1_2 & C^2_2 & C^3_2 \\ C^1_3 & C^2_3 & C^3_3 \end{bmatrix}^t \\ &= -\frac{1}{4} \begin{bmatrix} 2 & -2 & -6 \\ -1 & -1 & 1 \\ -3 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{3}{4} \end{bmatrix}. \end{aligned}$$

Q.21.16. (Solution to 21.5.4) Since $Te^1 = T(1,0) = (1,0,2,-4)$ and $Te^2 = T(0,1) = (-3,7,1,5)$ the matrix representation of T is given by

$$[T] = \begin{bmatrix} 1 & -3 \\ 0 & 7 \\ 2 & 1 \\ -4 & 5 \end{bmatrix}.$$

Q.21.17. (Solution to 21.5.5) Let $a = [T]$. Then $a^j_k = (Te^k)_j$ for each j and k. Notice that the map

$$S\colon \mathbb{R}^n \to \mathbb{R}^m \colon x \mapsto ax$$

is linear by proposition 21.3.10 (a) and (b). We wish to show that $S = T$. According to problem 21.1.19 (b) it suffices to show that $Se^k = Te^k$ for $1 \le k \le n$. But this is essentially obvious: for each $j \in \mathbb{N}_m$

$$(Se^k)_j = (ae^k)_j = \sum_{l=1}^{n} a^j_l e^k_l = a^j_k = (Te^k)_j.$$

To prove the last assertion of the proposition, suppose that $Tx = ax$ for all x in $\mathbb{R}^n$. By the first part of the proposition $[T]x = ax$ for all x in $\mathbb{R}^n$. But then proposition 21.3.11 implies that $[T] = a$.

Q.21.18. (Solution to 21.5.6) First we show that the map $T \mapsto [T]$ is surjective. Given an $m \times n$-matrix a we wish to find a linear map T such that $a = [T]$. This is easy: let $T\colon \mathbb{R}^n \to \mathbb{R}^m \colon x \mapsto ax$. By proposition 21.3.10 (a) and (b) the map T is linear. By proposition 21.5.5

$$[T]x = Tx = ax \qquad \text{for all } x \in \mathbb{R}^n.$$

Then proposition 21.3.11 tells us that $[T] = a$.

Next we show that the map $T \mapsto [T]$ is injective. If $[T] = [S]$, then by proposition 21.5.5

$$Tx = [T]x = [S]x = Sx \qquad \text{for all } x \in \mathbb{R}^n.$$

This shows that $T = S$.

Q.21.19. (Solution to 21.5.7) By proposition 21.3.11 it suffices to show that $[S + T]x = ([S] + [T])x$ for all x in $\mathbb{R}^n$. By proposition 21.5.5

$$[S+T]x = (S+T)x = Sx + Tx = [S]x + [T]x = ([S] + [T])x\,.$$

The last step uses proposition 21.3.10(c).

(b) Show that $[\alpha T]x = (\alpha[T])x$ for all x in $\mathbb{R}^n$.

$$[\alpha T]x = (\alpha T)x = \alpha(Tx) = \alpha([T]x) = (\alpha[T])x\,.$$

The last step uses proposition 21.3.10(d).

Q.22. Exercises in chapter 22

Q.22.1. (Solution to 22.1.6) Here of course we use the usual norm on $\mathbb{R}^4$.

$$\begin{aligned}\|f(a+\lambda h)\| &= \|f\big((4,2,-4)+(-\frac{1}{2})(2,4,-4)\big)\| \\ &= \|f(3,0,-2)\| \\ &= \|(-6,9,3,-3\sqrt{2})\| \\ &= 3\big[(-2)^2+3^2+1^2+(-\sqrt{2})^2\big]^{1/2} \\ &= 12\end{aligned}$$

Q.22.2. (Solution to 22.1.7) Since

$$\begin{aligned}f(a+h)-f(a)-mh &= f(1+h_1,h_2,-2+h_3)-f(1,0,-2)-\begin{bmatrix}6&0&0\\0&1&-1\end{bmatrix}\begin{bmatrix}h_1\\h_2\\h_3\end{bmatrix} \\ &= \big(3(1+2h_1+{h_1}^2),h_2+h_1h_2+2-h_3\big)-(3,2)-(6h_1,h_2-h_3) \\ &= (3{h_1}^2,h_1h_2),\end{aligned}$$

we have

$$\|f(a+h)-f(a)-mh\| = (9{h_1}^4+{h_1}^2{h_2}^2)^{1/2} = |h_1|(9{h_1}^2+{h_2}^2)^{1/2}\,.$$

Q.22.3. (Solution to 22.1.10) Using the second derivative test from beginning calculus (and checking the value of $f+g$ at the endpoints of the interval) we see that $f(x)+g(x)$ assumes a maximum value of $\sqrt{2}$ at $x=\pi/4$ and a minimum value of $-\sqrt{2}$ at $x=5\pi/4$. So

$$\|f+g\|_u = \sup\{|f(x)+g(x)|\colon 0\le x\le 2\pi\} = \sqrt{2}\,.$$

Q.22.4. (Solution to 22.1.12) Let $x\in V$. Then

(a) $\|\mathbf{0}\| = \|0\cdot x\| = |0|\,\|x\| = 0$.

(b) $\|-x\| = \|(-1)x\| = |-1|\,\|x\| = \|x\|$.

(c) $0 = \|\mathbf{0}\| = \|x+(-x)\| \le \|x\|+\|-x\| = 2\|x\|$.

Q.22.5. (Solution to 22.2.2(a)) By proposition 22.1.12(b) it is clear that $\|x\|<r$ if and only if $\|-x\|<r$. That is, $d(x,\mathbf{0})<r$ if and only $d(-x,\mathbf{0})<r$. Therefore, $x\in B_r(\mathbf{0})$ if and only if $-x\in B_r(\mathbf{0})$, which in turn holds if and only if $x=-(-x)\in -B_r(\mathbf{0})$.

Q.22.6. (Solution to 22.3.2) Suppose $\|\ \ \|_1$ and $\|\ \ \|_2$ are equivalent norms on V. Let d_1 and d_2 be the metrics induced by $\|\ \ \|_1$ and $\|\ \ \|_2$, respectively. (That is, $d_1(x,y)=\|x-y\|_1$ and $d_2(x,y)=\|x-y\|_2$ for all $x,\ y\in V$.) Then d_1 and d_2 are equivalent metrics. Thus there exist $\alpha,\ \beta>0$ such that $d_1(x,y)\le\alpha d_2(x,y)$ and $d_2(x,y)\le\beta d_1(x,y)$ for all $x,\ y\in V$. Then in particular

$$\|x\|_1 = \|x-\mathbf{0}\|_1 = d_1(x,\mathbf{0}) \le \alpha d_2(x,\mathbf{0}) = \alpha\|x-\mathbf{0}\|_2 = \alpha\|x\|_2$$

and similarly

$$\|x\|_2 = d_2(x,\mathbf{0}) \le \beta d_1(x,\mathbf{0}) = \beta\|x\|_1$$

for all x in V. Conversely, suppose there exist α, $\beta > 0$ such that $\|x\|_1 \le \alpha\|x\|_2$ and $\|x\|_2 \le \beta\|x\|_1$ for all x in V. Then for all x, $y \in V$

$$d_1(x,y) = \|x-y\|_1 \le \alpha\|x-y\|_2 = \alpha d_2(x,y)$$

and similarly

$$d_2(x,y) \le \beta d_1(x,y)\,.$$

Thus d_1 and d_2 are equivalent metrics.

Q.22.7. (Solution to 22.3.5) Let $f\colon \mathbb{R}\times V \to V\colon (\beta,x)\mapsto \beta x$. We show that f is continuous at an arbitrary point (α,a) in $\mathbb{R}\times V$. Given $\epsilon>0$ let M be any number larger than both $|\alpha|$ and $\|a\|+1$. Choose $\delta=\min\{1,\epsilon/M\}$. Notice that

$$\begin{aligned}|\beta-\alpha| + \|x-a\| &= \|(\beta-\alpha, x-a)\|_1\\ &= \|(\beta,x)-(\alpha,a)\|_1.\end{aligned}$$

Thus whenever $\|(\beta,x)-(\alpha,a)\|_1 < \delta$ we have

$$\|x\| \le \|a\| + \|x-a\| < \|a\| + \delta \le \|a\| + 1 \le M$$

so that

$$\begin{aligned}\|f(\beta,x)-f(\alpha,a)\| &= \|\beta x - \alpha a\|\\ &\le \|\beta x-\alpha x\| + \|\alpha x - \alpha a\|\\ &= |\beta-\alpha|\,\|x\| + |\alpha|\,\|x-a\|\\ &\le M(|\beta-\alpha| + \|x-a\|)\\ &< M\,\delta\\ &\le \epsilon.\end{aligned}$$

Q.22.8. (Solution to 22.3.6) If $\beta_n\to\alpha$ in $\mathbb{R}$ and $x_n\to a$ in V, then $(\beta_n,x_n)\to(\alpha,a)$ in $\mathbb{R}\times V$ by proposition 12.3.4). According to the previous proposition

$$f\colon \mathbb{R}\times V\to V\colon (\beta,x)\mapsto \beta x$$

is continuous. Thus it follows immediately from proposition 14.1.26 that

$$\beta_n x_n = f(\beta_n,x_n)\to f(\alpha,a)=\alpha a\,.$$

Q.22.9. (Solution to 22.4.2) We know from example 20.1.11 that $\mathcal{F}(S,V)$ is a vector space. We show that $\mathcal{B}(S,V)$ is a vector space by showing that it is a subspace of $\mathcal{F}(S,V)$. Since $\mathcal{B}(S,V)$ is nonempty (it contains every constant function), we need only verify that $f+g$ and αf are bounded when f, $g\in\mathcal{B}(S,V)$ and $\alpha\in\mathbb{R}$. There exist constants M, $N>0$ such that $\|f(x)\|\le M$ and $\|g(x)\|\le N$ for all x in S. But then

$$\|(f+g)(x)\| \le \|f(x)+g(x)\| \le \|f(x)\| + \|g(x)\| \le M+N$$

and

$$\|(\alpha f)(x)\| = \|\alpha f(x)\| = |\alpha|\,\|f(x)\| \le |\alpha|\,M\,.$$

Thus the functions $f+g$ and αf are bounded.

Q.23. Exercises in chapter 23

Q.23.1. (Solution to 23.1.4)

(a) $\implies$ (b): Obvious.

(b) $\implies$ (c): Suppose T is continuous at a point a in V. Given $\epsilon > 0$ choose $\delta > 0$ so that $\|Tx - Ta\| < \epsilon$ whenever $\|x - a\| < \delta$. If $\|h - \mathbf{0}\| = \|h\| < \delta$ then $\|(a + h) - a\| < \delta$ and $\|Th - T\mathbf{0}\| = \|Th\| = \|T(a + h) - Ta\| < \epsilon$. Thus T is continuous at $\mathbf{0}$.

(c) $\implies$ (d): Argue by contradiction. Assume that T is continuous at $\mathbf{0}$ but is not bounded. Then for each $n \in \mathbb{N}$ there is a vector x_n in V such that $\|Tx_n\| > n\|x_n\|$. Let $y_n = \big(n\|x_n\|\big)^{-1} x_n$. Then $\|y_n\| = n^{-1}$; so $y_n \to \mathbf{0}$. Since T is continuous at $\mathbf{0}$ we conclude that $Ty_n \to \mathbf{0}$ as $n \to \infty$. On the other hand

$$\|Ty_n\| = \big(n\|x_n\|\big)^{-1}\|Tx_n\| > 1$$

for every n. This shows that $Ty_n \not\to \mathbf{0}$ as $n \to \infty$, which contradicts the preceding assertion.

(d) $\implies$ (a): If T is bounded, there exists $M > 0$ such that $\|Tx\| \le M\|x\|$ for all x in V. It is easy to see that T is continuous at an arbitrary point a in V. Given $\epsilon > 0$ choose $\delta = \epsilon/M$. If $\|x - a\| < \delta$, then

$$\|Tx - Ta\| = \|T(x - a)\| \le M\|x - a\| < M\delta = \epsilon\,.$$

Q.23.2. (Solution to 23.1.6) The first equality is an easy computation.

$$\begin{aligned}\sup\{\|x\|^{-1}\,\|Tx\|\colon x \ne \mathbf{0}\} &= \inf\{M > 0\colon M \ge \|x\|^{-1}\|Tx\| \text{for all } x \ne \mathbf{0}\}\\ &= \inf\{M > 0\colon \|Tx\| \le M\|x\| \text{for all } x\}\\ &= \|T\|\end{aligned}$$

The second is even easier.

$$\begin{aligned}\sup\{\|x\|^{-1}\,\|Tx\|\colon x \ne \mathbf{0}\} &= \sup\{\|T\big(\|x\|^{-1}x\big)\|\}\\ &= \sup\{\|Tu\|\colon \|u\| = 1\}.\end{aligned}$$

To obtain the last equality notice that since

$$\{\|Tu\|\colon \|u\| = 1\} \subseteq \{\|Tx\|\colon \|x\| \le 1\}$$

it is obvious that

$$\sup\{\|Tu\|\colon \|u\| = 1\} \le \sup\{\|Tx\|\colon \|x\| \le 1\}\,.$$

On the other hand, if $\|x\| \le 1$ and $x \ne 0$, then $v := \|x\|^{-1}x$ is a unit vector, and so

$$\begin{aligned}\|Tx\| &\le \|x\|^{-1}\|Tx\|\\ &= \|Tv\|\\ &\le \sup\{\|Tu\|\colon \|u\| = 1\}.\end{aligned}$$

Therefore

$$\sup\{\|Tx\|\colon \|x\| \le 1\} \le \sup\{\|Tu\|\colon \|u\| = 1\}\,.$$

Q.23.3. (Solution to 23.1.11)

(a) Let $I\colon V \to V : x \mapsto x$. Then (by lemma 23.1.6)

$$\|I\| = \sup\{\|Ix\| \colon \|x\| = 1\} = \sup\{\|x\| \colon \|x\| = 1\} = 1\,.$$

(b) Let $\widehat{\mathbf{0}}\colon V \to W : x \mapsto \mathbf{0}$. Then $\|\widehat{\mathbf{0}}\| = \sup\{\|\widehat{\mathbf{0}}x\| \colon \|x\| = 1\} = \sup\{0\} = 0$.

(c) We suppose $k = 1$. (The case $k = 2$ is similar.) Let x be a nonzero vector in V_1 and $u = \|x\|^{-1}x$. Since $\|(u,\mathbf{0})\|_1 = \|u\| + \|\mathbf{0}\| = \|u\| = 1$, we see (from lemma 23.1.6) that

$$\begin{aligned}\|\pi_1\| &= \sup\{\|\pi_1(x_1,x_2)\| \colon \|(x_1,x_2)\| = 1\}\\ &\ge \|\pi_1(u,\mathbf{0})\|\\ &= \|u\|\\ &= 1.\end{aligned}$$

On the other hand since $\|\pi_1(x_1,x_2)\| = \|x_1\| \le \|(x_1,x_2)\|_1$ for all (x_1,x_2) in $V_1 \times V_2$, it follows from the definition of the norm of a transformation that $\|\pi_1\| \le 1$.

Q.23.4. (Solution to 23.1.12) Let f, $g \in \mathcal{C}$ and $\alpha \in \mathbb{R}$. Then

$$J(f+g) = \int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = Jf + Jg$$

and

$$J(\alpha f) = \int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx = \alpha Jf.$$

Thus J is linear. If $f \in \mathcal{C}$, then

$$|Jf| = |\int_a^b f(x)\,dx| \le \int_a^b |f(x)|\,dx \le \int_a^b \|f\|_u\,dx = (b-a)\|f\|_u\,.$$

This shows that J is bounded and that $\|J\| \le b-a$. Let $g(x) = 1$ for all x in $[a,b]$. Then g is a unit vector in $\mathcal{C}$ (that is, $\|g\|_u = 1$) and $Jg = \int_a^b g(x)\,dx = b-a$. From lemma 23.1.6 we conclude that $\|J\| \ge b-a$. This and the preceding inequality prove that $\|J\| = b-a$.

Q.23.5. (Solution to 23.1.14) It was shown in proposition 21.2.1 that $\mathfrak{L}(V,W)$ is a vector space. Since $\mathfrak{B}(V,W)$ is a nonempty subset of $\mathfrak{L}(V,W)$ [it contains the zero transformation], we need only show that sums and scalar multiples of bounded linear maps are bounded in order to establish that $\mathfrak{B}(V,W)$ is a vector space. This is done below in the process of showing that the map $T \mapsto \|T\|$ is a norm.

Let S, $T \in \mathfrak{B}(V,W)$ and $\alpha \in \mathbb{R}$. To show that $\|S+T\| \le \|S\| + \|T\|$ and $\|\alpha T\| = |\alpha|\|T\|$ we make use of the characterization $\|T\| = \sup\{\|Tu\| \colon \|u\| = 1\}$ given in lemma 23.1.6. If v is a unit vector in V, then

$$\begin{aligned}\|(S+T)v\| &= \|Sv + Tv\|\\ &\le \|Sv\| + \|Tv\|\\ &\le \sup\{\|Su\| \colon \|u\| = 1\} + \sup\{\|Tv\| \colon \|v\| = 1\}\\ &= \|S\| + \|T\|.\end{aligned}$$

This shows that $S+T$ is bounded and that

$$\begin{aligned}\|S+T\| &= \sup\{\|(S+T)v\|\colon \|v\|=1\}\\ &\le \|S\|+\|T\|.\end{aligned}$$

Also

$$\begin{aligned}\sup\{\|\alpha Tv\|\colon \|v\|=1\} &= |\alpha|\sup\{\|Tv\|\colon \|v\|=1\}\\ &= |\alpha|\|T\|,\end{aligned}$$

which shows that αT is bounded and that $\|\alpha T\| = |\alpha|\|T\|$.

Finally, if $\sup\{\|x\|^{-1}\|Tx\|\colon x\neq \mathbf{0}\} = \|T\| = 0$, then $\|x\|^{-1}\|Tx\| = 0$ for all x in V, so that $Tx=\mathbf{0}$ for all x and therefore $T=\mathbf{0}$.

Q.23.6. (Solution to 23.1.15) The composite of linear maps is linear by proposition 21.1.11. From corollary 23.1.7 we have

$$\|TSx\| \le \|T\|\,\|Sx\| \le \|T\|\,\|S\|\,\|x\|$$

for all x in U. Thus TS is bounded and $\|TS\| \le \|T\|\,\|S\|$.

Q.23.7. (Solution to 23.2.1) First deal with the case $\|f\|_u \le 1$. Let (p_n) be a sequence of polynomials on $[0,1]$ that converges uniformly to the square root function (see 15.3.5). Given $\epsilon>0$, choose $n_0\in\mathbb{N}$ so that $n\ge n_0$ implies $|p_n((t)-\sqrt{t}|\le\epsilon$ for all $t\in[0,1]$. Since $\|f\|_u\le 1$

$$\left|p_n\big([f(x)]^2\big)-|f(x)|\right| = \left|p_n\big([f(x)]^2\big)-\sqrt{[f(x)]^2}\right| < \epsilon$$

whenever $x\in M$ and $n\ge n_0$. Thus $p_n\circ f^2 \to |f|\,(\text{unif})$. For every $n\in\mathbb{N}$ the function $p_n\circ f^2$ belongs to A. Consequently, $|f|$ is the uniform limit of functions in A and therefore belongs to $\overline{A}$.

If $\|f\|_u>1$ replace f in the argument above by $g=f/\|f\|_u$.

Q.23.8. (Solution to 23.2.5) Let $f\in\mathcal{C}(M,\mathbb{R})$, $a\in M$, and $\epsilon>0$. According to proposition 23.2.4 we can choose, for each $y\neq a$ in M, a function $\phi_y\in A$ such that

$$\phi_y(a)=f(a) \qquad\text{and}\qquad \phi_y(y)=f(y)\,.$$

And for $y=a$ let ϕ_y be the constant function whose value is $f(a)$. Since in either case ϕ_y and f are continuous functions that agree at y, there exists a neighborhood U_y of y such that

$$\phi_y(x) < f(x)+\epsilon$$

for all $x\in U_y$. Clearly $\{U_y\colon y\in M\}$ covers M. Since M is compact there exist points $y_1,\dots,y_n$ in M such that the family $\{U_{y_1},\dots,U_{y_n}\}$ covers M. Let $g=\phi_{y_1}\wedge\dots\wedge\phi_{y_n}$. By corollary 23.2.2 the function g belongs to $\overline{A}$. Now $g(a)=\phi_{y_1}(a)\wedge\dots\wedge\phi_{y_n}(a)=f(a)$. Furthermore, given any x in M there is an index k such that $x\in U_{y_k}$. Thus

$$g(x)\le\phi_{y_k}(x)<f(x)+\epsilon\,.$$

Q.23.9. (Solution to 23.3.6) Let (T_n) be a Cauchy sequence in the normed linear space $\mathfrak{B}(V,W)$. For each x in V

$$\|T_mx-T_nx\|\le\|T_m-T_n\|\,\|x\|\to 0$$

as $m,\ n \to \infty$. Thus $(T_n x)$ is a Cauchy sequence in W for each x. Since W is complete, there exists a vector Sx in W such that $T_n x \to Sx$. Define the map

$$S\colon V \to W\colon x \mapsto Sx\,.$$

It is easy to see that S is linear: $S(x+y) = \lim T_n(x+y) = \lim(T_n x + T_n y) = \lim T_n x + \lim T_n y = Sx + Sy$; $S(\alpha x) = \lim T_n(\alpha x) = \alpha \lim T_n x = \alpha Sx$. For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|T_m - T_n\| < \frac{1}{2}\epsilon$ whenever $m,\ n \geq N$. Then for all such m and n and for all x in V

$$\begin{aligned}
\|(S - T_n)x\| &= \|Sx - T_n x\| \\
&\leq \|Sx - T_m x\| + \|T_m x - T_n x\| \\
&\leq \|Sx - T_m x\| + \|T_m - T_n\|\|x\| \\
&\leq \|Sx - T_m x\| + \tfrac{1}{2}\epsilon\|x\|.
\end{aligned}$$

Taking limits as $m \to \infty$ we obtain

$$\|(S - T_n)x\| \leq \tfrac{1}{2}\epsilon\|x\|$$

for all $n \geq N$ and $x \in V$. This shows that $S - T_n$ is bounded and that $\|S - T_n\| \leq \frac{1}{2}\epsilon < \epsilon$ for $n \geq N$. Therefore the transformation

$$S = (S - T_n) + T_n$$

is bounded and

$$\|S - Tn\| \to 0$$

as $n \to \infty$. Since the Cauchy sequence (T_n) converges in the space $\mathfrak{B}(V, W)$, that space is complete.

Q.23.10. (Solution to 23.4.5) We wish to show that if $g \in W^*$, then $T^*g \in V^*$. First we check linearity: if $x,\ y \in V$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}
(T^*g)(x+y) &= gT(x+y) \\
&= g(Tx + Ty) \\
&= gTx + gTy \\
&= (T^*g)(x) + (T^*g)(y)
\end{aligned}$$

and

$$(T^*g)(\alpha x) = gT(\alpha x) = g(\alpha Tx) = \alpha gTx = \alpha(T^*g)(x)\,.$$

To see that T^*g is bounded use corollary 23.1.7. For every x in V

$$|(T^*g)(x)| = |gTx| \leq \|g\|\,\|Tx\| \leq \|g\|\,\|T\|\,\|x\|\,.$$

Thus T^*g is bounded and $\|T^*g\| \leq \|T\|\,\|g\|$.

Q.24. Exercises in chapter 24

Q.24.1. (Solution to 24.1.4) Given $\epsilon > 0$ choose $\delta = \epsilon$. Assume $|x - y| < \delta$. Then

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = \left|\frac{x - y}{xy}\right| \leq |x - y| < \delta = \epsilon\,.$$

Q.24.2. (Solution to 24.1.5) We must show that

$$(\exists\epsilon > 0)(\forall\delta > 0)(\exists x, y \in (0,1])|x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon\,.$$

Let $\epsilon = 1$. Suppose $\delta > 0$. Let $\delta_0 = \min\{1, \delta\}$, $x = \frac{1}{2}\delta_0$, and $y = \delta_0$. Then x, $y \in (0, 1]$, $|x - y| = \frac{1}{2}\delta_0 \leq \frac{1}{2}\delta < \delta$, and $|f(x) - f(y)| = |\frac{2}{\delta_0} - \frac{1}{\delta_0}| = \frac{1}{\delta\,0} \geq 1$.

Q.24.3. (Solution to 24.1.10) Since M is compact it is sequentially compact (by theorem 16.2.1). Thus the sequence (x_n) has a convergent subsequence (x_{n_k}). Let a be the limit of this subsequence. Since for each k

$$d\big(y_{n_k}, a\big) \leq d\big(y_{n_k}, x_{n_k}\big) + d\big(x_{n_k}, a\big)$$

and since both sequences on the right converge to zero, it follows that $y_{n_k} \to a$ as $k \to \infty$.

Q.24.4. (Solution to 24.1.11) Assume that f is not uniformly continuous. Then there is a number $\epsilon > 0$ such that for every n in $\mathbb{N}$ there correspond points x_n and y_n in M_1 such that $d(x_n, y_n) < 1/n$ but $d\big(f(x_n), f(y_n)\big) \geq \epsilon$. By lemma 24.1.10 there exist subsequences $\big(x_{n_k}\big)$ of (x_n) and $\big(y_{n_k}\big)$ of (y_n) both of which converge to some point a in M_1. It follows from the continuity of f that for some integer k sufficiently large, $d\big(f(x_{n_k}), f(a)\big) < \epsilon/2$ and $d\big(f(y_{n_k}), f(a)\big) < \epsilon/2$. This contradicts the assertion that $d\big(f(x_n), f(y_n)\big) \geq \epsilon$ for every n in $\mathbb{N}$.

Q.24.5. (Solution to 24.1.14) By hypothesis there exists a point a in M_1 such that $x_n \to a$ and $y_n \to a$. It is easy to see that the "interlaced" sequence $(z_n) = (x_1, y_1, x_2, y_2, x_3, y_3, \dots)$ also converges to a. By proposition 18.1.4 the sequence (z_n) is Cauchy (in M_1 and therefore) in S, and by proposition 24.1.12 (applied to the metric space S) the sequence $\big(f(z_n)\big)$ is Cauchy in M_2. The sequence $\big(f(x_n)\big)$ is a subsequence of $\big(f(z_n)\big)$ and is, by hypothesis, convergent. Therefore, according to proposition 18.1.5, the sequence $\big(f(z_n)\big)$ converges and

$$\lim f(x_n) = \lim f(z_n) = \lim f(y_n)\,.$$

Q.24.6. (Solution to 24.2.2) Just take the union of the sets of points in P and Q and put them in increasing order. Thus

$$P \vee Q = \big(0, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{5}{6}, 1\big)\,.$$

Q.24.7. (Solution to 24.2.5) (a) Either sketch the graph of σ or reduce the function algebraically to obtain

$$\sigma = -\chi_{\{2\}} - 2\chi_{(2,3)} - \chi_{\{5\}}\,.$$

Then the partition associated with σ is $P = (0, 2, 3, 5)$.

(b) $\sigma_Q = (0, 0, -2, 0, 0)$.

Q.24.8. (Solution to 24.2.7) The values of σ on the subintervals of P are given by $\sigma_P = (0, -2, 0)$. Multiply each of these by the length of the corresponding subinterval:

$$\int_0^5 \sigma = (2 - 0)(0) + (3 - 2)(-2) + (5 - 3)(0) = -2\,.$$

Q.24.9. (Solution to 24.2.9) Perhaps the simplest way to go about this is to observe first that we can get from the partition associated with σ to the refinement Q one point at a time. That is, there exist partitions

$$P_1 \preceq P_2 \preceq \cdots \preceq P_r = Q$$

where P_1 is the partition associated with σ and P_{j+1} contains exactly one point more than P_j (for $1 \le j \le r-1$).

Thus it suffices to prove the following: If σ is a step function on $[a,b]$, if $P = (s_0, \dots, s_n)$ is a refinement of the partition associated with σ, and if $P \preceq Q = (t_0, \dots, t_{n+1})$, then

$$\sum_{k=1}^{n+1} (\Delta t_k) y_k = \sum_{k=1}^{n} (\Delta s_k) x_k$$

where $\sigma_P = (x_1, \dots, x_n)$ and $\sigma_Q = (y_1, \dots, y_{n+1})$.

To prove this assertion, notice that since the partition Q contains exactly one point more than P, it must be of the form

$$Q = (s_0, \dots, s_{p-1}, u, s_p, \dots, s_n)$$

for some p such that $1 \le p \le n$. Thus

$$y_k = \begin{cases} x_k, & \text{for } 1 \le k \le p \\ x_{k-1}, & \text{for } p+1 \le k \le n+1. \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{n+1} (\Delta t_k) y_k &= \sum_{k=1}^{p-1} (\Delta t_k) y_k + (\Delta t_p) y_p + (\Delta t_{p+1}) y_{p+1} + \sum_{k=p+2}^{n+1} (\Delta t_k) y_k \\ &= \sum_{k=1}^{p-1} (\Delta s_k) x_k + (u - s_{p-1}) x_p + (s_p - u) x_p + \sum_{k=p+2}^{n+1} (\Delta s_{k-1}) x_{k-1} \\ &= \sum_{k=1}^{p-1} (\Delta s_k) x_k + (s_p - s_{p-1}) x_p + \sum_{k=p+1}^{n} (\Delta s_k) x_k \\ &= \sum_{k=1}^{n} (\Delta s_k) x_k \end{aligned}$$

Q.24.10. (Solution to 24.2.13) That σ is an E valued step function on $[a,b]$ is obvious. Let $Q = (u_0, \dots, u_m)$ and $R = (v_0, \dots, v_n)$ be the partitions associated with τ and ρ, respectively; and suppose that $\tau_Q = (y_1, \dots, y_m)$ and $\rho_R = (z_1, \dots, z_n)$. For $1 \le k \le m+n$, let

$$t_k = \begin{cases} u_k, & \text{for } 0 \le k \le m \\ v_{k-m}, & \text{for } m+1 \le k \le m+n \end{cases}$$

and $P = (t_0, \dots, t_{m+n})$. Also define

$$x_k = \begin{cases} y_k, & \text{for } 1 \le k \le m \\ z_{k-m}, & \text{for } m+1 \le k \le m+n. \end{cases}$$

Then P is a partition of $[a,b]$ and $\sigma_P = (x_1, \dots, x_{m+n})$. Furthermore,

$$\begin{aligned}\int_a^b \sigma &= \sum_{k=1}^{m} +n(\Delta t_k)x_k \\ &= \sum_{k=1}^{m}(\Delta t_k)x_k + \sum_{k=m+1}^{m+n}(\Delta t_k)x_k \\ &= \sum_{k=1}^{m}(\Delta u_k)y_k + \sum_{k=m+1}^{m+n}(\Delta v_{k-m})z_{k-m} \\ &= \sum_{k=1}^{m}(\Delta u_k)y_k + \sum_{k=1}^{n}(\Delta v_k)z_k \\ &= \int_a^c \tau + \int_c^b \rho.\end{aligned}$$

(The third equality requires the observation that $\Delta t_{m+1} = v_1 - u_m = v_1 - c = v_1 - v_0$.)

Q.24.11. (Solution to 24.3.2) Let $f\colon [a,b] \to E$ be continuous. Given $\epsilon > 0$ we find a step function σ such that $\|f - \sigma\|_u < \epsilon$. Since the domain of f is compact, proposition 24.1.11 guarantees that f is uniformly continuous. Thus there exists $\delta > 0$ such that $\|f(u) - f(v)\| < \epsilon/2$ whenever u and v are points in $[a,b]$ such that $|u - v| < \delta$. Choose a partition $(t_0, \dots, t_n)$ of $[a,b]$ so that $t_k - t_{k-1} < \delta$ for each $k = 1, \dots, n$.

Define $\sigma\colon [a,b] \to E$ by $\sigma(s) = f(t_{k-1})$ if $t_{k-1} \le s \le t_k$ $(1 \le k \le n)$ and define $\sigma(b) = f(b)$. It is easy to see that $\|f(s) - \sigma(s)\| < \epsilon/2$ for every s in $[a,b]$. Thus $\|f - \sigma\|_u < \epsilon$; so f belongs to the closure of the family of step functions.

Q.24.12. (Solution to 24.3.6) Let $\overline{\mathcal{S}}$ be the closure of $\mathcal{S}([a,b],E)$ in the space $\mathcal{B}([a,b],E)$. If $g,h \in \overline{\mathcal{S}}$, then there exist sequences (σ_n) and (τ_n) of step functions that converge uniformly to g and h, respectively. Then $(\sigma_n + \tau_n)$ is a sequence of step functions and $\sigma_n + \tau_n \to g + h$ (unif); so $g + h \in \overline{\mathcal{S}}$. Thus

$$\begin{aligned}\int (g+h) &= \lim \int (\sigma_n + \tau_n) \\ &= \lim \left(\int \sigma_n + \int \tau_n \right) \\ &= \lim \int \sigma_n + \lim \int \tau_n \\ &= \int g + \int h.\end{aligned}$$

Similarly, if $\alpha \in \mathbb{R}$, then $(\alpha\sigma_n)$ is a sequence of step functions that converges to αg. Thus $\alpha g \in \overline{\mathcal{S}}$ and

$$\int (\alpha g) = \lim \int (\alpha\sigma_n) = \lim \left(\alpha \int \sigma_n \right) = \alpha \lim \int \sigma_n = \alpha \int g\,.$$

The map $\int\colon \overline{\mathcal{S}} \to E$ is bounded since it is both linear and uniformly continuous (see 24.1.15 and 24.1.9).

Q.24.13. (Solution to 24.3.18) The function f, being regulated, is the uniform limit in $\mathcal{B}([a,b],E)$ of a sequence (σ_n) of step functions. Since $\int \sigma_n \to \int f$ in E and T is continuous, we see that

$$T\left(\int f\right) = \lim T\left(\int \sigma_n\right). \tag{108}$$

By problem 24.2.14 each $T \circ \sigma_n$ is an F valued step function and

$$\int (T \circ \sigma_n) = T\left(\int \sigma_n\right) \qquad \text{for each } n. \tag{109}$$

For every t in $[a,b]$

$$\begin{aligned} &\|(T\circ f - T\circ \sigma_n)(t)\| \\ &\qquad = \|T\big((f-\sigma_n)(t)\big)\| \\ &\qquad \le \|T\|\|(f-\sigma_n)(t)\| \\ &\qquad \le \|T\|\|f-\sigma_n\|_u \end{aligned}$$

so

$$\|T\circ f - T\circ\sigma_n\|_u \le \|T\|\|f-\sigma_n\|_u\,.$$

Since $\|f-\sigma_n\|_u \to 0$, we conclude that

$$T\circ\sigma_n \to T\circ f (\text{unif})$$

in $\mathcal{B}([a,b],F)$. Thus $T \circ f$ is regulated and

$$\int (T\circ f) = \lim \int (T\circ \sigma_n). \tag{110}$$

The desired conclusion follows immediately from (108), (109), and (110).

Q.25. Exercises in chapter 25

Q.25.1. (Solution to 25.1.5) Suppose that $T \in \mathfrak{B} \cap \mathfrak{o}$. Then given $\epsilon > 0$, we may choose $\delta > 0$ so that $\|Ty\| \le \epsilon\|y\|$ whenever $\|y\| < \delta$. Let x be an arbitrary unit vector. Choose $0 < t < \delta$. Then $\|tx\| = t < \delta$; so $\|Tx\| = \|T(\frac{1}{t}tx)\| = \frac{1}{t}\|T(tx)\| \le \frac{1}{t}\epsilon\|tx\| = \epsilon$. Since this last inequality holds for every unit vector x, $\|T\| \le \epsilon$. And since ϵ was arbitrary, $\|T\| = 0$. That is, $T = \mathbf{0}$.

Q.25.2. (Solution to 25.1.6) If f, $g \in \mathfrak{O}$, then there exist positive numbers M, N, δ, and η such that $\|f(x)\| \le M\|x\|$ whenever $\|x\| < \delta$ and $\|g(x)\| \le N\|x\|$ whenever $\|x\| < \eta$. Then $\|f(x)+g(x)\| \le (M+N)\|x\|$ whenever $\|x\| < \min\{\delta,\eta\}$. So $f+g \in \mathfrak{O}$.

If $\alpha \in \mathbb{R}$, then $\|\alpha f(x)\| \le |\alpha| M\|x\|$ whenever $\|x\| < \delta$; so $\alpha f \in \mathfrak{O}$.

Q.25.3. (Solution to 25.1.9) The domain of $f \circ g$ is taken to be the set of all x in V such that $g(x)$ belongs to the domain of f; that is, $\operatorname{dom}(f\circ g) = g^{\leftarrow}(\operatorname{dom} f)$. Since $f \in \mathfrak{O}$ there exist M, $\delta > 0$ such that $\|f(y)\| \le M\|y\|$ whenever $\|y\| < \delta$. Given $\epsilon > 0$, choose $\eta > 0$ so that $\|g(x)\| \le \frac{\epsilon}{M}\|x\|$ whenever $\|x\| < \eta$. If $\|x\| < \min\{\eta, \frac{M}{\epsilon}\delta\}$, then $\|g(x)\| \le \frac{\epsilon}{M}\|x\| < \delta$, so that $\|(f\circ g)(x)\|\| \le M\|g(x)\| \le \epsilon\|x\|$.

Q.25.4. (Solution to 25.1.11) Suppose $w \neq \mathbf{0}$. If $\epsilon > 0$, then there exists $\delta > 0$ such that $|\phi(x)| \leq \frac{\epsilon}{\|w\|}\|x\|$ whenever $\|x\| < \delta$. Thus $\|(\phi w)(x)\| = |\phi(x)|\|w\| \leq \epsilon\|x\|$ when $\|x\| < \delta$.

Q.25.5. (Solution to 25.1.12) There exist positive numbers M, N, δ, and η such that $\|\phi(x)\| \leq M\|x\|$ whenever $\|x\| < \delta$ and $\|f(x)\| \leq N\|x\|$ whenever $\|x\| < \eta$. Suppose that $\epsilon > 0$. If $x \in V$ and $\|x\| < \min\{\epsilon(MN)^{-1}, \delta, \eta\}$, then

$$\|(\phi f)(x)\| = |\phi(x)|\|f(x)\| \leq MN\|x\|^2 \leq \epsilon\|x\|\,.$$

Q.25.6. (Solution to 25.2.2) Reflexivity is obvious. Symmetry: If $f \simeq g$, then $f - g \in \mathfrak{o}$; so $g - f = (-1)(f - g) \in \mathfrak{o}$ by proposition 25.1.7. This proves $g \simeq f$. Transitivity: If $f \simeq g$ and $g \simeq h$, then both $f - g$ and $g - h$ belong to $\mathfrak{o}$; thus $f \simeq h$, since $f - h = (f - g) + (g - h) \in \mathfrak{o} + \mathfrak{o} \subseteq \mathfrak{o}$ (again by 25.1.7).

Q.25.7. (Solution to 25.2.3) By the preceding proposition $S \simeq T$. Thus $S - T \in \mathfrak{B} \cap \mathfrak{o} = \{\mathbf{0}\}$ by proposition 25.1.5.

Q.25.8. (Solution to 25.2.5) This requires only a simple computation: $\phi w - \psi w = (\phi - \psi)w \in \mathfrak{o}(V, \mathbb{R}) \cdot W \subseteq \mathfrak{o}(V, W)$ by proposition 25.1.11.

Q.25.9. (Solution to 25.3.10) The map $(x, y) \mapsto 7x - 9y$ is clearly continuous and linear. So all that needs to be verified is condition (iii) of remark 25.3:

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} \frac{\Delta f_{(1,-1)}(x, y) - (7x - 9y)}{\sqrt{x^2 + y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{3(x + 1)^2 - (x + 1)(y - 1) + 4(y - 1)^2 - 8 - 7x + 9y}{\sqrt{x^2 + y^2}} \\
&= \lim_{(x,y)\to(0,0)} \frac{3x^2 - xy + 4y^2}{\sqrt{x^2 + y^2}} = 0. \qquad \text{(See problem 14.3.15(a).)}
\end{aligned}$$

(The equation $z = 7x - 9y$ represents a plane through the origin.)

Q.25.10. (Solution to 25.3.11) Since

$$\begin{aligned}
\Delta f_{(1,-1,0)}(h, j, k) &= f(h + 1, j - 1, k) - f(1, -1, 0) \\
&= \big((h + 1)^2(j - 1) - 7, 3(h + 1)k + 4(j - 1)\big) - (-8, -4) \\
&= \big(h^2(j - 1) + 2hj - 2h + j, 3hk + 4j + 3k\big)
\end{aligned}$$

and

$$\begin{aligned}
M(h, j, k) &= \begin{bmatrix} r & s & t \\ u & v & w \end{bmatrix} (h, j, k) \\
&= (rh + sj + tk, uh + vj + wk),
\end{aligned}$$

we find that the first coordinate of the Newton quotient

$$\frac{\Delta f_{(1,-1,0)}(h, j, k) - M(h, j, k)}{\|(h, j, k)\|}$$

turns out to be

$$\frac{h^2(j - 1) + 2hj - (2 + r)h + (1 - s)j - tk}{\sqrt{h^2 + j^2 + k^2}}\,.$$

If we choose $r = -2$, $s = 1$, and $t = 0$, then the preceding expression approaches zero as $(h, j, k) \to (0, 0, 0)$. (See problem 14.3.15(a).) Similarly, the second coordinate of the Newton quotient is

$$\frac{3hk - uh + (4 - v)j + (3 - w)k}{\sqrt{h^2 + j^2 + k^2}},$$

which approaches zero as $(h, j, k) \to (0, 0, 0)$ if we choose $u = 0$, $v = 4$, and $w = 3$. We conclude from the uniqueness of differentials (proposition 25.3.9) that

$$\left[df_{(1,-1,0,)}\right] = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 4 & 3 \end{bmatrix}.$$

Equivalently we may write

$$df_{(1,-1,0)}(h, j, k) = (-2h + j, 4j + 3k).$$

Q.25.11. (Solution to 25.3.16) It is easy to check that $\phi(a)df_a + d\phi_a \cdot f(a)$ is bounded and linear. From our hypotheses $\Delta f_a \simeq df_a$ and $\Delta\phi_a \simeq d\phi_a$ we infer (using propositions 25.2.4 and 25.2.5) that $\phi(a)\Delta f_a \simeq \phi(a)df_a$ and that $\Delta\phi_a \cdot f(a) \simeq d\phi_a \cdot f(a)$. Then from corollary 25.3.13 and proposition 25.1.12 we conclude that $\Delta\phi_a \, \Delta f_a$ belongs to $\mathfrak{O}(V, \mathbb{R}) \cdot \mathfrak{O}(V, W)$ and therefore to $\mathfrak{o}(V, W)$. That is, $\Delta\phi_a \, \Delta f_a \simeq 0$. Thus by propositions 25.3.4 and 25.2.4

$$\begin{aligned}\Delta(\phi f)_a &= \phi(a) \cdot \Delta f_a + \Delta\phi_a \cdot f(a) + \Delta\phi_a \cdot \Delta f_a \\ &\simeq \phi(a)df_a + d\phi_a \cdot f(a) + 0 \\ &= \phi(a)df_a + d\phi_a \cdot f(a).\end{aligned}$$

Q.25.12. (Solution to 25.3.17) Our hypotheses are $\Delta f_a \simeq df_a$ and $\Delta g_{f(a)} \simeq dg_{f(a)}$. By proposition 25.3.12 $\Delta f_a \in \mathfrak{O}$. Then by proposition 25.2.7

$$\Delta g_{f(a)} \circ \Delta f_a \simeq dg_{f(a)} \circ \Delta f_a \tag{111}$$

and by proposition 25.2.6

$$dg_{f(a)} \circ \Delta f_a \simeq dg_{f(a)} \circ df_a . \tag{112}$$

According to proposition 25.3.5

$$\Delta(g \circ f)_a \simeq \Delta g_{f(a)} \circ \Delta f_a . \tag{113}$$

From (111), (112),(113), and proposition 25.2.2 it is clear that

$$\Delta(g \circ f)_a \simeq dg_{f(a)} \circ df_a .$$

Since $dg_{f(a)} \circ df_a$ is a bounded linear transformation, the desired conclusion is an immediate consequence of proposition 25.3.9.

Q.25.13. (Solution to 25.4.6)

(a) $Dc(\pi/3) = (-\sin(\pi/3), \cos(\pi/3)) = (-\sqrt{3}/2, 1/2)$.
(b) $l(t) = (1/2, \sqrt{3}/2) + t(-\sqrt{3}/2, 1/2) = \frac{1}{2}(1 - \sqrt{3}t, \sqrt{3} + t)$.
(c) $x + \sqrt{3}y = 2$.

Q.25.14. (Solution to 25.4.7) If c is differentiable at a, then there exists a bounded linear transformation $dc_a \colon \mathbb{R} \to V$ that is tangent to Δc_a at 0. Then

$$\begin{aligned} Dc(a) &= \lim_{h\to 0} \frac{\Delta c_a(h)}{h} \\ &= \lim_{h\to 0} \frac{\Delta c_a(h) - dc_a(h) + dc_a(h)}{h} \\ &= \lim_{h\to 0} \frac{\Delta c_a(h) - dc_a(h)}{h} + \lim_{h\to 0} \frac{h\, dc_a(1)}{h} \\ &= dc_a(1). \end{aligned}$$

Q.25.15. (Solution to 25.4.12) Let $\epsilon > 0$. Since f is continuous at c and the interval J is open, we may choose $\delta > 0$ so that $c + h \in J$ and $\|\Delta f_c(h)\| < \epsilon$ whenever $|h| < \delta$. Thus if $0 < |h| < \delta$,

$$\begin{aligned} \|\Delta F_c(h) - h\, f(c)\| &= \left\| \int_a^{c+h} f - \int_a^c f - h\, f(c) \right\| \\ &= \left\| \int_c^{c+h} f(t)\, dt - \int_c^{c+h} f(c)\, dt \right\| \\ &= \left\| \int_c^{c+h} \Delta f_c(t-c)\, dt \right\| \\ &\le \left| \int_c^{c+h} \|\Delta f_c(t-c)\|\, dt \right| \qquad \text{(by 24.3.10)} \\ &< \epsilon\, |h|\,. \end{aligned}$$

It follows immediately that

$$\left\| \frac{\Delta F_c(h)}{h} - f(c) \right\| < \frac{1}{|h|} \epsilon |h| = \epsilon$$

whenever $0 < |h| < \delta$; that is,

$$DF(c) = \lim_{h\to 0} \frac{\Delta F_c(h)}{h} = f(c)\,.$$

Q.25.16. (Solution to 25.5.2) If $l(t) = a + tv$, then

$$\begin{aligned} D(f\circ l)(0) &= \lim_{t\to 0} \frac{1}{t} \Delta(f\circ l)_0(t) \\ &= \lim_{t\to 0} \frac{1}{t} ((f\circ l)(0+t) - (f\circ l)(0)) \\ &= \lim_{t\to 0} \frac{1}{t} (f(a+tv) - f(a)) \\ &= \lim_{t\to 0} \frac{1}{t} \Delta f_a(tv) \\ &= D_v f(a). \end{aligned}$$

Q.25.17. (Solution to 25.5.3) Let $l(t) = a + tv$. Then

$$\begin{aligned}(f \circ l)(t) &= f(a+tv)\\ &= f((1,1) + t(\tfrac{3}{5}, \tfrac{4}{5}))\\ &= f(1 + \tfrac{3}{5}t, 1 + \tfrac{4}{5}t)\\ &= \tfrac{1}{2}\ln\big((1+\tfrac{3}{5}t)^2 + (1+\tfrac{4}{5}t)^2\big)\\ &= \tfrac{1}{2}\ln\big(2 + \tfrac{14}{5}t + t^2\big).\end{aligned}$$

It follows that $D(f\circ l)(t) = \frac{1}{2}\big(\frac{14}{5}+2t\big)\big(2+\frac{14}{5}t+t^2\big)^{-1}$, so that $D_vf(a) = D(f\circ l)(0) = \frac{7}{10}$.

Q.25.18. (Solution to 25.5.5) As usual let $l(t) = a + tv$. Then

$$\begin{aligned}(\phi \circ l)(t) &= \int_0^{\pi/2} (\cos x + Da(x) + tDv(x))^2\,dx\\ &= \int_0^{\pi/2} (2\cos x + t\sin x)^2\,dx\\ &= \int_0^{\pi/2} 4\cos^2 x\,dx + 4t\int_0^{\pi/2} \sin x\cos x\,dx + t^2\int_0^{\pi/2}\sin^2 x\,dx\,.\end{aligned}$$

Differentiating we obtain

$$D(\phi\circ l)(t) = 4\int_0^{\pi/2}\sin x\cos x\,dx + 2t\int_0^{\pi/2}\sin^2 x\,dx\,;$$

so

$$D_v\phi(a) = D(\phi\circ l)(0) = 4\int_0^{\pi/2}\sin x\cos x\,dx = 2\,.$$

Q.25.19. (Solution to 25.5.9) If $l = a + tv$, then, since $l(0) = a$ and $Dl(0) = v$, we have

$$\begin{aligned}D_vf(a) &= D(f\circ l)(0)\\ &= df_{l(0)}(Dl(0)) \quad \text{(by 25.4.11)}\\ &= df_a(v).\end{aligned}$$

Q.25.20. (Solution to 25.6.1) If $x \in \operatorname{dom} f^1 \cap \operatorname{dom} f^2$, then

$$\begin{aligned}\big((j_1\circ f^1) + (j_2\circ f^2)\big)(x) &= j_1\big(f^1(x)\big) + j_2\big(f^2(x)\big)\\ &= \big(f^1(x), 0\big) + \big(0, f^2(x)\big)\\ &= \big(f^1(x), f^2(x)\big)\\ &= f(x)\,.\end{aligned}$$

Being the sum of composites of differentiable functions, f is differentiable, and

$$\begin{aligned} df_a &= d\big((j_1 \circ f^1) + (j_2 \circ f^2)\big)_a \\ &= d\big(j_1 \circ f^1\big)_a + d\big(j_2 \circ f^2\big)_a && \text{(by 25.3.15)} \\ &= d\big(j_1\big)_{f^1(a)} \circ d\big(f^1\big)_a + d\big(j_2\big)_{f^2(a)} \circ d\big(f^2\big)_a && \text{(by 25.3.17)} \\ &= j_1 \circ d\big(f^1\big)_a + j_2 \circ d\big(f^2\big)_a && \text{(by 25.3.24)} \\ &= \big(d\big(f^1\big)_a\,,\, d\big(f^2\big)_a\big)\,. \end{aligned}$$

Q.25.21. (Solution to 25.6.3) By propositions 25.4.7 and 25.4.8 a curve has a derivative at t if and only if it is differentiable at t. Thus the desired result is an immediate consequence of the following easy computation:

$$\begin{aligned} Dc(t) &= dc_t(1) \\ &= \big(d\big(c^1\big)_t(1)\,,\, d\big(c^2\big)_t(1)\big) \\ &= \big(Dc^1(t)\,,\, Dc^2(t)\big)\,. \end{aligned}$$

Q.26. Exercises in chapter 26

Q.26.1. (Solution to 26.1.5) Let $f\colon [0, 2\pi] \to \mathbb{R}^2\colon t \mapsto (\cos t, \sin t)$. Then f is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$. Notice that $f(2\pi) - f(0) = (1, 0) - (1, 0) = (0, 0)$. But $Df(t) = (-\sin t, \cos t)$. Certainly there is no number c such that $2\pi(-\sin c, \cos c) = (0, 0)$.

Q.26.2. (Solution to 26.1.6) Given $\epsilon > 0$, define $h(t) = \|f(t)-f(a)\| - (t-a)(M+\epsilon)$ for $a \le t \le b$. Since f is continuous on $[a, b]$, so is h. Let $A = h^{\leftarrow}(-\infty, \epsilon]$. The set A is nonempty (it contains a) and is bounded above (by b). By the *least upper bound axiom* J.3.1 it has a supremum, say l. Clearly $a \le l \le b$. Since h is continuous and $h(a) = 0$, there exists $\eta > 0$ such that $a \le t < a + \eta$ implies $h(t) \le \epsilon$. Thus $[a, a + \eta) \subseteq A$ and $l > a$. Notice that since h is continuous the set A is closed (proposition 14.1.13); and since $l \in \overline{A}$ (see example 2.2.7), l belongs to A.

We show that $l = b$. Assume to the contrary that $l < b$. Since f is differentiable at l, there exists $\delta > 0$ such that if $t \in (l, l+\delta)$ then $\|(t-l)^{-1}(f(t)-f(l))\| < M+\epsilon$. Choose any point t in $(l, l + \delta)$. Then

$$\begin{aligned} h(t) &= \|f(t) - f(a)\| - (t - a)(M + \epsilon) \\ &\le \|f(t) - f(l)\| + \|f(l) - f(a)\| - (t - l)(M + \epsilon) - (l - a)(M + \epsilon) \\ &< (t - l)(M + \epsilon) + h(l) - (t - l)(M + \epsilon) \\ &= h(l) \\ &\le \epsilon\,. \end{aligned}$$

This says that $t \in A$, which contradicts the fact that l is an upper bound for A. Thus $l = b$ and $h(b) \le \epsilon$. That is,

$$\|f(b) - f(a)\| \le (M + \epsilon)(b - a) + \epsilon\,.$$

Since ϵ was arbitrary,

$$\|f(b) - f(a)\| \le M(b - a)\,.$$

Q.26.3. (Solution to 26.2.1)

(a) For every x in V_k

$$(\pi_k \circ j_k)(x) = \pi_k(0, \dots, 0, x, 0, \dots, 0) = x\,.$$

(b) For every x in V

$$\begin{aligned}\sum_{k=1}^{n}(j_k \circ \pi_k)(x) &= \sum_{k=1}^{n} j_k(x_k) \\ &= (x_1, 0, \dots, 0) + \dots + (0, \dots, 0, x_n) \\ &= (x_1, \dots, x_n) \\ &= x\,.\end{aligned}$$

Q.26.4. (Solution to 26.2.5) For every h in B_k

$$\begin{aligned}(\Delta f_a \circ j_k)(h) &= \Delta f_a(j_k(h)) \\ &= f(a + j_k(h)) - f(a) \\ &= g(h) - g(\mathbf{0}) \\ &= \Delta g_{\mathbf{0}}(h)\,.\end{aligned}$$

Q.26.5. (Solution to 26.2.10) Fix a point (a, b) in U. We make two observations about the notation introduced in the hint. First,

$$d(h^v)_z = d_1 f_{(a+z,b+v)}\,. \tag{114}$$

[Proof: Since $f(a+z+s, b+v) = h^v(z+s) = (h^v \circ T_z)(s)$, we see that $d_1 f_{(a+z,b+v)} = d(h^v \circ T_z)_{\mathbf{0}} = d(h^v)_{T_z(\mathbf{0})} \circ d(T_z)_{\mathbf{0}} = d(h^v)_z \circ I = d(h^v)_z$.]

Second,

$$\Delta f_{(a,b)}(u, v) = \Delta(h^v)_{\mathbf{0}}(u) + \Delta g_{\mathbf{0}}(v)\,. \tag{115}$$

[Proof:

$$\begin{aligned}\Delta f_{(a,b)}(u, v) &= f(a + u, b + v) - f(a, b) \\ &= f(a + u, b + v) - f(a, b + v) + f(a, b + v) - f(a, b) \\ &= h^v(u) - h^v(\mathbf{0}) + g(v) - g(\mathbf{0}) \\ &= \Delta(h^v)_{\mathbf{0}}(u) + \Delta g_{\mathbf{0}}(v)\,.]\end{aligned}$$

Let $\epsilon > 0$. By hypothesis the second partial differential of f exists at (a, b). That is, the function g is differentiable at $\mathbf{0}$ and

$$\Delta g_{\mathbf{0}} \simeq dg_{\mathbf{0}} = d_2 f_{(a,b)} = T\,.$$

Thus there exists $\delta_1 > 0$ such that

$$\|\Delta g_{\mathbf{0}}(v) - Tv\| \le \epsilon\|v\| \tag{116}$$

whenever $\|v\| < \delta_1$.

Since $d_1 f$ is assumed to (exist and) be continuous on U, there exists $\delta_2 > 0$ such that

$$\|d_1 f_{(a+s,b+t)} - d_1 f_{(a,b)}\| < \epsilon \tag{117}$$

whenever $\|(s,t)\|_1 < \delta_2$. Suppose then that (u,v) is a point in U such that $\|(u,v)\|_1 < \delta_2$. For each z in the segment $[\mathbf{0},u]$

$$\|(z,v)\|_1 = \|z\| + \|v\| \le \|u\| + \|v\| = \|(u,v)\|_1 < \delta_2$$

so by (114) and (117)

$$\|d(h^v)_z - S\| = \|d_1f_{(a+z,b+v)} - d_1f_{(a,b)}\| < \epsilon\,.$$

Thus according to the version of the *mean inequality* given in corollary 26.1.8

(118) $$\|\Delta(h^v)_{\mathbf{0}}(u) - Su\| \le \epsilon\|u\|$$

whenever $\|(u,v)\|_1 < \delta_2$.

Now let $\delta = \min\{\delta_1,\delta_2\}$. Suppose $\|(u,v)\|_1 < \delta$. Then since $\|v\| \le \|u\| + \|v\| = \|(u,v)\|_1 < \delta \le \delta_1$ inequality (116) holds, and since $\|(u,v)\|_1 < \delta \le \delta_2$ inequality (118) holds. Making use of these two inequalities and (115) we obtain

$$\begin{aligned}\|\Delta f_{(a,b)}(u,v) - R(u,v)\| &= \|\Delta(h^v)_{\mathbf{0}}(u) + \Delta g_{\mathbf{0}}(v) - Su - Tv\|\\ &\le \|\Delta(h^v)_{\mathbf{0}}(u) - Su\| + \|\Delta g_{\mathbf{0}}(v) - Tv\|\\ &\le \epsilon\,\|u\| + \epsilon\,\|v\|\\ &= \epsilon\,\|(u,v)\|_1\,.\end{aligned}$$

Thus $\Delta f_{(a,b)} \simeq R$ showing that f is differentiable at (a,b) and that its differential is given by

(119) $$df_{(a,b)} = R = d_1f_{(a,b)} \circ \pi_1 + d_2f_{(a,b)} \circ \pi_2\,.$$

That df is continuous is clear from (119) and the hypothesis that d_1f and d_2f are continuously differentiable.

Q.26.6. (Solution to 26.2.12) First we compute df_a. A straightforward calculation gives

$$\frac{\Delta f_a(h)}{\|h\|_1} = \frac{h_1 + 2\,h_2 - 3\,h_3 + 6\,h_4 + {h_2}^2 + 2\,h_1h_2 + h_1{h_2}^2 + 3\,h_3h_4}{\|h\|_1}\,.$$

From this it is clear that the desired differential is given by

$$df_a(h) = h_1 + 2\,h_2 - 3\,h_3 + 6\,h_4$$

for then

$$\frac{\Delta f_a(h) - df_a(h)}{\|h\|_1} = \frac{{h_2}^2 + 2\,h_1h_2 + h_1{h_2}^2 + 3\,h_3h_4}{\|h\|_1} \to 0 \qquad \text{as } h \to 0$$

Note. In the preceding computation the use of the product norm $\|\ \ \|_1$ for $\mathbb{R}^4$ rather than the usual Euclidean norm is both arbitrary and harmless (see problem 22.3.21.

(a) Compose df_a with the injection

$$j_1 \colon \mathbb{R} \to \mathbb{R} \times R \times \mathbb{R} \times \mathbb{R} \colon x \mapsto (x,0,0,0)\,.$$

Then

$$d_1f_a(x) = df_a(j_1(x)) = df_a(x,0,0,0) = x$$

for all x in $\mathbb{R}$.

(b) This has exactly the same answer as part (a)—although the rationale is slightly different. The appropriate injection map is

$$j_1 \colon \mathbb{R} \to \mathbb{R}^3 \colon x \mapsto (x, \mathbf{0})$$

(where $\mathbf{0}$ is the zero vector in $\mathbb{R}^3$). We may rewrite (43) in this case as

$$f(x,y) = x{y_1}^2 + 3\,y_2y_3$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}^3$. Also write $a = (b,c)$ where $b = 1 \in \mathbb{R}$ and $c = (1,2,-1) \in \mathbb{R}^3$, and write $h = (r,s)$ where $r \in \mathbb{R}$ and $s \in \mathbb{R}^3$. Then

$$df_a(h) = df_{(b,c)}(r,s) = r + 2\,s_1 - 3\,s_2 + 6\,s_3$$

so that

$$d_1f(x) = df_a(j_1(x)) = df_a(x,\mathbf{0}) = x$$

for all x in $\mathbb{R}$.

(c) Here the appropriate injection is

$$j_1 \colon \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2 \colon x \mapsto (x, \mathbf{0})$$

where $\mathbf{0}$ is the zero vector in $\mathbb{R}^2$. Rewrite (43) as

$$f(x,y) = x_1{x_2}^2 + 3\,y_1y_2$$

for all x, $y \in \mathbb{R}^2$. Let $a = (b,c)$ where $b = (1,1)$ and $c = (2,-1)$; and let $h = (r,s)$ where r, $s \in \mathbb{R}^2$. Then

$$df_a(h) = df_{(b,c)}(r,s) = r_1 + 2\,r_2 - 3\,s_1 + 6\,s_2$$

so that

$$d_1f_a(x) = df_a(j_1(x)) = df_a(x,\mathbf{0}) = x_1 + 2\,x_2$$

for all x in $\mathbb{R}^2$.

(d) As far as the partial differential d_1 is concerned, this is essentially the same problem as (c). However, in this case the injection j_1 is given by

$$j_1 \colon \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \colon x \mapsto (x,0,0)\,.$$

Equation (43) may be written

$$f(x,y,z) = x_1{x_2}^2 + 3\,yz$$

for all $x \in \mathbb{R}^2$ and y, $z \in \mathbb{R}$. Let $a = (b,c,d)$ where $b = (1,1)$, $c = 2$, and $d = -1$; and let $h = (q,r,s)$ where $q \in \mathbb{R}^2$ and r, $s \in \mathbb{R}$. Then

$$df_a(h) = df_{(b,c,d)}(q,r,s) = q_1 + 2\,q_2 - 3\,r + 6\,s$$

so that

$$d_1f_a(x) = df_a(j_1(x)) = df_{(b,c,d)}(x,0,0) = x_1 + 2\,x_2\,.$$

(e) Here $j_1 \colon \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R} \colon x \mapsto (x,0)$. Rewrite (43) as

$$f(x,y) = x_1{x_2}^2 + 3\,x_3y$$

for all $x \in \mathbb{R}^3$ and $y \in \mathbb{R}$. Let $a = (b,c)$ with $b = (1,1,2)$ and $c = -1$; and let $h = (r,s)$ where $r \in \mathbb{R}^3$ and $s \in \mathbb{R}$. Then

$$df_a(h) = df_{(b,c)}(r,s) = r_1 + 2\,r_2 - 3\,r_3 + 6\,s$$

so that

$$d_1f_a(x) = df_a(j_1(x)) = df_{(b,c)}(x,0) = x_1 + 2\,x_2 - 3\,x_3\,.$$

Q.26.7. (Solution to 26.2.17) Just do what you have always done: hold two of the variables constant and differentiate with respect to the other. (See the paragraph after equation (44).)

$$f_1(x, y, z) = (3\,x^2y^2 \sin z, 2\,x);\ \text{ so } f_1(a) = (12, 2).$$
$$f_2(x, y, z) = (2\,x^3y \sin z, \cos z);\ \text{ so } f_2(a) = (-4, 0).$$
$$f_3(x, y, z) = (x^3y^2 \cos z, -y \sin z);\ \text{ so } f_3(a) = (0, 2).$$

Q.26.8. (Solution to 26.3.2) Let $\epsilon > 0$. Since $[a, b]\times[c, d]$ is compact, the continuous function f must be uniformly continuous (see proposition 24.1.11). Thus there exists $\delta > 0$ such that $\|(x, y)-(u, v)\|_1 < \delta$ implies $\|f(x, y)-f(u, v)\| < \epsilon\,(b-a)^{-1}$. Suppose that y and v lie in $[c, d]$ and that $|y - v| < \delta$. Then $\|(x, y) - (x, v)\|_1 < \delta$ for all x in $[a, b]$; so $\|f(x, y) - f(x, v)\| < \epsilon\,(b - a)^{-1}$ from which it follows that

$$\begin{aligned}
\|g(y) - g(v)\| &= \left\| \int_a^b f^y - \int_a^b f^v \right\| \\
&= \left\| \int_a^b (f^y - f^v) \right\| \\
&\le \int_a^b \|f^y(x) - f^v(x)\|\,dx \\
&= \int_a^b \|f(x, y) - f(x, v)\|\,dx \\
&< \int_a^b \epsilon\,(b - a)^{-1}\,dx \\
&= \epsilon\,.
\end{aligned}$$

Thus g is uniformly continuous.

Q.26.9. (Solution to 26.3.4) Let $h(y) = \int_a^b f_2(x, y)\,dx$. By lemma 26.3.2 the function h is continuous and therefore integrable on every interval of the form $[c, z]$ where $c \le z \le d$. Then by proposition 26.3.3 we have

$$\begin{aligned}
\int_c^z h &= \int_c^z \int_a^b f_2(x, y)\,dx\,dy \\
&= \int_a^b \int_c^z f_2(x, y)\,dy\,dx \\
&= \int_a^b \int_c^z \frac{d}{dy}\big({}^x f(y)\big)\,dy\,dx \\
&= \int_a^b \big({}^x f(z) - {}^x f(c)\big)\,dx \\
&= \int_a^b \big(f(x, z) - f(x, c)\big)\,dx \\
&= g(z) - g(c)\,.
\end{aligned}$$

Differentiating we obtain

$$h(z) = g'(z)$$

for $c < z < d$. This shows that g is continuously differentiable on (c, d) and that

$$\begin{aligned}\frac{d}{dy}\int_a^b f(x,y)\,dx &= g'(y)\\ &= h(y)\\ &= \int_a^b f_2(x,y)\,dx\\ &= \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx\,.\end{aligned}$$

Q.27. Exercises in chapter 27

Q.27.1. (Solution to 27.1.4) If x, y, and $z \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}\langle x, y+z\rangle &= \langle y+z, x\rangle && \text{(by (c))}\\ &= \langle y, x\rangle + \langle z, x\rangle && \text{(by (a))}\\ &= \langle x, y\rangle + \langle x, z\rangle && \text{(by (c))}\end{aligned}$$

and

$$\begin{aligned}\langle x, \alpha y\rangle &= \langle \alpha y, x\rangle && \text{(by (c))}\\ &= \alpha\langle y, x\rangle && \text{(by (b))}\\ &= \alpha\langle x, y\rangle && \text{(by (c))}\end{aligned}$$

Q.27.2. (Solution to 27.1.8) The domain of the arccosine function is the closed interval $[-1, 1]$. According to the *Schwarz inequality* $|\langle x, y\rangle| \le \|x\|\|y\|$; equivalently,

$$-1 \le \frac{\langle x, y\rangle}{\|x\|\|y\|} \le 1$$

for nonzero vectors x and y. This shows that $\langle x, y\rangle\|x\|^{-1}\|y\|^{-1}$ is in the domain of arccosine.

Q.27.3. (Solution to 27.1.10) If $x = (1, 0, 1)$ and $y = (0, -1, 1)$, then $\langle x, y\rangle = 1$ and $\|x\| = \|y\| = \sqrt{2}$. So

$$\measuredangle(x, y) = \arccos\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right) = \arccos\frac{1}{2} = \frac{\pi}{3}\,.$$

Q.27.4. (Solution to 27.2.1) The computations

$$\psi_b(x+y) = \langle x+y, b\rangle = \langle x, b\rangle + \langle y, b\rangle = \psi_b(x) + \psi_b(y)$$

and

$$\psi_b(\alpha x) = \langle \alpha x, b\rangle = \alpha\langle x, b\rangle = \alpha\psi_b(x)$$

show that ψ_b is linear. Since

$$|\psi_b(x)| = |\langle x, b\rangle| \le \|b\|\|x\|$$

for every x in $\mathbb{R}^n$, we conclude that ψ_b is bounded and that $\|\psi_b\| \le \|b\|$. On the other hand, if $b \ne \mathbf{0}$, then $\|b\|^{-1}b$ is a unit vector, and since

$$|\psi_b(\|b\|^{-1}b)| = \langle \|b\|^{-1}b, b\rangle = \|b\|^{-1}\langle b, b\rangle = \|b\|$$

we conclude (from lemma 23.1.6) that $\|\psi_b\| \ge \|b\|$.

Q.27.5. (Solution to 27.2.6) By proposition 25.5.9 we have for every unit vector u in $\mathbb{R}^n$

$$\begin{aligned} D_u\phi(a) &= d\phi_a(u) \\ &= \langle u, \nabla\phi(a)\rangle \\ &= \|\nabla\phi(a)\| \cos\theta \end{aligned}$$

where $\theta = \measuredangle(u, \nabla\phi(a))$. Since ϕ and a are fixed we maximize the directional derivative $D_u\phi(a)$ by maximizing $\cos\theta$. But $\cos\theta = 1$ when $\theta = 0$; that is, when u and $\nabla\phi(a)$ point in the same direction. Similarly, to minimize $D_u\phi(a)$ choose $\theta = \pi$ so that $\cos\theta = -1$.

Q.27.6. (Solution to 27.2.13) It suffices, by proposition 26.1.9, to show that the derivative of the total energy TE is zero.

$$\begin{aligned} D(TE) &= D(KE) + D(PE) \\ &= \tfrac{1}{2}mD\langle v, v\rangle + D(\phi \circ x) \\ &= \tfrac{1}{2}m(2\langle v, Dv\rangle) + \langle Dx, (\nabla\phi) \circ x\rangle \\ &= m\langle v, a\rangle + \langle v, -F \circ x\rangle \\ &= m\langle v, a\rangle - m\langle v, a\rangle \\ &= 0. \end{aligned}$$

(The third equality uses 27.1.17 and 27.2.7; the second last uses *Newton's second law*.)

Q.27.7. (Solution to 27.2.14) Using the hint we compute

$$\begin{aligned} \nabla\phi(a) &= \sum_{k=1}^{n} \langle \nabla\phi(a), e^k\rangle e^k \qquad \text{(by 27.1.3)} \\ &= \sum_{k=1}^{n} d\phi_a(e^k) e^k \\ &= \sum_{k=1}^{n} D_{e^k}\phi(a) e^k \qquad \text{(by 25.5.9)} \\ &= \sum_{k=1}^{n} \phi_k(a) e^k. \end{aligned}$$

Q.27.8. (Solution to 27.2.15) By proposition 25.5.9

$$D_u\phi(a) = d\phi_a(u) = \langle u, \nabla\phi(a)\rangle .$$

Since

$$\begin{aligned} \nabla\phi(w,x,y,z) &= \sum_{k=1}^{4} \phi_k(w,x,y,z) e^k \\ &= (z, -y, -x, w) \end{aligned}$$

we see that

$$\nabla\phi(a) = (4, -3, -2, 1) .$$

Thus

$$D_u\phi(a) = \langle(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}), (4, -3, -2, 1)\rangle = 2\,.$$

Q.27.9. (Solution to 27.2.16) As suggested in the hint, let $c\colon t \mapsto \big(x(t), y(t)\big)$ be the desired curve and set

$$c(0) = \big(x(0), y(0)\big) = a = (2, -1)\,.$$

At each point $c(t)$ on the curve set the tangent vector $Dc(t)$ equal to $-\big(\nabla\phi\big)\big(c(t)\big)$. Then for every t we have

$$\begin{aligned}\big(Dx(t), Dy(t)\big) &= -\big(\nabla\phi\big)\big(x(t), y(t)\big)\\ &= \big(-4x(t), -12y(t)\big).\end{aligned}$$

The two resulting equations

$$Dx(t) = -4x(t) \qquad \text{and} \qquad Dy(t) = -12y(t)$$

have as their only nonzero solutions

$$x(t) = x(0)e^{-4t} = 2e^{-4t}$$

and

$$y(t) = y(0)e^{-12t} = -e^{-12t}\,.$$

Eliminating the parameter we obtain

$$y(t) = -e^{-12t} = -\big(e^{-4t}\big)^3 = -\Big(\frac{1}{2}x(t)\Big)^3 = -\frac{1}{8}\big(x(t)\big)^3\,.$$

Thus the path of steepest descent (in the xy-plane) follows the curve $y = -\frac{1}{8}x^3$ from $x = 2$ to $x = 0$ (where ϕ obviously assumes its minimum).

Q.27.10. (Solution to 27.3.2) By proposition 21.3.11 it suffices to show that

$$[df_a]e^l = [f^j_k(a)]e^l$$

for $1 \le l \le n$. Since the i^{th} coordinate ($1 \le i \le m$) of the vector that results from the action of the matrix $[f^j_k(a)]$ on the vector e^l is

$$\sum_{k=1}^{n} f^i_k(a)(e^l)_k = f^i_l(a)$$

we see that

$$\begin{aligned}[f^j_k(a)]e^l &= \sum_{i=1}^{m} f^i_l(a)\hat{e}^i\\ &= f_l(a) && \text{(by proposition 26.2.15)}\\ &= df_a(e^l)\\ &= [df_a]e^l.\end{aligned}$$

Q.27.11. (Solution to 27.3.3)

(a) By proposition 27.3.2

$$[df_{(w,x,y,z)}] = \begin{bmatrix} xz & wz & 0 & wx \\ 0 & 2x & 4y & 6z \\ y \arctan z & 0 & w \arctan z & wy(1+z^2)^{-1} \end{bmatrix}.$$

Therefore

$$[df_a] = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 6 \\ \pi/4 & 0 & \pi/4 & 1/2 \end{bmatrix}.$$

(b)

$$\begin{aligned} df_a(v) &= [df_a]v \\ &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 4 & 6 \\ \pi/4 & 0 & \pi/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{bmatrix} \\ &= \left(3, -2, \tfrac{1}{4}(2-3\pi)\right). \end{aligned}$$

Q.27.12. (Solution to 27.4.3) Let

$$g \colon \mathbb{R}^4 \to \mathbb{R}^2 \colon (u,v,w,x) \mapsto \big(y(u,v,w,x), z(u,v,w,x)\big)$$

and

$$f \colon \mathbb{R}^2 \to \mathbb{R}^4 \colon (s,t) \mapsto \big(u(s,t), v(s,t), w(s,t), x(s,t)\big).$$

Here it is appropriate to think of the variables and functions as being arranged in the following fashion.

$$\begin{array}{ccccc} & & u & & \\ s & \xrightarrow{\ f\ } & v & \xrightarrow{\ g\ } & y \\ t & & w & & z \\ & & x & & \end{array}$$

The expression $\frac{\partial u}{\partial t}$ is then taken to represent the function f_2^1. The expression $\frac{\partial z}{\partial u}$ appearing in the statement of the exercise represents $g_1^2 \circ f$. [One's first impulse might be to let $\frac{\partial z}{\partial u}$ be just g_1^2. But this cannot be correct. The product of $\frac{\partial z}{\partial u}$ and $\frac{\partial u}{\partial t}$ is defined only at points where both are defined. The product of g_1^2 (whose domain lies in $\mathbb{R}^4$) and f_2^1 (whose domain is in $\mathbb{R}^2$) is never defined.] On the left side of the equation the expression $\frac{\partial z}{\partial t}$ is the partial derivative with respect to t of the composite function $f \circ g$. Thus it is expressed functionally as $(g \circ f)_2^2$.

Using proposition 27.4.1 we obtain

$$\begin{aligned}\frac{\partial z}{\partial t} &= (g \circ f)^2_2 \\ &= \sum_{i=1}^{4} (g^2_i \circ f) f^i_2 \\ &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\end{aligned}$$

This equation is understood to hold at all points a in $\mathbb{R}^2$ such that f is differentiable at a and g is differentiable at $f(a)$.

Q.27.13. (Solution to 27.4.5) Since

$$\left[df_{(x,y,z)}\right] = \begin{bmatrix} y^2 & 2xy & 0 \\ 3 & 0 & -2z \\ yz & xz & xy \\ 2x & 2y & 0 \\ 4z & 0 & 4x \end{bmatrix}$$

we see that

$$\left[df_a\right] = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \\ -4 & 0 & 4 \end{bmatrix}.$$

And since

$$\left[dg_{(s,t,u,v,w)}\right] = \begin{bmatrix} 2s & 0 & 2u & 2v & 0 \\ 2sv & -2w^2 & 0 & s^2 & -4tw \end{bmatrix}$$

we see that

$$\left[dg_{f(a)}\right] = \left[dg_{(0,2,0,1,1)}\right] = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 & -8 \end{bmatrix}.$$

Thus by equation (47)

$$\begin{aligned}
\left[d(g\circ f)_a\right] &= \left[dg_{f(a)}\right]\left[df_a\right] \\
&= \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \\ -4 & 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 & 0 \\ 26 & 0 & -36 \end{bmatrix}.
\end{aligned}$$

Q.27.14. (Solution to 27.4.8) Use formula (52). It is understood that $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ must be evaluated at the point $(1,1)$; and since $x(1,1)=2$ and $y(1,1)=\pi/4$, the partials $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, and $\left(\frac{\partial w}{\partial t}\right)_{x,y}$ are to be evaluated at the point $(2,\pi/4,1)$. Calculate the terms appearing on the right hand side of (52):

$$\begin{aligned}
\frac{\partial x}{\partial t} &= 2t; &\quad &\text{so } \frac{\partial x}{\partial t}(1,1) = 2, \\
\frac{\partial y}{\partial t} &= \frac{s}{1+t^2}; & &\text{so } \frac{\partial y}{\partial t}(1,1) = 1/2, \\
\frac{\partial w}{\partial x} &= -\frac{2y}{x^2}; & &\text{so } \frac{\partial w}{\partial x}(2,\pi/4,1) = -\pi/8, \\
\frac{\partial w}{\partial y} &= \frac{2}{x}; & &\text{so } \frac{\partial w}{\partial y}(2,\pi/4,1) = 1, \text{ and} \\
\left(\frac{\partial w}{\partial t}\right)_{x,y} &= 3t^2; & &\text{so } \left(\frac{\partial w}{\partial t}\right)_{x,y}(2,\pi/4,1) = 3\,.
\end{aligned}$$

Therefore

$$\left(\frac{\partial w}{\partial t}\right)_s(1,1) = -\frac{\pi}{8}\cdot 2 + 1\cdot\frac{1}{2} + 3 = \frac{7}{2} - \frac{\pi}{4}.$$

Q.27.15. (Solution to 27.4.9) We proceed through steps (a)–(g) of the hint.

(a) Define $g(x,y) = y/x$ and compute its differential

$$\begin{aligned}
\left[dg_{(x,y)}\right] &= \left[g_1(x,y) \quad g_2(x,y)\right] \\
&= \left[-yx^{-2} \quad x^{-1}\right].
\end{aligned}$$

(b) Then compute the differential of $\phi\circ g$

$$\begin{aligned}
\left[d(\phi\circ g)_{(x,y)}\right] &= \left[d\phi_{g(x,y)}\right]\left[dg_{(x,y)}\right] \\
&= \phi'\big(g(x,y)\big)\left[dg_{(x,y)}\right] \\
&= \phi'(yx^{-1})\left[-yx^{-2} \quad x^{-1}\right] \\
&= \left[-yx^{-2}\phi'(yx^{-1}) \quad x^{-1}\phi'(yx^{-1})\right].
\end{aligned}$$

(c) Let $G(x,y) = \big(x, \phi(y/x)\big)$ and use (b) to calculate $\big[dG_{(x,y)}\big]$

$$\begin{aligned}\big[dG_{(x,y)}\big] &= \begin{bmatrix} G^1_1(x,y) & G^1_2(x,y) \\ G^2_1(x,y) & G^2_2(x,y)\end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -yx^{-2}\phi'(yx^{-1}) & x^{-1}\phi'(yx^{-1})\end{bmatrix}\end{aligned}$$

(d) Let $m(x,y) = xy$ and compute its differential

$$\big[dm_{(x,y)}\big] = \big[m_1(x,y) \qquad m_2(x,y)\big] = \big[y \qquad x\big]\,.$$

(e) Since $h(x,y) = x\phi(yx^{-1}) = m(G(x,y))$ we see that $h = m \circ G$ and therefore

$$\begin{aligned}\big[dh_{(x,y)}\big] &= \big[d(m \circ G)_{(x,y)}\big] \\ &= \big[dm_{G(x,y)}\big]\big[dG_{(x,y)}\big] \\ &= \big[G^2(x,y) \qquad G^1(x,y)\big]\big[dG_{(x,y)}\big] \\ &= \big[\phi(yx^{-1}) \qquad x\big]\begin{bmatrix} 1 & 0 \\ -yx^{-2}\phi'(yx^{-1}) & x^{-1}\phi'(yx^{-1})\end{bmatrix} \\ &= \big[\phi(yx^{-1} - yx^{-1}\phi'(yx^{-1}) \qquad \phi'(yx^{-1})\big]\,.\end{aligned}$$

(f) Since $j(x,y) = xy + x\phi(yx^{-1}) = m(x,y) + h(x,y)$, we see that

$$\begin{aligned}\big[dj_{(x,y)}\big] &= \big[dm_{(x,y)}\big] + \big[dh_{(x,y)}\big] \\ &= \big[y + \phi(yx^{-1} - yx^{-1}\phi'(yx^{-1}) \qquad x + \phi'(yx^{-1})\big]\,.\end{aligned}$$

(g) Then finally,

$$\begin{aligned}xj_1(x,y) + yj_2(x,y) &= x\big(y + \phi(yx^{-1} - yx^{-1}\phi'(yx^{-1})\big) + y\big(x + \phi'(yx^{-1})\big) \\ &= xy + x\phi(yx^{-1}) + yx \\ &= xy + j(x,y)\end{aligned}$$

Q.27.16. (Solution to 27.4.10) Let h be as in the hint. Then

$$\big[dh_{(x,y)}\big] = \begin{pmatrix} 2x & -2y \\ 2y & 2x\end{pmatrix}$$

so

$$\begin{aligned}\big[dg_{(x,y)}\big] &= \big[d(f \circ h)_{(x,y)}\big] \\ &= \big[df_{h(x,y)}\big]\big[dh_{(x,y)}\big] \\ &= \big[f_1(h(x,y)) \qquad f_2(h(x,y))\big]\begin{bmatrix} 2x & -2y \\ 2y & 2x\end{bmatrix} \\ &= \big[2xf_1(h(x,y)) + 2yf_2(h(x,y)) \qquad -2yf_1(h(x,y)) + 2xf_2(h(x,y))\big]\,.\end{aligned}$$

Therefore

$$\begin{aligned} yg_1(x,y) - xg_2(x,y) &= 2xyf_1(h(x,y)) + 2y^2 f_2(h(x,y)) + 2xyf_1(h(x,y)) - 2x^2 f_2(h(x,y)) \\ &= 4xyf_1(h(x,y)) - 2(x^2 - y^2) f_2(h(x,y)) \\ &= 2h^2(x,y) f_1(h(x,y)) - 2h^1(x,y) f_2(h(x,y))\,. \end{aligned}$$

This computation, incidentally, gives one indication of the attractiveness of notation that omits evaluation of partial derivatives. If one is able to keep in mind the points at which the partials are being evaluated, less writing is required.

Q.28. Exercises in chapter 28

Q.28.1. (Solution to 28.1.2) If n is odd then the n^{th} partial sum s_n is 1; if n is even then $s_n = 0$.

Q.28.2. (Solution to 28.1.3) Use problem 28.1.8. The n^{th} partial sum of the sequence $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right)$ is

$$\begin{aligned} s_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} \\ &= \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \\ &= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k - 1 \\ &= \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} - 1 \\ &= 2 - \left(\frac{1}{2}\right)^n - 1 \\ &= 1 - 2^{-n}. \end{aligned}$$

Q.28.3. (Solution to 28.1.5) For the sequence given in exercise 28.1.2, the corresponding series $\sum_{k=1}^{\infty} a_k$ is the sequence $(1, 0, 1, 0, 1, \dots)$ (of partial sums). For the sequence in exercise 28.1.3, the series $\sum_{k=1}^{\infty} a_k$ is the sequence $\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\right)$ (of partial sums).

Q.28.4. (Solution to 28.1.7) For the sequence (a_k) given in 28.1.2 the corresponding sequence of partial sums $(1, 0, 1, 0, 1, \dots)$ does not converge. Thus the sequence $(1, -1, 1, -1, \dots)$ is not summable. Equivalently, the series $\sum_{k=1}^{\infty} (-1)^{k+1}$ diverges.

For the sequence (a_k) of 28.1.3, the n^{th} partial sum is $1 - 2^{-n}$ (see 28.1.3). Since $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - 2^{-n}\right) = 1$ (see proposition 4.3.8), we conclude that the sequence $(1/2, 1/4, 1/8, \dots)$ is summable; in other words, the series $\sum_{k=1}^{\infty} 2^{-k}$ converges. The sum of this series is 1; that is

$$\sum_{k=1}^{\infty} 2^{-k} = 1\,.$$

Q.28.5. (Solution to 28.1.10) Suppose that $\sum_{k=1}^{\infty} a_k = b$. If $s_n = \sum_{k=1}^{n} a_k$, then it is easy to see that for each n we may write a_n as $s_n - s_{n-1}$ (where we let $s_0 = 0$). Take limits as $n \to \infty$ to obtain

$$a_n = s_n - s_{n-1} \to b - b = 0\,.$$

Q.28.6. (Solution to 28.1.11) Assume that the series $\sum_{k=1}^{\infty} k^{-1}$ converges. Let $s_n = \sum_{k=1}^{n} k^{-1}$. Since the sequence (s_n) of partial sums is assumed to converge, it is Cauchy (by proposition 18.1.4). Thus there exists an index p such that $|s_n - s_p| < \frac{1}{2}$ whenever $n \geq p$. We obtain a contradiction by noting that

$$\begin{aligned} |s_{2p} - s_p| &= \sum_{k=p+1}^{2p} \frac{1}{k} \\ &\geq \sum_{k=p+1}^{2p} \frac{1}{2p} \\ &= \frac{p}{2p} \\ &= \frac{1}{2}\,. \end{aligned}$$

Q.28.7. (Solution to 28.1.17) Let $\sum a_k$ be a convergent series in the normed linear space V. For each n in $\mathbb{N}$ let $s_n = \sum_{k=1}^{n} a_k$. Then (s_n) is a convergent sequence. By proposition 18.1.4 it is Cauchy. Thus given $\epsilon > 0$ we may choose n_0 in $\mathbb{N}$ so that $n > m \geq n_0$ implies

$$\left\| \sum_{k=m+1}^{n} a_k \right\| = \|s_n - s_m\| < \epsilon\,. \tag{120}$$

For the second assertion of the proposition, suppose that V is complete. Suppose further that (a_k) is a sequence in V for which there exists $n_0 \in \mathbb{N}$ such that (120) holds whenever $n > m \geq n_0$. (As above, $s_n = \sum_{k=1}^{n} a_k$.) This says that the sequence (s_n) of partial sums is Cauchy, and since V is complete, the sequence (s_n) converges. That is, the series $\sum a_k$ converges.

Q.28.8. (Solution to 28.1.19) Let $f_n(x) = x^n\big(1 + x^n\big)^{-1}$ for every $n \in \mathbb{N}$ and $x \in [-\delta, \delta]$. Also let $M_n = \delta^n(1-\delta)^{-1}$. Since $0 < \delta < 1$, the series $\sum M_n = \sum \delta^n(1-\delta)^{-1}$ converges (by problem 28.1.8). For $|x| \leq \delta$, we have $-x^n \leq |x|^n \leq \delta^n \leq \delta$; so $x^n \geq -\delta$ and $1 + x^n \geq 1 - \delta$. Thus

$$|f_n(x)| = \frac{|x|^n}{1 + x^n} \leq \frac{\delta^n}{1 - \delta} = M_n\,.$$

Thus

$$\|f_n\|_u = \sup\{|f_n(x)| \colon |x| \leq \delta\} \leq M_n$$

By the *Weierstrass M-test* (proposition 28.1.18), the series $\sum_{k=1}^{\infty} f_k$ converges uniformly.

Q.28.9. (Solution to 28.2.3) As was remarked after proposition 28.1.17, the convergence of a series is not affected by altering any finite number of terms. Thus without loss of generality we suppose that $a_{k+1} \leq \delta a_k$ for all k. Notice that $a_2 \leq \delta a_1$, $a_3 \leq \delta a_2 \leq \delta^2 a_1$, $a_4 \leq \delta^3 a_1$, *etc.* In general, $a_k \leq \delta^{k-1} a_1$ for all k.

The geometric series $\sum \delta^{k-1}$ converges by problem 28.1.8. Thus by the *comparison test* (proposition 28.2.2), the series $\sum a_k$ converges. The second conclusion follows similarly from the observations that $a_k \geq M^{k-1}a_1$ and that $\sum M^{k-1}$ diverges.

Q.28.10. (Solution to 28.3.2) Suppose that V is complete and that (a_k) is an absolutely summable sequence in V. We wish to show that (a_k) is summable. Let $\epsilon > 0$. Since $\sum\|a_k\|$ converges in $\mathbb{R}$ and $\mathbb{R}$ is complete, we may invoke the *Cauchy criterion* (proposition 28.1.17) to find an integer n_0 such that $n > m \geq n_0$ implies $\sum_{k=m+1}^{n}\|a_k\| < \epsilon$. But for all such m and n

$$\left\|\sum_{k=m+1}^{n} a_k\right\| \leq \sum_{k=m+1}^{n} \|a_k\| < \epsilon\,.$$

This, together with the fact V is complete, allows us to apply for a second time the *Cauchy criterion* and to conclude that $\sum a_k$ converges. That is, the sequence (a_k) is summable.

For the converse suppose that every absolutely summable sequence in V is summable. Let (a_k) be a Cauchy sequence in V. In order to prove that V is complete we must show that (a_k) converges. For each k in $\mathbb{N}$ we may choose a natural number p_k such that $\|a_n - a_m\| \leq 2^{-k}$ whenever $n > m \geq p_k$. Choose inductively a sequence (n_k) in $\mathbb{N}$ as follows. Let n_1 be any integer such that $n_1 \geq p_1$. Having chosen integers $n_1 < n_2 < \cdots < n_k$ in $\mathbb{N}$ so that $n_j \geq p_j$ for $1 \leq j \leq k$, choose n_{k+1} to be the larger of p_{k+1} and $n_k + 1$. Clearly, $n_{k+1} > n_k$ and $n_{k+1} \geq p_{k+1}$. Thus $\left(a_{n_k}\right)$ is a subsequence of (a_n) and (since $n_{k+1} > n_k \geq p_k$ for each k) $\left\|a_{n_{k+1}} - a_{n_k}\right\| < 2^{-k}$ for each k in $\mathbb{N}$. Let $y_k = a_{n_{k+1}} - a_{n_k}$ for each k. Then (y_n) is absolutely summable since $\sum\|y_k\| < \sum 2^{-k} = 1$. Consequently (y_k) is summable in V. That is, there exists b in V such that $\sum_{k=1}^{j} y_k \to b$ as $j \to \infty$. However, since $\sum_{k=1}^{j} y_k = a_{n_{j+1}} - a_{n_1}$, we see that

$$a_{n_{j+1}} \to a_{n_1} + b \qquad \text{as } j \to \infty\,.$$

This shows that $\left(a_{n_k}\right)$ converges. Since (a_n) is a Cauchy sequence having a convergent subsequence it too converges (proposition 18.1.5). But this is what we wanted to show.

Q.28.11. (Solution to 28.4.4)

(a) It follows immediately from

$$|(fg)(x)| = |f(x)||g(x)| \leq \|f\|_u\|g\|_u \qquad \text{for every } x \in S$$

that

$$\|fg\|_u = \sup\{|(fg)(x)|\colon x \in S\} \leq \|f\|_u\,\|g\|_u\,.$$

(b) Define f and g on $[0,2]$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 < x \leq 2 \end{cases}$$

and $g(x) = 1 - f(x)$. Then $\|f\|_u = \|g\|_u = 1$, but $\|fg\|_u = 0$.

Q.28.12. (Solution to 28.4.11) Since $\|x\| < 1$, the series $\sum_{k=0}^{\infty} \|x\|^k$ converges by problem 28.1.8. Condition (e) in the definition of normed algebras is that $\|xy\| \leq \|x\|\,\|y\|$. An easy inductive argument shows that $\|x^n\| \leq \|x\|^n$ for all n in $\mathbb{N}$. We know that $\|x\|^n \to 0$ (by proposition 4.3.8); so $\|x^n\| \to 0$ also. Thus (by proposition 22.2.3(d)) $x^n \to 0$ as $n \to \infty$. Furthermore, comparing $\sum_0^{\infty} \|x^k\|$ with the series $\sum_0^{\infty} \|x\|^k$ shows that the former converges (see proposition 28.2.2). But this says just that the series $\sum_0^{\infty} x^k$ converges absolutely. It then follows from proposition 28.3.2 that $\sum_0^{\infty} x^k$ converges. Letting $s_n = \sum_{k=0}^{n} x^k$ we see that

$$\begin{aligned} (\mathbf{1} - x) \sum_{k=0}^{\infty} x^k &= (\mathbf{1} - x) \lim s_n \\ &= \lim \big((\mathbf{1} - x) s_n\big) \\ &= \lim \big(\mathbf{1} - x^{n+1}\big) \\ &= \mathbf{1}. \end{aligned}$$

Similarly, $\big(\sum_0^{\infty} x^k\big)(\mathbf{1} - x) = \mathbf{1}$. This shows that $\mathbf{1} - x$ is invertible and that its inverse $(\mathbf{1} - x)^{-1}$ is the geometric series $\sum_0^{\infty} x^k$.

Q.28.13. (Solution to 28.4.14) Let $a \in \operatorname{Inv} A$. We show that r is continuous at a. Given $\epsilon > 0$ choose δ to be the smaller of the numbers $\frac{1}{2}\|a^{-1}\|^{-1}$ and $\frac{1}{2}\|a^{-1}\|^{-2}\epsilon$. Suppose that $\|y - a\| < \delta$ and prove that $\|r(y) - r(a)\| < \epsilon$. Let $x = \mathbf{1} - a^{-1}y$. Since

$$\|x\| = \|a^{-1}a - a^{-1}y\| \leq \|a^{-1}\|\|y - a\| < \|a^{-1}\|\delta \leq \|a^{-1}\|\tfrac{1}{2}\|a^{-1}\|^{-1} = \tfrac{1}{2}$$

we conclude from 28.4.11 and 28.4.12 that $\mathbf{1} - x$ is invertible and that

$$\|(\mathbf{1} - x)^{-1} - \mathbf{1}\| \leq \frac{\|x\|}{1 - \|x\|} \tag{121}$$

Thus

$$\begin{aligned} \|r(y) - r(a)\| &= \|y^{-1}(a - y)a^{-1}\| && \text{(by 28.4.10(e))} \\ &\leq \|y^{-1}a - \mathbf{1}\|\,\|a^{-1}\| \\ &= \|(a^{-1}y)^{-1} - \mathbf{1}\|\,\|a^{-1}\| && \text{(by 28.4.10(d))} \\ &= \|(\mathbf{1} - x)^{-1} - \mathbf{1}\|\,\|a^{-1}\| \\ &\leq \frac{\|x\|}{1 - \|x\|}\|a^{-1}\| && \text{(by inequality (121))} \\ &\leq 2\|x\|\,\|a^{-1}\| && \text{(because } \|x\| \leq \tfrac{1}{2}) \\ &< 2\|a^{-1}\|^2 \\ &\leq \epsilon. \end{aligned}$$

Q.28.14. (Solution to 28.4.17) Throughout the proof we use the notation introduced in the hint. To avoid triviality we suppose that (a_k) is not identically zero. Since u_n is defined to be $\sum_{k=0}^{n} c_k = \sum_{k=0}^{n}\sum_{j=0}^{k} a_j b_{k-j}$ it is clear that u_n can be obtained by finding the sum of each column of the matrix $\big[d_{jk}\big]$ and then adding these sums. On the other hand the expression

$$\sum_{k=0}^{n} a_{n-k}t_k = \sum_{k=0}^{n}\sum_{j=0}^{k} a_{n-k}b_j$$

is obtained by finding the sum of each row of the matrix $[d_{jk}]$ and then adding the sums. It is conceivable that someone might find the preceding argument too "pictorial", depending as it does on looking at a "sketch" of the matrix $[d_{jk}]$. It is, of course, possible to carry out the proof in a purely algebraic fashion. And having done so, it is also quite conceivable that one might conclude that the algebraic approach adds more to the amount of paper used than to the clarity of the argument. In any event, here, for those who feel more comfortable with it, is a formal verification of the same result.

$$\begin{aligned}
u_n &= \sum_{k=0}^{n} c_k \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k} a_j b_{k-j} \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k} d_{jk} \\
&= \sum_{k=0}^{n}\sum_{j=0}^{n} d_{jk} \\
&= \sum_{j=0}^{n}\sum_{k=0}^{n} d_{jk} \\
&= \sum_{j=0}^{n}\sum_{k=j}^{n} d_{jk} \\
&= \sum_{j=0}^{n}\sum_{k=j}^{n} a_j b_{k-j} \\
&= \sum_{j=0}^{n} a_j \sum_{r=0}^{n-j} b_r \\
&= \sum_{j=0}^{n} a_j t_{n-j} \\
&= \sum_{k=0}^{n} a_{n-k} t_k .
\end{aligned}$$

Now that equation (63) has been established we see that

$$u_n = \sum_{k=0}^{n} a_{n-k}b + \sum_{k=0}^{n} a_{n-k}(t_k - b)$$

$$= s_n b + \sum_{k=0}^{n} a_{n-k}(t_k - b).$$

Since $s_n b \to ab$, it remains only to show that the last term on the right approaches 0 as $n \to \infty$. Since

$$\begin{aligned}\left\|\sum_{k=0}^{n} a_{n-k}(t_k - b)\right\| &\le \sum_{k=0}^{n}\|a_{n-k}(t_k - b)\| \\ &\le \sum_{k=0}^{n}\|a_{n-k}\|\,\|(t_k - b)\| \\ &= \sum_{k=0}^{n}\alpha_{n-k}\beta_k\end{aligned}$$

it is sufficient to prove that given any $\epsilon > 0$ the quantity $\sum_{k=0}^{n}\alpha_{n-k}\beta_k$ is less than ϵ whenever n is sufficiently large.

Let $\alpha = \sum_{k=0}^{\infty}\|a_k\|$. Then $\alpha > 0$. Since $\beta_k \to 0$ there exists n_1 in $\mathbb{N}$ such that $k \ge n_1$ implies $\beta_k < \epsilon/(2\alpha)$. Choose $\beta > \sum_{k=0}^{n_1}\beta_k$. Since $\alpha_k \to 0$, there exists n_2 in $\mathbb{N}$ such that $k \ge n_2$ implies $\alpha_k < \epsilon/(2\beta)$.

Now suppose that $n \ge n_1 + n_2$. If $0 \le k \le n_1$, then $n - k \ge n - n_1 \ge n_2$, so that $\alpha_{n-k} < \epsilon/(2\beta)$. This shows that

$$\begin{aligned}p &= \sum_{k=0}^{n_1}\alpha_{n-k}\beta_k \\ &\le \epsilon(2\beta)^{-1}\sum_{k=0}^{n_1}\beta_k \\ &< \epsilon/2.\end{aligned}$$

Furthermore,

$$\begin{aligned}q &= \sum_{k=n_1+1}^{n}\alpha_{n-k}\beta_k \\ &\le \epsilon(2\alpha)^{-1}\sum_{k=n_1+1}^{n}\alpha_{n-k} \\ &\le \epsilon(2\alpha)^{-1}\sum_{j=0}^{\infty}\|a_j\| \\ &= \epsilon/2.\end{aligned}$$

Thus

$$\sum_{k=0}^{n}\alpha_{n-k}\beta_k = p + q < \epsilon\,.$$

Q.28.15. (Solution to 28.4.25) Let $0 < s < r$, let $M > 0$ be such that $\|a_k\| r^k \le M$ for every k in $\mathbb{N}$, and let $\rho = s/r$. Let $f_k(x) = a_k x^k$ for each k in $\mathbb{N}$ and x in $B_s(0)$.

For each such k and x

$$\begin{aligned}\|f_k(x)\| = \|a_k x^k\| &\leq \|a_k\|\|x\|^k \leq \|a_k\| s^k \\ &= \|a_k\| r^k \rho^k \leq M\rho^k .\end{aligned}$$

Thus $\|f_k\|_u \leq M\rho^k$ for each k. Since $0 < \rho < 1$, the series $\sum M\rho^k$ converges. Then, according to the *Weierstrass M-test* (proposition 28.1.18), the series $\sum a_k x^k = \sum f_k(x)$ converges uniformly on $B_s(0)$. The parenthetical comment in the statement of the proposition is essentially obvious: For $a \in B_r(0)$ choose s such that $\|a\| < s < r$. Since $\sum a_k x^k$ converges uniformly on $B_s(0)$, it converges at a (see problem 22.4.7).

Q.28.16. (Solution to 28.4.26) Let a be an arbitrary point of U. Let $\phi = \lim_{n\to\infty} d(f_n)$. We show that $\Delta F_a \simeq T$ where $T = \phi(a)$. We are supposing that $d(f_n) \to \phi$ (unif) on U. Thus given $\epsilon > 0$ we may choose N in $\mathbb{N}$ so that

$$\sup\{\|d(f_n)_x - \phi(x)\| : x \in U\} < \tfrac{1}{8}\epsilon$$

whenever $x \in U$ and $n \geq N$. Let $g_n = f_n - f_N$ for all $n \geq N$. Then for all such n and all $x \in U$ we have

$$\|d(g_n)_x\| \leq \|d(f_n)_x - \phi(x)\| + \|\phi(x) - d(f_N)_x\| < \tfrac{1}{4}\epsilon .$$

Also it is clear that

$$\|d(g_n)_x - d(g_n)_a\| \leq \|d(g_n)_x\| + \|d(g_n)_a\| < \tfrac{1}{2}\epsilon$$

for $x \in U$ and $n \geq N$. According to corollary 26.1.8

$$\|\Delta(g_n)_a(h) - d(g_n)_a(h)\| \leq \tfrac{1}{2}\epsilon\|h\|$$

whenever $n \geq N$ and h is a vector such that $a + h \in U$. Thus

$$\|\Delta(f_n)_a(h) - d(f_n)_a(h) - \Delta(f_N)_a(h) + d(f_N)_a(h)\| \leq \tfrac{1}{2}\epsilon\|h\|$$

when $n \geq N$ and $a + h \in U$. Taking the limit as $n \to \infty$ we obtain

$$\|(\Delta F_a(h) - Th) - (\Delta(f_N)_a(h) - d(f_N)_a(h))\| \leq \frac{1}{2}\epsilon\|h\| \tag{122}$$

for h such that $a + h \in U$. Since f_N is differentiable, $\Delta(f_N)_a \simeq d(f_N)_a$; thus there exists $\delta > 0$ such that $B_\delta(a) \subseteq U$ and

$$\|\Delta(f_N)_a(h) - d(f_N)_a(h)\| < \tfrac{1}{2}\epsilon\|h\| \tag{123}$$

for all h such that $\|h\| < \delta$. From (122) and (123) it is clear that

$$\|\Delta F_a(h) - Th\| < \epsilon\|h\|$$

whenever $\|h\| < \delta$. Thus $\Delta F_a \simeq T$, which shows that F is differentiable at a and

$$dF_a = T = \lim_{n\to\infty} d(f_n)_a .$$

Q.29. Exercises in chapter 29

Q.29.1. (Solution to 29.1.2) Let $U = V = \mathbb{R}$ and $f(x) = x^3$ for all x in $\mathbb{R}$. Although f is continuously differentiable and does have an inverse, it is not $\mathcal{C}^1$-invertible. The inverse function $x \mapsto x^{\frac{1}{3}}$ is not differentiable at 0.

Q.29.2. (Solution to 29.1.4) Set $y = x^2 - 6x + 5$ and solve for x in terms of y. After completing the square and taking square roots we have

$$|x - 3| = \sqrt{y+4}\,.$$

Thus there are two solutions $x = 3 + \sqrt{y+4}$ and $x = 3 - \sqrt{y+4}$. The first of these produces values of x no smaller than 3 and the second produces values no larger than 3. Thus for $x = 1$ we choose the latter. A local $\mathcal{C}^1$-inverse of f is given on the interval $f^{\rightarrow}(0, 2) = (-3, 5)$ by

$$f_{\text{loc}}^{-1}(y) = 3 - \sqrt{y+4}\,.$$

Q.29.3. (Solution to 29.1.7) In order to apply the chain rule to the composite function $f_{\text{loc}}^{-1} \circ f$ we need to know that *both* f and f_{loc}^{-1} are differentiable. But differentiability of f_{loc}^{-1} was not a hypothesis. Indeed, the major difficulty in proving the *inverse function theorem* is showing that a local $\mathcal{C}^1$-inverse of a $\mathcal{C}^1$-function is in fact differentiable (at points where its differential does not vanish). Once that is known, the argument presented in 29.1.7 correctly derives the formula for $Df_{\text{loc}}^{-1}(b)$.

Q.29.4. (Solution to 29.2.2) Let $f(x, y) = x^2 y + \sin(\frac{\pi}{2} x y^2) - 2$ for all x and y in $\mathbb{R}$.

(a) There exist a neighborhood V of 1 and a function $h\colon V \to \mathbb{R}$ that satisfy

$$\begin{aligned} &\text{(i)} \quad h(1) = 2; \quad \text{and} \\ &\text{(ii)} \quad f(x, h(x)) = 0 \quad \text{for all } x \text{ in } V. \end{aligned}$$

(b) Let $G(x, y) = (x, f(x, y))$ for all $x, y \in \mathbb{R}$. Then G is continuously differentiable and

$$\left[dG_{(1,2)}\right] = \begin{bmatrix} 1 & 0 \\ 4 + 2\pi & 1 + 2\pi \end{bmatrix}.$$

Thus $dG_{(1,2)}$ is invertible, so by the *inverse function theorem* G has a local $\mathcal{C}^1$-inverse, say H, defined on some neighborhood W of $(1, 0) = G(1, 2)$. Let $V = \{x \colon (x, 0) \in W\}$ and $h(x) = H^2(x, 0)$ for all x in V. The function h is in $\mathcal{C}^1$ because H is. Condition (i) is satisfied by h since

$$\begin{aligned} (1, 2) &= H(G(1, 2)) \\ &= H(1, f(1, 2)) \\ &= H(1, 0) \\ &= (H^1(1, 0)\,,\, H^2(1, 0)) \\ &= (H^1(1, 0)\,,\, h(1)) \end{aligned}$$

and (ii) holds because

$$\begin{aligned}(x,0) &= G(H(x,0)) \\ &= G(H^1(x,0)\,,\,H^2(x,0)) \\ &= (H^1(x,0)\,,\,f(H^1(x,0),H^2(x,0))) \\ &= (x\,,\,f(x,h(x)))\end{aligned}$$

for all x in V.

(c) Let G, H, and h be as in (b). By the *inverse function theorem*

$$\begin{aligned}\left[dH_{(1,0)}\right] &= \left[dH_{G(1,2)}\right] \\ &= \left[dG_{(1,2)}\right]^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 4+2\pi & 1+2\pi \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{4+2\pi}{1+2\pi} & \frac{1}{1+2\pi} \end{bmatrix}.\end{aligned}$$

Then $\frac{dy}{dx}$ at $(1,2)$ is just $h'(1)$ and

$$h'(1) = H^2_1(1,0) = -\frac{4+2\pi}{1+2\pi}\,.$$

Q.29.5. (Solution to 29.2.4) Let $f(x,y,z) = x^2z + yz^2 - 3z^3 - 8$ for all x, y, $z \in \mathbb{R}$.

(a) There exist a neighborhood V of $(3,2)$ and a function $h\colon V \to \mathbb{R}$ that satisfy

(i) $h(3,2) = 1$; and

(ii) $f(x,y,h(x,y)) = 0$ for all x, $y \in V$.

(b) Let $G(x,y,z) := (x,y,f(x,y,z))$ for all x, y, $z \in \mathbb{R}$. Then

$$\left[dG_{(3,2,1)}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 1 & 4 \end{bmatrix}$$

so $dG_{(3,2,1)}$ is invertible. By the *inverse function theorem* G has a local $\mathcal{C}^1$-inverse H defined on some neighborhood W of $(3,2;0) = G(3,2,1)$. Write $H = (H^1, H^2)$ where ran $H^1 \subseteq \mathbb{R}^2$ and ran $H^2 \subseteq \mathbb{R}$. Let $V = \{(x,y) : (x,y,0) \in W\}$ and $h(x,y) = H^2(x,y;0)$. The function h belongs to $\mathcal{C}^1$ because H does. Now condition (i) holds because

$$\begin{aligned}(3,2;1) &= H(G(3,2;1)) \\ &= H(3,2;f(3,2,1)) \\ &= H(3,2;0) \\ &= (H^1(3,2;0)\,;\,H^2(3,2;0)) \\ &= (H^1(3,2;0)\,;\,h(3,2))\end{aligned}$$

and condition (ii) follows by equating the third components of the first and last terms of the following computation

$$\begin{aligned}(x, y; 0) &= G(H(x, y; 0)) \\ &= G(H^1(x, y; 0); H^2(x, y; 0)) \\ &= (H^1(x, y; 0)\,;\, f(H^1(x, y; 0); H^2(x, y; 0))) \\ &= (x, y\,;\, f(x, y; h(x, y)))\,.\end{aligned}$$

(c) We wish to find $\left(\frac{\partial z}{\partial x}\right)_y$ and $\left(\frac{\partial z}{\partial y}\right)_x$ at $(3, 2, 1)$; that is, $h_1(3, 2)$ and $h_2(3, 2)$, respectively. The *inverse function theorem* tells us that

$$\begin{aligned}\left[dH_{(3,2,0)}\right] &= \left[dH_{G(3,2,1)}\right] \\ &= \left[dG_{(3,2,1)}\right]^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 1 & 4 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.\end{aligned}$$

Thus at $(3, 2, 1)$

$$\left(\frac{\partial z}{\partial x}\right)_y = h_1(3, 2) = \frac{\partial H^2}{\partial x}(3, 2) = -\frac{3}{2}$$

and

$$\left(\frac{\partial z}{\partial y}\right)_x = h_2(3, 2) = \frac{\partial H^2}{\partial y}(3, 2) = -\frac{1}{4}\,.$$

Q.29.6. (Solution to 29.2.14) Let $f = (f^1, f^2)$ where

$$f^1(u, v; x, y) = 2u^3vx^2 + v^2x^3y^2 - 3u^2y^4$$

and

$$f^2(u, v; x, y) = 2uv^2y^2 - uvx^2 + u^3xy - 2\,.$$

(a) There exist a neighborhood V of (a, b) in $\mathbb{R}^2$ and a function $h\colon V \to \mathbb{R}^2$ that satisfy

(i) $h(a, b) = (c, d)$; and

(ii) $f(u, v; h(u, v)) = (0, 0)$ for all u, $v \in V$.

(b) Let $G(u,v;x,y) := (u,v;f(u,v;x,y))$ for all u, v, x, $y \in \mathbb{R}$. Then G is continuously differentiable and

$$\left[dG_{(1,1)}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 7 & -10 \\ 4 & 3 & -1 & 3 \end{bmatrix} .$$

Since $\det\left[dG_{(1,1)}\right] = 11 \neq 0$, we know from the *inverse function theorem* that G is locally $\mathcal{C}^1$-invertible at $(1,1)$. That is, there exist a neighborhood W of $G(1,1;1,1) = (1,1;0,0)$ in $\mathbb{R}^4$ and a local $\mathcal{C}^1$-inverse $H : W \to \mathbb{R}^4$ of G. Write H in terms of its component functions, $H = (H^1, H^2)$ where $\operatorname{ran} H^1$ and $\operatorname{ran} H^2$ are contained in $\mathbb{R}^2$, and set $h(u,v) = H^2(u,v;0,0)$ for all (u,v) in $V := \{(u,v) : (u,v;0,0) \in W\}$. Then V is a neighborhood of $(1,1)$ in $\mathbb{R}^2$ and the function h is continuously differentiable because H is. We conclude that $h(1,1) = (1,1)$ from the following computation.

$$\begin{aligned}
(1,1;1,1) &= H(G(1,1;1,1)) \\
&= H(1,1;f(1,1;1,1)) \\
&= (H^1(1,1;f(1,1;1,1))\,;\,H^2(1,1;f(1,1;1,1))) \\
&= (1,1\,;\,H^2(1,1;0,0)) \\
&= (1,1\,;\,h(1,1))\,.
\end{aligned}$$

And from

$$\begin{aligned}
(u,v;0,0) &= G(H(u,v;0,0)) \\
&= G(H^1(u,v;0,0);H^2(u,v;0,0)) \\
&= (H^1(u,v;0,0)\,;\,f(H^1(u,v;0,0);H^2(u,v;0,0))) \\
&= (u,v\,;\,f(u,v;h(u,v)))
\end{aligned}$$

we conclude that (ii) holds; that is,

$$f(u,v;h(u,v)) = (0,0)$$

for all u, $v \in V$.

Q.30. Exercises in chapter 30

Q.30.1. (Solution to 30.1.2)

(a) If B is bilinear, then

$$\begin{aligned}
B(u+v,x+y) &= B(u,x+y) + B(v,x+y) \\
&= B(u,x) + B(u,y) + B(v,x) + B(v,y)
\end{aligned}$$

(b) If B is linear, then

$$\begin{aligned}
B(u+v,x+y) &= B\big((u,x)+(v,y)\big) \\
&= B(u,x) + B(v,y)
\end{aligned}$$

(c) If B is bilinear, then

$$\begin{aligned} B(\alpha u, \alpha x) &= \alpha\, B(u, \alpha x) \\ &= \alpha^2 B(u, x) \end{aligned}$$

(d) If B is linear, then

$$\begin{aligned} B(\alpha u, \alpha x) &= B\big(\alpha(u, x)\big) \\ &= \alpha B(u, x) \end{aligned}$$

Q.30.2. (Solution to 30.1.15) There are a lot of details. The first thing to check is that the notation $F\colon \mathfrak{B}(V, \mathfrak{B}(V, W)) \to \mathfrak{B}^2(V, W)$ appearing in (78) is not overly optimistic. That is, show that if ϕ belongs to $\mathfrak{B}(V, \mathfrak{B}(V, W))$, then $\widehat{\phi} = F(\phi)$ really does belong to $\mathfrak{B}^2(V, W)$. The computations are utterly routine: if t, u, $v \in V$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} \widehat{\phi}(t + u, v) &= \big(\phi(t + u)\big)(v) \\ &= \big(\phi(t) + \phi(u)\big)(v) \\ &= \big(\phi(t)\big)(v) + \big(\phi(u)\big)(v) \\ &= \widehat{\phi}(t, v) + \widehat{\phi}(u, v) \end{aligned}$$

$$\begin{aligned} \widehat{\phi}(\alpha u, v) &= \big(\phi(\alpha u)\big)(v) \\ &= \big(\alpha\phi(u)\big)(v) \\ &= \alpha\,\big(\phi(u)\big)(v) \\ &= \alpha\,\widehat{\phi}(u, v) \end{aligned}$$

$$\begin{aligned} \widehat{\phi}(t, u + v) &= \big(\phi(t)\big)(u + v) \\ &= \big(\phi(t)\big)(u) + \big(\phi(t)\big)(v) \\ &= \widehat{\phi}(t, u) + \widehat{\phi}(t, v) \end{aligned}$$

$$\begin{aligned} \widehat{\phi}(u, \alpha v) &= \big(\phi(u)\big)(\alpha v) \\ &= \alpha\big(\phi(u)\big)(v) \\ &= \alpha\,\widehat{\phi}(u, v) \end{aligned}$$

$$\begin{aligned} \|\widehat{\phi}(u, v)\| &= \|\big(\phi(u)\big)(v)\| \\ &\le \|\phi(u)\|\,\|v\| \\ &\le \|\phi\|\|u\|\|v\| \end{aligned}$$

Two things are perhaps worth mentioning. First, although the computations above that prove linearity in the first variable may *look* like those that demonstrate linearity in the second variable, at most places the reasons are quite different.

Second, the last chain of inequalities established not only that $\widehat{\phi}$ is bounded but also that $\|\widehat{\phi}\| \le \|\phi\|$.

Having verified that $\widehat{\phi}$ is indeed a member of $\mathfrak{B}^2(V,W)$, we next establish that the mapping $F\colon \phi \mapsto \widehat{\phi}$ is linear. Again the computations require no deep thinking: for all u, $v \in V$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}(\phi+\psi)\widehat{\ }\,(u,v) &= \big((\phi+\psi)(u)\big)(v)\\ &= \big(\phi(u)+\psi(u)\big)(v)\\ &= \big(\phi(u)\big)(v) + \big(\psi(u)\big)(v)\\ &= \widehat{\phi}(u,v) + \widehat{\psi}(u,v)\end{aligned}$$

$$\begin{aligned}(\alpha\,\phi)\widehat{\ }\,(u,v) &= \big((\alpha\,\phi)(u)\big)(v)\\ &= \big(\alpha\,(\phi)(u)\big)(v)\\ &= \alpha\big((\phi)(u)\big)(v)\\ &= \alpha\widehat{\phi}(u,v)\end{aligned}$$

We have already shown that $\|F(\phi)\| = \|\widehat{\phi}\| \le \|\phi\|$, so we know that F is a *bounded* linear map—and that its norm is no greater than 1. To prove that F is an isometry we need the reverse inequality. Since

$$\|\big(\phi(u)\big)(v)\| = \|\widehat{\phi}(u,v)\| \le \|\widehat{\phi}\|\|u\|\|v\|$$

holds for all u, $v \in V$ and $\phi \in \mathfrak{B}(V,\mathfrak{B}(V,V))$, we conclude that

$$\|\phi(u)\| \le \|\widehat{\phi}\|\|u\|$$

for all $u \in V$ and $\phi \in \mathfrak{B}(V,\mathfrak{B}(V,V))$; and this in turn implies that

$$\|\phi\| \le \|\widehat{\phi}\|$$

for all $\phi \in \mathfrak{B}(V,\mathfrak{B}(V,V))$.

All that remains is to show that F is surjective. Given $\psi \in \mathfrak{B}^2(V,W)$, we must find $\phi \in \mathfrak{B}(V,\mathfrak{B}(V,W))$ such that $\widehat{\phi} = \psi$. For each $u \in V$ define a map $\phi(u)\colon v \mapsto \psi(u,v)$ from V into W. Each $\phi(u)$ is linear since ψ is linear in its second variable. Since

$$\|\big(\phi(u)\big)(v)\| = \|\psi(u,v)\| \le \|\psi\|\|u\|\|v\|$$

for all u, $v \in V$, we see that each $\phi(u)$ is bounded. Thus, $\phi(u) \in \mathfrak{B}(V,W)$ for all u. Next we show that the mapping

$$\phi\colon V \to \mathfrak{B}(V,W)\colon u \mapsto \phi(u)$$

is linear and bounded.

$$\begin{aligned}\big(\phi(t+u)\big)(v) &= \psi(t+u,v)\\ &= \psi(t,v)+\psi(u,v)\\ &= \big(\phi(t)\big)(v)+\big(\phi(u)\big)(v)\\ &= \big(\phi(t)+\phi(u)\big)(v)\end{aligned}$$

for all t, u, $v \in V$; and so, $\phi(t+u) = \phi(t)+\phi(u)$ for all t, $u \in V$.

$$\begin{aligned}\big(\phi(\alpha u)\big)(v) &= \psi(\alpha u,v)\\ &= \alpha\,\psi(u,v)\\ &= \alpha\,\big(\phi(u)\big)(v)\end{aligned}$$

for all u, $v \in V$ and $\alpha \in \mathbb{R}$; and so, $\phi(\alpha u) = \alpha\,\phi(u)$ for all $u \in V$ and $\alpha \in \mathbb{R}$. Thus ϕ is linear.

To establish the boundedness of ϕ, notice that

$$\|\big(\phi(u)\big)(v)\| = \|\psi(u,v)\| \le \|\psi\|\|u\|\|v\|$$

for all u, $v \in V$, so that $\|\phi(u)\| \le \|\psi\|\|u\|$ for all $u \in V$. Thus ϕ is bounded (and $\|\phi\| \le \|\psi\|$). So now we know that ϕ belongs to $\mathfrak{B}(V,\mathfrak{B}(V,W))$.

Finally, we notice that

$$\widehat{\phi}(u,v) = \big(\phi(u)\big)(v) = \psi(u,v)$$

for all u, $v \in V$, so that $F(\phi) = \widehat{\phi} = \psi$.

Q.30.3. (Solution to 30.2.2) For each $u \in V$ let E_u be the 'evaluate at u' function on $\mathfrak{B}(V,W)$; that is

$$E_u \colon \mathfrak{B}(V,W) \to W : T \mapsto Tu$$

Then, by hypothesis, the function $D_u f = E_u \circ df$ is differentiable on U. Then, using the *chain rule* we see that for all u, $v \in V$ and $a \in U$

$$\begin{aligned}\big(D_v(D_u(f)\big)(a) &= d(D_u f)_a(v)\\ &= d(E_u \circ df)_a(v)\\ &= (E_u \circ d^2 f_a)(v)\\ &= \big(d^2 f_a(v)\big)(u)\\ &= d^2 f_a(v,u)\end{aligned}$$

Q.30.4. (Solution to 30.2.15) Let $\alpha \in \mathbb{R}$. For this exercise it will be convenient to introduce a notation for the operator "multiplication by α" on $\mathbb{R}$.

$$M_\alpha \colon \mathbb{R} \to \mathbb{R} \colon x \mapsto \alpha x$$

Clearly M_α belongs to $\mathfrak{B}(\mathbb{R},\mathbb{R})$ and has norm α.

(a) For all $x \in \mathbb{R}$

$$\begin{aligned}\Delta f_a(x) &= f(a+x) - f(a)\\ &= x^3 + (3a-2)x^2 + (3a^2-4a+5)x\end{aligned}$$

(b) We already know (from 8.4.16) that $df_a(x) = M_{f'(a)}\, x = (3a^2 - 4a + 5)x$; so, as we expect,

$$\lim_{x\to 0} \frac{\Delta f_a(x) - df_a(x)}{x} = \lim_{x\to 0} x^2 - (3a-2)x = 0$$

(c) For all $x,\ y \in \mathbb{R}$

$$\Delta(df)_a(x)(y) = df_{a+x}(y) - df_a(y) = 3x^2y + (6a-4)xy$$

(d) This is an unpleasant, but entirely elementary, exercise in high school algebra. For all $x,\ y \in \mathbb{R}$ we have (by (81))

$$\begin{aligned}\Delta^2 f_a(x)(y) &= f(a+x+y) - f(a+x) - f(a+y) + f(a)\\ &= 3xy^2 + 3x^2y + (6a-4)xy\end{aligned}$$

(e) It is clear from (c) and (d) that

$$\frac{|\Delta(df)_a(x)(y) - \Delta^2 f_a(x)(y)|}{|x|\,|y|} = 3|y| \longrightarrow 0 \text{ as } (x,y) \longrightarrow \mathbf{0}$$

(f) In discovering the identity of $d^2 f_a$ it may be helpful to simplify notation just a bit. Let $g = df$. Since $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$, the function df_a belongs to $\mathfrak{B}(\mathbb{R},\mathbb{R})$, and so $g \in \mathcal{F}(\mathbb{R},\mathfrak{B}(\mathbb{R},\mathbb{R}))$. According to part (c)

$$\Delta g_a(x) = M_{3x^2+(6a-4)x}\,.$$

The differential of g at a should be a bounded linear map from $\mathbb{R}$ into $\mathfrak{B}(\mathbb{R},\mathbb{R})$ whose difference from Δg_a should lie in $\mathfrak{o}(\mathbb{R},\mathfrak{B}(\mathbb{R},\mathbb{R}))$. Isn't it almost obvious from the equality above that

$$dg_a \colon x \mapsto M_{(6a-4)x}$$

will work? We check:

$$\frac{\left\|\Delta g_a(x) - M_{(6a-4)x}\right\|}{|x|} = \frac{\|M_{3x^2}\|}{|x|} = \frac{3x^2}{|x|} = 3|x| \longrightarrow 0 \text{ as } x \longrightarrow 0\,.$$

To show that $d^2 f_a$ and $\Delta^2 f_a$ are bitangent, write them as functions of two variables and compute:

$$\begin{aligned}\frac{\Delta^2 f_a(x,y) - d^2 f_a(x,y)}{xy} &= \frac{\big((6a-4)xy + 3x^2y + 3xy^2\big) - \big((6a-4)xy + 3x^2y\big)}{xy}\\ &= 3y \longrightarrow 0 \text{ as } (x,y) \longrightarrow \mathbf{0}.\end{aligned}$$

Q.30.5. (Solution to 30.2.16.) To simplify notation let $g = df$. Use the unit square to parametrize the parallelogram $p(s,t) := a + sh + tj$ for $0 \le s,t \le 1$. Then invoke the (first order) *mean value theorem* 26.1.14 to obtain

$$\left(\int_0^1 d^2 f_{p(s,t)}\, ds\right)(h) = \left(\int_0^1 dg_{(a+tj)+sh}\, ds\right)(h) = \Delta g_{a+tj}(h) \tag{124}$$

Use the same theorem a second time to get

$$\begin{aligned}
\int_0^1 \int_0^1 d^2 f_{p(s,t)} ds\, dt\, (h)(j) &= \int_0^1 \left(\int_0^1 d^2 f_{p(s,t)} ds\, (h) \right) dt\, (j) \\
&= \int_0^1 \Delta g_{a+tj}(h)\, dt\, (j) \qquad \text{(by (124))} \\
&= \int_0^1 df_{a+h+tj} dt\, (j) - \int_0^1 df_{a+tj} dt\, (j) \\
&= \Delta f_{a+h}(j) - \Delta f_a(j) \\
&= \Delta^2 f_a(h)(j)
\end{aligned}$$

Q.31. Exercises in appendix D

Q.31.1. (Solution to D.1.1)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Q.31.2. (Solution to D.3.2) First observe that the operation $\wedge$ is commutative and associative. (The former is obvious and the latter may be easily checked by means of a truth table.) Therefore if A, B, and C are propositions

$$\begin{aligned} A \wedge (B \wedge C) &\text{ iff } (A \wedge B) \wedge C \\ &\text{ iff } (B \wedge A) \wedge C \\ &\text{ iff } B \wedge (A \wedge C). \end{aligned} \tag{125}$$

It then follows that

$$\begin{aligned} (\exists x \in S)(\exists y \in T)\, P(x,y) &\text{ iff } (\exists x \in S)(\exists y)((y \in T) \wedge P(x,y)) \\ &\text{ iff } (\exists x)\big((x \in S) \wedge (\exists y)((y \in T) \wedge P(x,y))\big) \\ &\text{ iff } (\exists x)(\exists y)\big((x \in S) \wedge ((y \in T) \wedge P(x,y))\big) \\ &\text{ iff } (\exists x)(\exists y)\big((y \in T) \wedge ((x \in S) \wedge P(x,y))\big) \quad \text{(by (125))} \\ &\text{ iff } (\exists y)(\exists x)\big((y \in T) \wedge ((x \in S) \wedge P(x,y))\big) \\ &\text{ iff } (\exists y)\big((y \in T) \wedge (\exists x)((x \in S) \wedge P(x,y))\big) \\ &\text{ iff } (\exists y)((y \in T) \wedge (\exists x \in S)\, P(x,y)) \\ &\text{ iff } (\exists y \in T)(\exists x \in S)\, P(x,y). \end{aligned}$$

Notice that at the third and sixth steps we used the remark made in the last paragraph of section D.1.

Q.31.3. (Solution to D.4.4)

(1) P	(2) Q	(3) $P \Rightarrow Q$	(4) $\sim P$	(5) $Q \vee (\sim P)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The third and fifth columns have the same truth values.

Q.32. Exercises in appendix F

Q.32.1. (Solution to F.2.4) If S, T, and U are sets, then

$$\begin{aligned} x \in S \cup (T \cap U) \quad &\text{iff} \quad x \in S \text{ or } x \in T \cap U \\ &\text{iff} \quad x \in S \text{ or } (x \in T \text{ and } x \in U) \\ &\text{iff} \quad (x \in S \text{ or } x \in T) \text{ and } (x \in S \text{ or } x \in U) \\ &\text{iff} \quad x \in S \cup T \text{ and } x \in S \cup U \\ &\text{iff} \quad x \in (S \cup T) \cap (S \cup U). \end{aligned}$$

Problem D.1.4 was used to get the third line.

Q.32.2. (Solution to F.2.9) If T is a set and $\mathfrak{S}$ is a family of sets, then

$$\begin{aligned} x \in T \cup \left(\bigcap \mathfrak{S}\right) \quad &\text{iff} \quad x \in T \text{ or } x \in \bigcap \mathfrak{S} \\ &\text{iff} \quad x \in T \text{ or } (\forall S \in \mathfrak{S})\, x \in S \\ &\text{iff} \quad (\forall S \in \mathfrak{S})(x \in T \text{ or } x \in S) \\ &\text{iff} \quad (\forall S \in \mathfrak{S})\, x \in T \cup S \\ &\text{iff} \quad x \in \bigcap \{T \cup S \colon S \in \mathfrak{S}\}. \end{aligned}$$

To obtain the third line we used the principle mentioned in the last paragraph of section D.1 of appendix D.

Q.32.3. (Solution to F.3.3) Here is one proof: A necessary and sufficient condition for an element x to belong to the complement of $S \cup T$ is that it not belong to S or to T. This is the equivalent to its belonging to both S^c and T^c, that is, to the intersection of the complements of S and T.

A second more “formalistic” proof looks like this :

$$\begin{aligned} x \in (S \cup T)^c \quad &\text{iff} \quad x \notin S \cup T \\ &\text{iff} \quad \sim (x \in S \cup T) \\ &\text{iff} \quad \sim (x \in S \text{ or } x \in T) \\ &\text{iff} \quad \sim (x \in S) \text{ and } \sim (x \in T) \\ &\text{iff} \quad x \notin S \text{ and } x \notin T \\ &\text{iff} \quad x \in S^c \text{ and } x \in T^c \\ &\text{iff} \quad x \in S^c \cap T^c. \end{aligned}$$

This second proof is not entirely without merit: at each step only one definition or fact is used. (For example, the result presented in example D.4.1 justifies the fourth “iff”.) But on balance most readers, unless they are very unfamiliar with the material, would probably prefer the first version. After all, it’s easier to read English than to translate code.

Q.32.4. (Solution to F.3.5) Here is another formalistic proof. It is a good idea to try and rewrite it in ordinary English.

$$\begin{aligned} x \in (\bigcup \mathfrak{S})^c \quad &\text{iff} \quad x \notin \bigcup \mathfrak{S} \\ &\text{iff} \quad \sim (x \in \bigcup \mathfrak{S}) \\ &\text{iff} \quad \sim (\exists S \in \mathfrak{S})(x \in S) \\ &\text{iff} \quad (\forall S \in \mathfrak{S}) \sim (x \in S) \\ &\text{iff} \quad (\forall S \in \mathfrak{S})(x \notin S) \\ &\text{iff} \quad (\forall S \in \mathfrak{S})(x \in S^c) \\ &\text{iff} \quad x \in \bigcap \{S^c \colon S \in \mathfrak{S}\}. \end{aligned}$$

Q.32.5. (Solution to F.3.9) To see that $S \setminus T$ and T are disjoint, notice that

$$\begin{aligned} (S \setminus T) \cap T &= S \cap T^c \cap T \\ &= S \cap \emptyset \\ &= \emptyset. \end{aligned}$$

Furthermore,

$$\begin{aligned} (S \setminus T) \cup T &= (S \cap T^c) \cup T \\ &= (S \cup T) \cap (T^c \cup T) \\ &= S \cup T. \end{aligned}$$

As usual S and T are regarded as belonging to some universal set, say U. Then $T^c \cup T$ is all of U and its intersection with $S \cup T$ (which is contained in U) is just $S \cup T$.

Q.32.6. (Solution to F.3.10) We know from proposition F.1.2 (e) that $S \cup T = S$ if and only if $T \subseteq S$. But proposition F.3.9 tells us that $S \cup T = (S \setminus T) \cup T$. Thus $(S \setminus T) \cup T = S$ if and only if $T \subseteq S$.

Q.33. Exercises in appendix G

Q.33.1. (Solution to G.1.10) If $x + x = x$, then

$$\begin{aligned} x &= x + 0 \\ &= x + (x + (-x)) \\ &= (x + x) + (-x) \\ &= x + (-x) \\ &= 0. \end{aligned}$$

Q.33.2. (Solution to G.1.12) Use associativity and commutativity of addition.

$$\begin{aligned}(w+x)+(y+z) &= ((w+x)+y)+z\\ &= (w+(x+y))+z\\ &= ((x+y)+w)+z\\ &= z+((x+y)+w)\\ &= z+(x+(y+w)).\end{aligned}$$

The first, second, and last equalities use associativity of addition; steps 3 and 4 use its commutativity.

Q.34. Exercises in appendix H

Q.34.1. (Solution to H.1.5) By definition $x > 0$ holds if and only if $0 < x$, and this holds (again by definition) if and only if $x - 0 \in \mathbb{P}$. Since $-0 = 0$ (which is obvious from $0 + 0 = 0$ and the fact that the additive identity is unique), we conclude that $x > 0$ if and only if

$$x = x + 0 = x + (-0) = x - 0 \in \mathbb{P}\,.$$

Q.34.2. (Solution to H.1.6) By the preceding exercise $x > 0$ implies that $x \in \mathbb{P}$; and $y < z$ implies $z - y \in \mathbb{P}$. Since $\mathbb{P}$ is closed under multiplication, $x(z-y)$ belongs to $\mathbb{P}$. Thus

$$\begin{aligned}xz - xy &= xz + (-(xy))\\ &= xz + x(-y) && \text{by problem G.4.4}\\ &= x(z + (-y))\\ &= x(z-y) \in \mathbb{P}\,.\end{aligned}$$

This shows that $xy < xz$.

Q.34.3. (Solution to H.1.12) Since $0 < w < x$ and $y > 0$, we may infer from exercise H.1.6 that $yw < yx$. Similarly, we obtain $xy < xz$ from the conditions $0 < y < z$ and $x > 0$ (which holds by the transitivity of $<$, proposition H.1.3). Then

$$wy = yw < yx = xy < xz\,.$$

Thus the desired inequality $wy < xz$ follows (again by transitivity of $<$).

Q.35. Exercises in appendix I

Q.35.1. (Solution to I.1.3) Since 1 belongs to A for every $A \in \mathfrak{A}$, it is clear that $1 \in \cap\mathfrak{A}$. If $x \in \cap\mathfrak{A}$, then $x \in A$ for every $A \in \mathfrak{A}$. Since each set A in $\mathfrak{A}$ is inductive, $x + 1 \in A$ for every $A \in \mathfrak{A}$. That is, $x + 1 \in \cap\mathfrak{A}$.

Q.35.2. (Solution to I.1.10) Let S be the set of all natural numbers for which the assertion is true. Certainly 1 belongs to S. If $n \in S$, then $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Therefore

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1)$$
$$= \frac{1}{2}n(n+1) + (n+1)$$
$$= \frac{1}{2}(n+1)(n+2),$$

which shows that $n+1 \in S$. Thus S is an inductive subset of $\mathbb{N}$. We conclude from corollary I.1.8 that $S = \mathbb{N}$. In other words, the assertion holds for all $n \in \mathbb{N}$.

Q.35.3. (Solution to I.1.18) Let K be a subset of $\mathbb{N}$ that has no smallest member. We show $K = \emptyset$. Let

$$J = \{n \in \mathbb{N} \colon n < k \text{ for all } k \in K\}.$$

Certainly 1 belongs to J. [If not, there would exist $c \in K$ such that $1 \geq c$. From proposition I.1.6 we see that $c = 1$. Thus 1 belongs to K and is the smallest member of K, contrary to our assumption.]

Now suppose that $n \in J$ and prove that $n+1 \in J$. If $n+1 \notin J$, then there exists $k \in K$ such that $n+1 \geq k$. By the inductive hypothesis $n < k$. Thus $n < k \leq n+1$. We conclude from problem I.1.16(b) that $k = n+1$. But, since n is smaller than every member of K, this implies that $n+1$ is the smallest member of K. But K has no smallest member. Therefore we conclude that $n+1 \in J$.

We have shown that J is an inductive subset of $\mathbb{N}$. Then $J = \mathbb{N}$ (by theorem I.1.7). If K contains any element at all, say j, then $j \in J$; so in particular $j < j$. Since this is not possible, we conclude that $K = \emptyset$.

Q.36. Exercises in appendix J

Q.36.1. (Solution to J.2.7)

(a) A number x belongs to the set A if $x^2 - 4x + 3 < 3$; that is, if $x(x-4) < 0$. This occurs if and only if $x > 0$ and $x < 4$. Thus $A = (0, 4)$; so $\sup A = 4$ and $\inf A = 0$.

(b) Use beginning calculus to see that $f'(x) = 2x - 4$. Conclude that the function f is decreasing on the interval $(-\infty, 2)$ and is increasing on $(2, 3)$. Thus f assumes a minimum at $x = 2$. Since $f(2) = -1$, we see that $B = [-1, \infty)$. Thus $\sup B$ does not exist and $\inf B = -1$.

Q.36.2. (Solution to J.3.7) As in the hint let $\ell = \sup A$ and $m = \sup B$, and suppose that ℓ, $m > 0$. If $x \in AB$, then there exist $a \in A$ and $b \in B$ such that $x = ab$. From $a \leq \ell$ and $b \leq m$ it is clear that $x \leq \ell m$; so ℓm is an upper bound for AB.

Since AB is bounded above it must have a least upper bound, say c. Clearly $c \leq \ell m$; we show that $\ell m \leq c$. Assume, to the contrary, that $c < \ell m$. Let $\epsilon = \ell m - c$. Since $\epsilon > 0$ and ℓ is the least upper bound for A we may choose an element a of A such that $a > \ell - \epsilon(2m)^{-1}$. Similarly, we may choose $b \in B$ so that

$b > m - \epsilon(2\ell)^{-1}$. Then

$$\begin{aligned} ab &> \big(\ell - \epsilon(2m)^{-1}\big)\big(m - \epsilon(2\ell)^{-1}\big) \\ &= \ell m - \epsilon + \epsilon^2(4\ell m)^{-1} \\ &> \ell m - \epsilon \\ &= c. \end{aligned}$$

This is a contradiction, since ab belongs to AB and c is an upper bound of AB. We have shown

$$\sup(AB) = c = \ell m = (\sup A)(\sup B)$$

as required.

Remark. It is not particularly difficult to follow the details of the preceding proof. But that is *not* the same thing as *understanding* the proof! It is easy to see, for example, that *if* we choose $a > \ell - \epsilon(2m)^{-1}$ and $b > m - \epsilon(2\ell)^{-1}$, *then* $ab > c$. But that still leaves room to be puzzled. You might reasonably say when shown this proof, "Well, that certainly *is* a proof. And it looks very clever. But what I don't understand is how did you know to choose a and b in just that particular (or should I say 'peculiar'?) way? Do you operate by fits of inspiration, or a crystal ball, or divination of entrails, or what?" The question deserves an answer. Once we have assumed c to be an upper bound smaller than ℓm (and set $\epsilon = \ell m - c$), our hope is to choose $a \in A$ close to ℓ and $b \in B$ close to m in such a way that their product ab exceeds c. It is difficult to say immediately *how* close a should be to ℓ (and b to m). Let's just say that $a > \ell - \delta_1$ and $b > m - \delta_2$, where δ_1 and δ_2 are small positive numbers. We will figure out *how* small they should be in a moment. Then

$$ab > (\ell - \delta_1)(m - \delta_2) = \ell m - m\delta_1 - \ell\delta_2 + \delta_1\delta_2 \,.$$

Since $\delta_1\delta_2$ is positive, we can simplify the preceding inequality and write

$$ab > \ell m - m\delta_1 - \ell\delta_2 \,. \tag{126}$$

What we *want* to get at the end of our computation is

$$ab > c = \ell m - \epsilon. \tag{127}$$

Now comparing what we have (126) with what we want (127), we see that all we need to do is choose δ_1 and δ_2 in such a way that

$$m\delta_1 + \ell\delta_2 < \epsilon \tag{128}$$

(for then $\ell m - (m\delta_1 + \ell\delta_2) > \ell m - \epsilon = c$, and we are done). To guarantee that the sum of two numbers is less than ϵ it suffices to choose both of them to be less than $\epsilon/2$. Clearly, we have $m\delta_1 < \epsilon/2$ if we choose $\delta_1 < \epsilon(2m)^{-1}$; and we have $\ell\delta_2 < \epsilon/2$ if we choose $\delta_2 < \epsilon(2\ell)^{-1}$. And that's all we need.

Q.36.3. (Solution to J.4.2) Let $A = \{t > 0 \colon t^2 < a\}$. The set A is not empty since it contains $a(1+a)^{-1}$. [$a^2(1+a)^{-2} < a(1+a)^{-1} < a$.] It is easy to see that A is bounded above by $M := \max\{1, a\}$. [If $t \in A$ and $t \le 1$, then $t \le M$; on the other hand, if $t \in A$ and $t > 1$, then $t < t^2 < a \le M$.] By the *least upper bound axiom* (J.3.1) A has a supremum, say x. It follows from the *axiom of trichotomy* (H.1.2) that exactly one of three things must be true: $x^2 < a$, $x^2 > a$, or $x^2 = a$. We show that $x^2 = a$ by eliminating the first two alternatives.

First assume that $x^2 < a$. Choose ϵ in $(0,1)$ so that $\epsilon < 3^{-1}x^{-2}(a-x^2)$. Then

$$(1+\epsilon)^2 = 1 + 2\epsilon + \epsilon^2 \tag{129}$$
$$< 1 + 3\epsilon \tag{130}$$

so that

$$x^2(1+\epsilon)^2 < x^2(1+3\epsilon) < a\,.$$

Thus $x(1+\epsilon)$ belongs to A. But this is impossible since $x(1+\epsilon) > x$ and x is the supremum of A.

Now assume $x^2 > a$. Choose ϵ in $(0,1)$ so that $\epsilon < (3a)^{-1}(x^2 - a)$. Then by (129)

$$a < x^2(1+3\epsilon)^{-1} < x^2(1+\epsilon)^{-2}\,. \tag{131}$$

Now since $x = \sup A$ and $x(1+\epsilon)^{-1} < x$, there must exist $t \in A$ such that $x(1+\epsilon)^{-1} < t < x$. But then

$$x^2(1+\epsilon)^{-2} < t^2 < a\,,$$

which contradicts (131). Thus we have demonstrated the existence of a number $x \geq 0$ such that $x^2 = a$. That there is only one such number has already been proved: see problem H.1.16.

Q.37. Exercises in appendix K

Q.37.1. (Solution to K.1.2) Suppose that $(x,y) = (u,v)$. Then

$$\big\{\{x,y\},\{x\}\big\} = \big\{\{u,v\},\{u\}\big\}.$$

We consider two cases.

Case 1: $\{x,y\} = \{u,v\}$ and $\{x\} = \{u\}$. The second equality implies that $x = u$. Then from the first equality we infer that $y = v$.

Case 2: $\{x,y\} = \{u\}$ and $\{x\} = \{u,v\}$. We derive $x = u = y$ from the first equality and $u = x = v$ from the second. Thus $x = y = u = v$. In either case $x = u$ and $y = v$. The converse is obvious.

Q.37.2. (Solution to K.3.7)

(a) $f(\frac{1}{2}) = 3$;

(b) Notice that $(1-x)^{-1}$ does not exist if $x = 1$, $(1 + (1-x)^{-1})^{-1}$ does not exist if $x = 2$, and $(1 - 2(1 + (1-x)^{-1})^{-1})^{-1}$ does not exist if $x = 0$; so $\operatorname{dom} f = \mathbb{R} \setminus \{0,1,2\}$.

Q.37.3. (Solution to K.3.8) We can take the square root of $g(x) = -x^2 - 4x - 1$ only when $g(x) \geq 0$, and since we take its reciprocal, it should not be zero. But $g(x) > 0$ if and only if $x^2 + 4x + 1 < 0$ if and only if $(x+2)^2 < 3$ if and only if $|x+2| < \sqrt{3}$ if and only if $-2-\sqrt{3} < x < -2+\sqrt{3}$. So $\operatorname{dom} f = (-2-\sqrt{3}, -2+\sqrt{3})$.

Q.38. Exercises in appendix L

Q.38.1. (Solution to L.1.2) We may write A as the union of three intervals

$$A = (-4,4) = (-4,-2) \cup [-2,1) \cup [1,4)\,.$$

Then

$$f^{\rightarrow}(A) = f^{\rightarrow}\big((-4,4)\big) = f^{\rightarrow}\big((-4,-2)\big) \cup f^{\rightarrow}\big([-2,1)\big) \cup f^{\rightarrow}\big([1,4)\big)\,. \tag{132}$$

(This step is justified in the next section by M.1.25.) Since f is constant on the interval $(-4,-2)$ we see that $f^{\rightarrow}\big((-4,-2)\big) = \{-1\}$. On the interval $[-2,1)$ the function increases from $f(-2) = 3$ to $f(0) = 7$ and then decreases to $f(1) = 6$ so that $f^{\rightarrow}\big([-2,1)\big) = [3,7]$. (This interval is closed because both -2 and 0 belong to $[-2,1)$.) Finally, since f is decreasing on $[1,4)$ we see that $f^{\rightarrow}\big([1,4)\big) = \big(f(4), f(1)\big] = (\frac{1}{4}, 1]$. Thus from equation (132) we conclude that

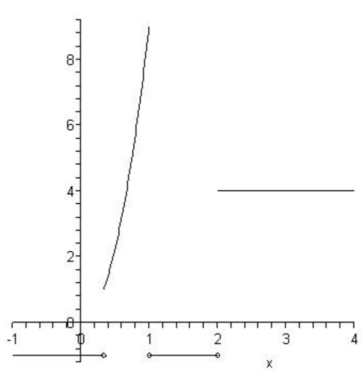
$$f^{\rightarrow}(A) = \{-1\} \cup (\tfrac{1}{4}, 1] \cup [3,7].$$

Q.38.2. (Solution to L.1.3) Use techniques from beginning calculus. The function is a fourth degree polynomial, so $f(x) \to \infty$ as $x \to -\infty$ and as $x \to \infty$. Thus the range of f is not bounded above. The minimum value of the range will occur at a critical point, that is, at a point where $f'(x) = 0$. But this occurs at $x = -3$, $x = 0$, and $x = 2$. The values of f at these points are, respectively -188, 0, and -63. We conclude that $\operatorname{ran} f = [-188, \infty)$.

Q.38.3. (Solution to L.1.5) Notice that the arctangent function is strictly increasing (its derivative at each x is $(1+x^2)^{-1}$). Its range is $(-\pi/2, \pi/2)$. Thus $f^{\leftarrow}(B) = f^{\leftarrow}\big((\pi/4, 2)\big) = f^{\leftarrow}\big((\pi/4, \pi/2)\big) = (1, \infty)$.

Q.38.4. (Solution to L.1.6) For $-\sqrt{9-x^2}$ to lie between 1 and 3, we would need $-3 < \sqrt{9-x^2} < -1$. But since the square root function on $\mathbb{R}$ takes on only positive values, this is not possible. So $f^{\leftarrow}(B) = \emptyset$.

Q.38.5. (Solution to L.2.2) For $x \le \frac{1}{3}$, $f(x) \le 1$, which implies $g(f(x)) = -1$. For $x \in (\frac{1}{3}, 1)$, $f(x) \in (1,3)$, so $g(f(x)) = 9x^2$. For $1 \le x \le 2$, $f(x) = 2$, so $g(f(x)) = -1$. Finally, for $x > 2$, $f(x) = 2$ and therefore $g(f(x)) = 4$.



Q.38.6. (Solution to L.2.3) Associativity: for every x

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = ((h \circ g) \circ f)(x)\,;$$

so $h \circ (g \circ f) = (h \circ g) \circ f$.

To see that composition is not commutative take, for example, $f(x) = x + 1$ and $g(x) = x^2$. Since $(g \circ f)(1) = 4$ and $(f \circ g)(1) = 2$, the functions $g \circ f$ and $f \circ g$ cannot be equal.

Q.39. Exercises in appendix M

Q.39.1. (Solution to M.1.2) If $f(x) = f(y)$, then $(x+2)(3y-5) = (y+2)(3x-5)$. Thus $6y - 5x = 6x - 5y$, which implies $x = y$.

Q.39.2. (Solution to M.1.3) Suppose that m and n are positive integers with no common prime factors. Let $f\big(\frac{m}{n}\big) = 2^m 3^n$. Then f is injective by the *unique factorization theorem* (see, for example, [BL53], page 21).

Q.39.3. (Solution to M.1.10) Let $f(x) = \frac{1}{x} - 1$ for $x \neq 0$ and $f(0) = 3$.

Q.39.4. (Solution to M.1.12) Define $f\colon \mathbb{Z} \to \mathbb{N}$ by

$$f(n) = \begin{cases} 2n+1, & \text{for } n \geq 0 \\ -2n, & \text{for } n < 0. \end{cases}$$

Q.39.5. (Solution to M.1.13) Define $f\colon \mathbb{R} \to (0,1)$ by $f(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$.

Q.39.6. (Solution to M.1.14) Let $\mathbb{S}^1$ be $\{(x,y)\colon x^2 + y^2 = 1\}$. Define $f\colon [0,1) \to \mathbb{S}^1$ by $f(t) = (\cos(2\pi t), \sin(2\pi t))$.

Q.39.7. (Solution to M.1.15) Let

$$g(x) = \begin{cases} 3 - 2x, & \text{for } 0 \leq x < 1 \\ f(x), & \text{for } 1 \leq x \leq 2 \\ \frac{1}{2}(3 - x), & \text{for } 2 < x \leq 3. \end{cases}$$

Q.39.8. (Solution to M.1.16) $\big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$.

Q.39.9. (Solution to M.1.22)

(a) We show that if $y \in f^{\rightarrow}(f^{\leftarrow}(B))$, then $y \in B$. Suppose that $y \in f^{\rightarrow}(f^{\leftarrow}(B))$. Then (by the definition of $f^{\rightarrow}$) there exists $x \in f^{\leftarrow}(B)$ such that $y = f(x)$. From $x \in f^{\leftarrow}(B)$ we infer (using the definition of $f^{\leftarrow}$) that $f(x) \in B$. That is, $y \in B$.

(b) Let $f(x) = x^2$ and $B = \{-1\}$. Then $f^{\rightarrow}(f^{\leftarrow}(B)) = f^{\rightarrow}(f^{\leftarrow}\{-1\}) = f^{\rightarrow}(\emptyset) = \emptyset \neq B$.

(c) Suppose that f is surjective. Show that $B \subseteq f^{\rightarrow}(f^{\leftarrow}(B))$ by showing that $y \in B$ implies $y \in f^{\rightarrow}(f^{\leftarrow}(B))$. If $y \in B$, then (since f is surjective) there exists $x \in S$ such that $y = f(x)$. Since $f(x) \in B$, we see that $x \in f^{\leftarrow}(B)$ (by the definition of $f^{\leftarrow}$). From this it follows (using the definition of $f^{\rightarrow}$) that $y = f(x) \in f^{\rightarrow}(f^{\leftarrow}(B))$.

Q.39.10. (Solution to M.1.25) This requires nothing other than the definitions of $\cup$ and $f^{\rightarrow}$:

$$\begin{aligned} y \in f^{\rightarrow}(A \cup B) &\text{ iff there exists } x \in A \cup B \text{ such that } y = f(x) \\ &\text{ iff there exists } x \in A \text{ such that } y = f(x) \text{ or} \\ &\qquad \text{there exists } x \in B \text{ such that } y = f(x) \\ &\text{ iff } y \in f^{\rightarrow}(A) \text{ or } y \in f^{\rightarrow}(B) \\ &\text{ iff } y \in f^{\rightarrow}(A) \cup f^{\rightarrow}(B). \end{aligned}$$

Q.39.11. (Solution to M.1.27) Here the definitions of $\cap$ and $f^{\leftarrow}$ are used:

$$\begin{aligned} x \in f^{\leftarrow}(C \cap D) &\text{ iff } f(x) \in C \cap D \\ &\text{ iff } f(x) \in C \text{ and } f(x) \in D \\ &\text{ iff } x \in f^{\leftarrow}(C) \text{ and } x \in f^{\leftarrow}(D) \\ &\text{ iff } x \in f^{\leftarrow}(C) \cap f^{\leftarrow}(D). \end{aligned}$$

Q.39.12. (Solution to M.1.31)

(a) Show that if $y \in f^{\rightarrow}(\bigcap \mathfrak{A})$, then $y \in \bigcap\{f^{\rightarrow}(A) \colon A \in \mathfrak{A}\}$. Suppose that $y \in f^{\rightarrow}(\bigcap \mathfrak{A})$. Then there exists $x \in \bigcap \mathfrak{A}$ such that $y = f(x)$. Since x belongs to the intersection of the family $\mathfrak{A}$ it must belong to every member of $\mathfrak{A}$. That is, $x \in A$ for every $A \in \mathfrak{A}$. Thus $y = f(x)$ belongs to $f^{\rightarrow}(A)$ for every $A \in \mathfrak{A}$; and so $y \in \bigcap\{f^{\rightarrow}(A) \colon A \in \mathfrak{A}\}$.

(b) Suppose f is injective. If $y \in \bigcap\{f^{\rightarrow}(A) \colon A \in \mathfrak{A}\}$, then $y \in f^{\rightarrow}(A)$ for every $A \in \mathfrak{A}$. Choose a set $A_0 \in \mathfrak{A}$. Since $y \in f^{\rightarrow}(A_0)$, there exists $x_0 \in A_0$ such that $y = f(x_0)$. The point x_0 belongs to every member of $\mathfrak{A}$. To see this, let A be an arbitrary set belonging to $\mathfrak{A}$. Since $y \in f^{\rightarrow}(A)$, there exists $x \in A$ such that $y = f(x)$; and since $f(x) = y = f(x_0)$ and f is injective, we conclude that $x_0 = x \in A$. Thus we have shown that $x_0 \in \bigcap \mathfrak{A}$ and therefore that $y = f(x_0) \in f^{\rightarrow}(\bigcap \mathfrak{A})$.

(c) If $y \in f^{\rightarrow}(\bigcup \mathfrak{A})$, then there exists $x \in \bigcup \mathfrak{A}$ such that $y = f(x)$. Since $x \in \bigcup \mathfrak{A}$ there exists $A \in \mathfrak{A}$ such that $x \in A$. Then $y = f(x) \in f^{\rightarrow}(A)$ and so $y \in \bigcup\{f^{\rightarrow}(A) : A \in \mathfrak{A}\}$. Conversely, if y belongs to $\bigcup\{f^{\rightarrow}(A) : A \in \mathfrak{A}\}$, then it must be a member of $f^{\rightarrow}(A)$ for some $A \in \mathfrak{A}$. Then $y = f(x)$ for some $x \in A \subseteq \bigcup \mathfrak{A}$ and therefore $y = f(x) \in f^{\rightarrow}(\bigcup \mathfrak{A})$.

Q.39.13. (Solution to M.2.1) Let $f \colon S \to T$ and suppose that g and h are inverses of f. Then

$$g = g \circ I_T = g \circ (f \circ h) = (g \circ f) \circ h = I_S \circ h = h\,.$$

Q.39.14. (Solution to M.2.3) *Arcsine* is the inverse of the restriction of the *sine* function to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The *arccosine* is the inverse of the restriction of *cosine* to $[0, \pi]$. And *arctangent* is the inverse of the restriction of *tangent* to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Q.39.15. (Solution to M.2.4) Suppose that f has a right inverse f_r. For each $y \in T$ it is clear that $y = I_T(y) = f\big(f_r(y)\big) \in \operatorname{ran} f$; so $\operatorname{ran} f = T$ and f is surjective.

Conversely, suppose that f is surjective. Then for every $y \in T$ the set $f^{\leftarrow}(\{y\})$ is nonempty. For each $y \in T$ let x_y be a member of $f^{\leftarrow}(\{y\})$ and define

$$f_r \colon T \to S \colon y \mapsto x_y\,.$$

Then $f\big(f_r(y)\big) = f(x_y) = y$, showing that f_r is a right inverse of f. (The reader who has studied a bit of set theory will likely have noticed the unadvertised use of the *axiom of choice* in this proof. It is used in this fashion throughout the text.)

Q.40. Exercises in appendix N

Q.40.1. (Solution to N.1.4) The existence of the function has already been demonstrated: if $f = (f^1, f^2)$, then $(\pi_k \circ f)(t) = \pi_k\big(f^1(t), f^2(t)\big) = f^k(t)$ for $k = 1, 2$ and $t \in T$.

To prove uniqueness suppose that there is a function $g \in \mathcal{F}(T, S_1 \times S_2)$ such that $\pi_k \circ g = f^k$ for $k = 1, 2$. Then $g(t) = \big(\pi_1(g(t)), \pi_2(g(t))\big) = \big(f^1(t), f^2(t)\big) = \big(f^1, f^2\big)(t)$ for $k = 1, 2$ and $t \in T$. So $g = \big(f^1, f^2\big)$.

Q.41. Exercises in appendix O

Q.41.1. (Solution to O.1.4) We wish to demonstrate that for all natural numbers m and n if there is a bijection from $\{1, \dots, m\}$ onto $\{1, \dots, n\}$, then $m = n$. To accomplish this use induction on n.

First, suppose that for an arbitrary natural number m we have $\{1, \dots, m\} \sim \{1\}$. That is, we suppose that there exists a bijection f from $\{1, \dots, m\}$ onto $\{1\}$. Then since $f(1) = 1 = f(m)$ and f is injective, we conclude that $m = 1$. This establishes the proposition in the case $n = 1$.

Next, we assume the truth of the result for some particular $n \in \mathbb{N}$: for every $m \in \mathbb{N}$ if $\{1, \dots, m\} \sim \{1, \dots, n\}$, then $m = n$. This is our inductive hypothesis. What we wish to show is that for an arbitrary natural number m if $\{1, \dots, m\} \sim \{1, \dots, n+1\}$, then $m = n + 1$. Suppose then that $m \in \mathbb{N}$ and $\{1, \dots, m\} \sim \{1, \dots, n+1\}$. Then there is a bijection f from $\{1, \dots, m\}$ onto $\{1, \dots, n+1\}$. Let $k = f^{-1}(n+1)$. The restriction of f to the set $\{1, \dots, k-1, k\ +\ 1\ , \dots, m\}$ is a bijection from that set onto $\{1, \dots, n\}$. Thus

$$\{1, \dots, k-1, k+1, \dots, m\} \sim \{1, \dots, n\}. \tag{133}$$

Furthermore, it is easy to see that

$$\{1, \dots, m-1\} \sim \{1, \dots, k-1, k+1, \dots, m\}. \tag{134}$$

(The required bijection is defined by $g(j) = j$ if $1 \le j \le k-1$ and $g(j) = j+1$ if $k \le j \le m-1$.) From (133), (134), and proposition O.1.2 we conclude that

$$\{1, \dots, m-1\} \sim \{1, \dots, n\}\,.$$

By our inductive hypothesis, $m - 1 = n$. This yields the desired conclusion $m = n + 1$.

Q.41.2. (Solution to O.1.7) The result is trivial if S or T is empty; so we suppose they are not. Let $m = \operatorname{card} S$ and $n = \operatorname{card} T$. Then $S \sim \{1, \dots, m\}$ and $T \sim \{1, \dots, n\}$. It is clear that

$$\{1, \dots, n\} \sim \{m+1, \dots, m+n\}\,.$$

(Use the map $j \mapsto j + m$ for $1 \le j \le n$.) Thus $T \sim \{m+1, \dots, m+n\}$. Let $f\colon S \to \{1, \dots, m\}$ and $g\colon T \to \{m+1, \dots, m+n\}$ be bijections. Define $h\colon S \cup T \to \{1, \dots, m+n\}$ by

$$h(x) = \begin{cases} f(x), & \text{for } x \in S \\ g(x), & \text{for } x \in T. \end{cases}$$

Then clearly h is a bijection. So $S \cup T$ is finite and $\operatorname{card}(S \cup T) = m + n = \operatorname{card} S + \operatorname{card} T$.

Q.41.3. (Solution to O.1.8) Proceed by mathematical induction. If $C \subseteq \{1\}$, then either $C = \emptyset$, in which case $\operatorname{card} C = 0$, or else $C = \{1\}$, in which case $\operatorname{card} C = 1$. Thus the lemma is true if $n = 1$.

Suppose then that the lemma holds for some particular $n \in \mathbb{N}$. We prove its correctness for $n + 1$. So we assume that $C \subseteq \{1, \dots, n+1\}$ and prove that C is finite and that $\operatorname{card} C \le n + 1$. It is clear that $C \setminus \{n+1\} \subseteq \{1, \dots, n\}$. By the inductive hypothesis $C \setminus \{n+1\}$ is finite and $\operatorname{card}(C \setminus \{n+1\}) \le n$. There are two possibilities: $n + 1 \notin C$ and $n + 1 \in C$. In case $n + 1$ does not belong to C, then $C = C \setminus \{n+1\}$; so C is finite and $\operatorname{card} C \le n < n + 1$. In the other case, where $n + 1$ does belong to C, it is clear that C is finite (because $C \setminus \{n+1\}$ is) and we have (by proposition O.1.7)

$$\begin{aligned} \operatorname{card} C &= \operatorname{card}\big((C \setminus \{n+1\}) \cup \{n+1\}\big) \\ &= \operatorname{card}(C \setminus \{n+1\}) + \operatorname{card}(\{n+1\}) \\ &\le n + 1. \end{aligned}$$

Q.41.4. (Solution to O.1.11) Suppose that S is infinite. We prove that there exists a proper subset T of S and a bijection f from S onto T. We choose a sequence of distinct elements a_k in S, one for each $k \in \mathbb{N}$. Let a_1 be an arbitrary member of S. Then $S \setminus \{a_1\} \ne \emptyset$. (Otherwise $S \sim \{a_1\}$ and S is finite.) Choose $a_2 \in S \setminus \{a_1\}$. Then $S \setminus \{a_1, a_2\} \ne \emptyset$. (Otherwise $S \sim \{a_1, a_2\}$ and S is finite.) In general, if distinct elements $a_1, \dots, a_n$ have been chosen, then $S \setminus \{a_1, \dots, a_n\}$ cannot be empty; so we may choose $a_{n+1} \in S \setminus \{a_1, \dots, a_n\}$. Let $T = S \setminus \{a_1\}$, and define $f\colon S \to T$ by

$$f(x) = \begin{cases} a_{k+1}, & \text{if } x = a_k \text{ for some } k \\ x, & \text{otherwise.} \end{cases}$$

Then f is a bijection from S onto the proper subset T of S.

For the converse construct a proof by contradiction. Suppose that $S \sim T$ for some proper subset $T \subseteq S$, and assume further that S is finite, so that $S \sim \{1, \dots, n\}$ for some $n \in \mathbb{N}$. Then by proposition O.1.9 the set $S \setminus T$ is finite and, since it is nonempty, is therefore cardinally equivalent to $\{1, \dots, p\}$ for some $p \in \mathbb{N}$. Thus

$$\begin{aligned} n &= \operatorname{card} S \\ &= \operatorname{card} T \\ &= \operatorname{card}(S \setminus (S \setminus T)) \\ &= \operatorname{card} S - \operatorname{card}(S \setminus T) \qquad \text{(by problem O.1.10)} \\ &= n - p. \end{aligned}$$

Therefore $p = 0$, which contradicts the earlier assertion that $p \in \mathbb{N}$.

Q.41.5. (Solution to O.1.13) The map $x \mapsto \frac{1}{2}x$ is a bijection from the interval $(0, 1)$ onto the interval $(0, \frac{1}{2})$, which is a proper subset of $(0, 1)$.

Q.41.6. (Solution to O.1.15) Since f is surjective it has a right inverse f_r (see proposition M.2.4). This right inverse is injective, since it has a left inverse (see proposition M.2.5). Let $A = \operatorname{ran} f_r$. The function f_r establishes a bijection between T and A. Thus $T \sim A \subseteq S$. If S is finite, so is A (by proposition O.1.9) and therefore so is T.

Q.41.7. (Solution to O.1.16) Let $B = \operatorname{ran} f$. Then $S \sim B \subseteq T$. If T is finite, so is B (by proposition O.1.9) and therefore so is S.

Q.42. Exercises in appendix P

Q.42.1. (Solution to P.1.4) If S is finite there is nothing to prove; so we suppose that S is an infinite subset of T. Then T is countably infinite. Let $f\colon \mathbb{N} \to T$ be an enumeration of the members of T. The restriction of f to the set $f^{\leftarrow}(S) \subseteq \mathbb{N}$ is a bijection between $f^{\leftarrow}(S)$ and S; so we may conclude that S is countable provided we can prove that $f^{\leftarrow}(S)$ is. Therefore it suffices to show that *every* subset of $\mathbb{N}$ is countable.

Let A be an infinite subset of $\mathbb{N}$. Define inductively elements $a_1 < a_2 < \dots$ in A. (Let a_1 be the smallest member of A. Having chosen $a_1 < a_2 < \dots < a_n$ in A, notice that the set $A \setminus \{a_1, \dots, a_n\}$ is not empty and choose a_{n+1} to be the smallest element of that set.) Let $a\colon \mathbb{N} \to A$ be the function $n \mapsto a_n$. It is clear that $a_k \geq k$ for all k and, since $a_k < a_{k+1}$ for all k, that a is injective. To see that a is surjective, assume that it is not and derive a contradiction. If a is not surjective, then the range of a is a proper subset of A. Let p be the smallest element of $A \setminus \operatorname{ran} a$. Since $p \in A \setminus \operatorname{ran} a \subseteq A \setminus \{a_1, \dots, a_p\}$, we see from the definition of a_{p+1} that $a_{p+1} \leq p$. On the other hand we know that $a_{p+1} \geq p + 1 > p$. This contradiction shows that a is a surjection. Thus $A \sim \mathbb{N}$ proving that A is countable.

Q.42.2. (Solution to P.1.7) To see that the map

$$f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}\colon (m, n) \mapsto 2^{m-1}(2n - 1)$$

is a bijection, we construct its inverse (see propositions M.2.4 and M.2.5). If $p \in \mathbb{N}$ let m be the largest member of $\mathbb{N}$ such that 2^{m-1} divides p. (If p is odd, then $m = 1$.) Then $p/2^{m-1}$ is odd and can be written in the form $2n-1$ for some $n \in \mathbb{N}$. The map $g\colon p \mapsto (m, n)$ is clearly the inverse of f.

Q.42.3. (Solution to P.1.11) If $\mathfrak{A}$ is infinite let

$$\mathfrak{A} = \{A_1, A_2, A_3, \dots\} ;$$

while if $\mathfrak{A}$ is finite, say $\operatorname{card} \mathfrak{A} = m$, let

$$\mathfrak{A} = \{A_1, \dots, A_m\}$$

and let $A_n = A_m$ for all $n > m$. For each $j \in \mathbb{N}$ the set A_j is either infinite, in which case we write

$$A_j = \{a_{j1}, a_{j2}, a_{j3}, \dots\} ,$$

or else it is finite, say card $A_j = p$, in which case we write

$$A_j = \{a_{j1}, \dots, a_{jp}\}$$

and let $a_{jq} = a_{jp}$ for all $q > p$. Then the map

$$a \colon \mathbb{N} \times \mathbb{N} \to \bigcup \mathfrak{A} \colon (j,k) \mapsto a_{jk}$$

is surjective. Thus $\bigcup \mathfrak{A} = \bigcup_{j,k=1}^{\infty} A_{j,k} = \operatorname{ran} a$ is countable by lemma P.1.7 and proposition P.1.6.

Bibliography

[Apo74] Tom M. Apostol, *Mathematical analysis*, second ed., Addison-Wesley Pub. Co., Reading, Mass., 1974. 242

[BL53] Garrett Birkhoff and Saunders Mac Lane, *A survey of modern algebra*, revised ed., Macmillan, New York, 1953. 431

[Die62] Jean Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1962. 49, 183

[Hal58] Paul R. Halmos, *Finite-dimensional vector spaces*, D. Van Nostrand, Princeton, 1958. 149

[Hal67] ———, *A hilbert space problem book*, D. Van Nostrand, Princeton, Toronto, London, 1967. xix, xx

[HK71] Kenneth Hoffman and Ray Kunze, *Linear algebra*, Prentice Hall, Englewood Cliffs, 1971. 149

[Kap77] Irving Kaplansky, *Set theory and metric spaces*, Chelsea Publ. Co., New York, 1977. 345

[LS90] Lynn H. Loomis and Shlomo Sternberg, *Advanced calculus*, Jones and Bartlett, Boston, 1990. xvi, 278

[Moo32] C. N. Moore, *Infinite series and sequences*, Amer. Mathematical Monthly **39** (1932), 62–71. 235

[Som56] Arnold Sommerfeld, *Thermodynamics and statistical mechanics*, Academic Press, New York, 1956. 263

[Spi67] Michael Spivak, *Calculus*, W. A. Benjamin, New York, 1967. 313

[Str81] Karl R. Stromberg, *An introduction to classical real analysis*, Wadsworth, Belmont, 1981. 347

Index