Chapter 1

Introduction

1.1. Design of this book

It has often been said that "mathematics is not a spectator sport". This truism is very much evident in the writing of this book. It is written so as to guide you through the entire story, yet permit you whenever appropriate to construct the mathematical story for yourself. The idea is to do some mathematics yourself rather than just observe it done by others. This "doing mathematics oneself" takes the form of exercises with enough help (hints) provided so that the "doing" is not so onerous as to get in the way of the story itself.

Strong evidence has been provided by students of mathematics over many centuries that such guided "doing" is indispensable for full understanding and retention. When the book is used as a course text, the instructor is encouraged to make available the exercise statements, one to a page, so that students can write out solutions (in correctable form), correct any errors, and insert the corrected exercise solution at the end of the appropriate section in the book.

Parts I–III of the book presuppose a working knowledge of high school geometry and basic trigonometry. They contain the mathematical material appropriate for a one-quarter geometry course for preservice or inservice middle school mathematics teachers. Parts I–III followed by Part IV contain appropriate mathematical material for a one-semester geometry course for preservice or inservice high school mathematics teachers. For more advanced students, a rapid review of the material in Parts I–III followed by Parts IV–VII can comprise a one-semester upper division Advanced Geometry course. Parts V–VII tell the unified story of all the two-dimensional geometries, one for each real number K. It is in this part of the story that the notion of the *group* of rigid motions enters and manifests some of its power.

1.1.1. One and only one two-dimensional geometry for each real number. In the end, a geometry will be seen to be completely characterized by its definition of distance, called its "K-dot-product", and the group of rigid motions that preserve that K-dot-product. As we have already mentioned, there will turn out to be one and only one geometry for each real number K. If $K > 0$, the geometry will be that of the sphere of radius $K^{-1/2}$. If $K = 0$, the geometry will be the Euclidean geometry that you were introduced to in high school. If $K < 0$, the geometry will be one of the so-called hyperbolic geometries, the one whose distance and area formulas have scaling factors of $|K|^{-1/2}$ and $|K|^{-1}$, respectively.

Besides telling their own beautiful story, Parts IV–VII can also serve as a bridge to Curves and Surfaces in Euclidean Three-space, the standard first course in elementary Riemannian geometry. Parts IV–VII treat the special case of two-dimensional geometries that are homogeneous, that is, that look the same at all their points and in all directions.

To treat these geometries efficiently, we introduce the notion of changing coordinates for the geometry without changing the geometry itself. The idea is to use a particular coordinate system, for example, spherical coordinates, to calculate a particular quantity or property, for example, distance. A different coordinate system for the same geometry, for example, stereographic projection coordinates, may be more useful for calculating, for example, measure of angles. That same notion of "different coordinates to treat different properties" also allowed geometers to treat surfaces and higher-dimensional smooth spaces that look different at different points, ones that often cannot be treated at all their points using a single set of coordinates.

For our purposes in this book, though, a *geometry* will be a two-dimensional set or space that "looks the same" at all its points. That is, the set admits a distance-preserving transformation that takes any given point to any other given point and takes any given direction at the first point to any given direction at the other point.

It is my hope and intention in writing this little book that you engage with and enjoy this uniform way of understanding all two-dimensional geometries as much as I did!

1.1.2. Special message to current or future teachers of high school geometry. As mentioned above, Parts I–III of this book are especially relevant to your teaching of the subject. Look especially closely at the treatment of congruence (rigid motion), similarity (dilation), circles, expressing geometric properties with equations, and geometric measurement and dimension, and compare them with the high school geometry sections of the Common Core State Standards in Mathematics. The latter can be found at

`http://www.corestandards.org/Math/Content/HSG/introduction`.

A useful companion course to one based on this book, one that might be called Geometry for Teaching, would explicitly make the connections between the material covered as in this book and what you do (or will do) in your high school geometry classroom. The idea is not that the material we will cover in Parts I–III will tell you how to teach that material, but rather that the treatment given here will give you the depth and breadth of geometric understanding that will allow you to design what you teach and bring it into your classroom in ways that those who lack that understanding cannot.

1.1.3. Acknowledgments. It is my pleasure to thank Dr. Bart Snapp of the Mathematics Department of Ohio State University for his contributions to this book, especially to the exercises, where his many suggestions have greatly improved their precision and richness. Finally I wish to thank the Ohio State Mathematics Department itself for the opportunity to regularly teach the advanced geometry course from which this book evolved over a 15-year period.

1.2. Parts V–VII: How many two-dimensional geometries are there?

As a young mathematician I was introduced to the classic *Leçons sur la Géométrie des Espaces de Riemann*, written by the great French geometer Élie Cartan. Early in his treatise on geometries in all dimensions, Cartan presents the case of two-dimensional geometries, in particular, those two-dimensional geometries that look the same at all points and in all directions. (For example, a cylinder looks the same at each of its points but not in all directions emanating from any one of its points, whereas a sphere looks the same at all points and in all directions.) As we mentioned above, it turns out that there is one and only one such geometry for each real number K, called the *curvature* of the geometry. The case $K = 0$ is the (flat) Euclidean geometry that you learned in high school.

These 25 pages of Cartan's book (Chapter VI, §i–v) so captivated me that I have returned to them regularly throughout my career and have adapted and taught them many times at the advanced undergraduate level. They form the basis for Parts V–VII of this book. To me they tell one of the most beautiful and satisfying stories in all of geometry, one which exemplifies a fundamental principle of all great mathematics, namely that, using the tools at hand but in a slightly novel way, the clouds part and one sees that objects and relationships that seemed so different are in fact parts of a single elegant story!

When $K > 0$ it turns out that the K-geometry is the geometry of the sphere of radius $R = 1/K^{1/2}$ that we can "see" as a subset of Euclidean three-space $\mathbb{R}^3$. But the geometries with $K < 0$ are not so easy to visualize. They are the so-called "hyperbolic" geometries. In fact it took mathematicians a couple thousand years to realize that they existed at all! It turns out that the secret to understanding all the two-dimensional geometries, including the ones with $K < 0$, in a unified way is to simply rescale the third coordinate in $\mathbb{R}^3$ and, using these "unusual" coordinates (x, y, z), to look at each two-dimensional geometry as the solution set to the equation

$$K\left(x^2 + y^2\right) + z^2 = 1.$$

However, the idea of changing coordinates without changing the underlying geometry described by those coordinates is a challenging one that did not come into mathematics until a couple of centuries ago. In Chapter 9 we introduce the main coordinate change that we are going to use, namely the above-mentioned rescaling of the third coordinate in Euclidean three-space. This change is central to the second half of the book. However, before we begin our unified study of two-dimensional geometries in the second half of this book, we devote Chapter 8 to a review of the concepts from several variable calculus and linear algebra that we will need. We briefly indicate these next.

1.3. Parts IV–VII: Some needed multivariable calculus and linear algebra facts

Parts IV–VII of this book will suppose familiarity with several variable calculus and the linear algebra of matrices and how they are used to define linear transformations. Also it will often be useful to consider a vector, for example, $V = (a, b, c)$, as a 1×3 matrix, writing

$$(V) = \begin{pmatrix} a & b & c \end{pmatrix},$$

or as a 3×1 matrix, writing

$$(V)^t = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This will allow us, for example, to write the scalar product of two vectors

$$\begin{aligned} V \bullet V' &= (a, b, c) \bullet (a', b', c') \\ &= aa' + bb' + cc' \end{aligned}$$

as a product of matrices

$$(V) \cdot (V')^t = \begin{pmatrix} a & b & c \end{pmatrix} \cdot \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}.$$

Also you will need basic facts about 2×2 and 3×3 matrices, such as how to add and multiply them and basic properties of their determinants.

From several variable calculus you will in particular need to remember and apply two rules, the Chain Rule (Theorem 1) for differentiable functions of several variables, written in matrix notation, and the Substitution Rule (Theorem 2).

Theorem 1 (Chain Rule)**.** *Given differentiable mappings*

$$(y_1 (x_1, \ldots, x_m), \ldots, y_n (x_1, \ldots, x_m))$$

and

$$(z_1 (y_1, \ldots, y_n), \ldots, z_p (y_1, \ldots, y_n)),$$

partial derivatives satisfy the matrix equation

$$(\partial z_k / \partial x_i) = (\partial z_k / \partial y_j) \cdot (\partial y_j / \partial x_i).$$

Proof. The rule comes from the fact that if I pick one of the functions $z_k(y_1, \ldots, y_n)$ and pick one of the variables x_i and vary it (and leave all the other $x_{i'}$ constant), I am left with finding the formula for

$$\frac{dz (y_1 (x), \ldots, y_n (x))}{dx}$$

that turns out to be

$$\sum_{j=1}^{n} \frac{\partial z}{\partial y_j} (y_1 (x), \ldots, y_n (x)) \cdot \frac{dy_j}{dx} (x).$$

This last expression can be written in matrix form as

$$\left(\frac{\partial z}{\partial y_1}\left(y_1\left(x\right),\ldots,y_n\left(x\right)\right) \quad \ldots \quad \frac{\partial z}{\partial y_n}\left(y_1\left(x\right),\ldots,y_n\left(x\right)\right) \right)\cdot\left(\begin{array}{c} \frac{dy_1}{dx}\left(x\right) \\ \ldots \\ \frac{dy_n}{dx}\left(x\right) \end{array} \right).$$

Setting $z = z_k$ and $x = x_i$, we have exactly the matrix multiplication formula for $\partial z_k/\partial x_i$ in the theorem. □

Exercise 1. a) Write out Theorem 1 in the case in which $n = m = p = 1$ to be sure that you recognize how it is just a several variable formulation of the most important theorems from your introductory course in calculus.

b) Write the full 2×2 matrix form of the Chain Rule when $n = m = p = 2$.

Theorem 2 (Substitution Rule)**.** *Given a mapping*

$$\left(y_1\left(x_1,\ldots,x_n\right),\ldots,y_n\left(x_1,\ldots,x_n\right)\right)$$

between corresponding regions R_y *in* $(y_1,\ldots,y_n)$*-space and* R_x *in* $(x_1,\ldots,x_n)$*-space, and given a function* $f\left(y_1,\ldots,y_n\right)$ *on* R_y*,*

$$\begin{array}{c} \int_{R_y} f\left(y_1,\ldots,y_n\right)\cdot dy_1\ldots dy_n \\ = \int_{R_x} f\left(y_1\left(x_1,\ldots,x_n\right),\ldots,y_n\left(x_1,\ldots,x_n\right)\right)\cdot D_{yx}\cdot dx_1\ldots dx_n, \end{array}$$

where

$$D_{yx} = \det\left(\partial y_j/\partial x_i\right).$$

Exercise 2. a) Write out Theorem 2 in the case in which $n = 1$ to be sure that you recognize how it is just a several variable formulation of the most important theorem from your introductory course in integral calculus.

b) Integrate $\sin\left(y_1^2 + y_2^2\right)$ over the region in the (y_1, y_2)-plane bounded by the unit circle.

Hint: Use the Substitution Rule to change (y_1, y_2) to polar coordinates.

1.4. References and notation

As help, at some points in the text and in some of the exercises, a more complete treatment of a particular topic can be found in [**MJG**] or [**DS**]. The corresponding topics in these texts are referenced. For example, [**MJG**, 311] refers to page 311 in the Greenberg book and [**DS**, 59ff] refers to page 59 and those pages just following page 59 in the Davis–Snider book.

Some final remarks about notation in this book: The letters **EG** will always mean Euclidean (usually plane but occasionally three-dimensional) geometry, the letters **SG** will always mean spherical geometry, and the letters **HG** will always mean hyperbolic geometry. One further kind of geometry, which we call neutral geometry, will be explained in the book and denoted by **NG**.