Chapter 11

The curvature K becomes negative

11.1. The world sheet and the light cone

We now turn to the case in which the radius R of the Euclidean R-sphere goes to infinity and beyond! Of course that doesn't make any sense in $(\hat{x}, \hat{y}, \hat{z})$-space but if we look at the R-sphere in (x, y, z)-coordinates, it makes perfect sense because there the equation of the R-sphere is

$$K\left(x^2 + y^2\right) + z^2 = 1 \tag{11.1.1}$$

for $K = \frac{1}{R^2}$, so that R going to infinity means that K goes to zero, and "beyond" simply means that K becomes negative. We have seen that all we need to have a geometry with lengths, angles, areas, and congruences is to have a smooth set and a dot-product between vectors tangent to that set. Now if K becomes negative, our geometry becomes a hyperboloid of two sheets (obtained by rotating a hyperbola in the (x, z)-plane with major axis the z-axis around that axis). We will still have a "K-geometry" so all the calculations of Part V are still valid. This means that we have already done much of the work for hyperbolic geometry. So this final chapter will be shorter, and the work that we do will so closely parallel that for spherical geometry that it will be presented as a sequence of exercises, where the hints are the analogies with what we have already done in the spherical geometry case.

So that we have a connected universe, we only consider the "top" sheet (where $z > 0$) as our K-geometry. (In special relativity, this sheet might be called something like the "world sheet".) If, instead of rotating a hyperbola around the z-axis which gives our K-geometry, we rotate the asymptotes of the hyperbola around the z-axis, we obtain a cone given by the equation

$$K\left(x^2 + y^2\right) + z^2 = 0.$$

(Again this might be called something like the "light cone".) This cone will play a major role in what follows. Indeed it is the role of this cone that is the source of the major differences arising in the case of hyperbolic geometry.

There is one potential problem we need to worry about when $K < 0$, and it concerns the lengths of tangent vectors. Namely, our formulas for lengths involve taking the square root of the dot-product of a tangent vector with itself, so that dot-product had better be non-negative (or zero only if the tangent vector itself is the zero-vector). Our K-dot-product is given by the formula

$$\begin{pmatrix} a & b & c \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

So, when $K < 0$, it seems entirely possible that some tangent vector V has the property that $V \bullet_K V < 0$. (Indeed that will always happen if c is sufficiently big and x and y are sufficiently small [**MJG**, 241–242].)

11.1.1. Nonzero tangent vectors in HG have positive lengths.

Exercise 142. Suppose that a vector V emanates from $(0, 0, 0)$ in (x, y, z)-space.

a) Show that $V \bullet_K V = 0$ if and only if V points in the direction of the light cone.

b) Show that $V \bullet_K V < 0$ if and only if V points in a direction inside the light cone.

c) Show that $V \bullet_K V > 0$ if and only if V points in a direction outside the light cone.

Hint: Use the fact that the (Euclidean) angle ϑ that the light cone makes with the plane $z = 0$ is given by taking any point (x, y, z) on the light cone with $z > 0$ and computing

$$\tan \vartheta = \frac{z}{\sqrt{x^2 + y^2}} = |K|^{1/2}.$$

Now our world sheet lies *inside* the light cone but tangent vectors to it point *outside* the light cone. That is what saves our K-dot-product, as we see in the next lemma.

Lemma 7 (HG). *Let $V = (a, b, c)$ denote a vector that is tangent to our K-geometry, that is, to the set* (11.1.1). *Then*

$$V \bullet_K V \geq 0$$

and $V \bullet_K V = 0$ if and only if $V = 0$.

Proof. If $c = 0$, then the assertion of the lemma is obviously true. So we can assume $c \neq 0$. Notice that since V is assumed to be a tangent vector at (x, y, z), this means that (x, y, z) is not the North Pole so that

$$x^2 + y^2 > 0.$$

Next, replacing V with $\frac{1}{c}(V)$ just multiplies $V \bullet_K V$ by $\frac{1}{c^2}$, so it suffices to consider the case in which

$$V = (a, b, 1),$$

and we must show that

$$\left(a^2+b^2\right)+K^{-1}>0.$$

Since V is tangent to our K-geometry at some point (x,y,z), we know by Exercise 101 that $(x,y,z)\bullet_K V=0$, that is,

$$ax+by+\frac{z}{K}=0.$$

On the other hand

$$K\left(x^2+y^2\right)+z^2=1.$$

Substituting the expression for z given by the former equation into the latter gives

$$K\left(x^2+y^2\right)+K^2\left(ax+by\right)^2=1.$$

On the other hand

$$\left(ay-bx\right)^2\geq 0$$
$$\left(ay\right)^2+\left(bx\right)^2\geq 2abxy$$

so that

$$K\left(x^2+y^2\right)+K^2\left(\left(ax\right)^2+\left(by\right)^2+\left(ay\right)^2+\left(bx\right)^2\right)\geq 1$$
$$K\left(x^2+y^2\right)+K^2\left(a^2+b^2\right)\left(x^2+y^2\right)\geq 1$$
$$K^{-1}+\left(a^2+b^2\right)\geq\frac{1}{K^2\left(x^2+y^2\right)}.$$

□

11.2. Hyperbolic geometry is homogeneous

11.2.1. Rigid motions in (x,y,z)-coordinates. Now **HG** is a K-geometry in the sense of Part V since, in (x,y,z)-coordinates, the equation for the K-geometry becomes

$$K\left(x^2+y^2\right)+z^2=1 \tag{11.2.1}$$

with

$$K<0,$$

and the K-dot-product is a valid dot-product. If we have a curve $X(t)=(x(t),y(t),z(t))$ on the K-geometry given (in K-coordinates) as

$$1=K\left(x^2+y^2\right)+z^2,$$

we have seen that we measure its length L by the formula

$$L=\int_b^e l(t)\,dt, \tag{11.2.2}$$

where

$$l(t)^2=\frac{dX}{dt}\bullet_K\frac{dX}{dt}, \tag{11.2.3}$$

and that we measure angles ϑ between tangent vectors $\hat{V}_1$ and $\hat{V}_2$ at a point by the formula

$$\vartheta=\arccos\left(\frac{V_1\bullet_K V_2}{\left|V_1\right|_K\cdot\left|V_2\right|_K}\right),$$

where

$$|V|_K^2 = V \bullet_K V.$$

We again want to explore the condition that a transformation

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot M$$

preserves the length of any curve $(x(t), y(t), z(t))$ lying on the K-geometry. Rewriting the transformation as

$$\underline{X} = X \cdot M,$$

all we have to worry about is that

$$\frac{d\underline{X}}{dt} \bullet_K \frac{d\underline{X}}{dt} = \frac{dX}{dt} \bullet_K \frac{dX}{dt}.$$

So, referring to Definition 19, a transformation given by a matrix M will preserve the length of any path and will preserve the measure of any angle if M is K-orthogonal.

Exercise 143 (HG). Show that the matrix

$$\begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is K-orthogonal. Describe geometrically what this transformation is doing to the K-geometry.

For our second K-rigid motion in **HG** we will need a pair of functions

$$\left(\cosh\varphi = \frac{e^{\varphi} + e^{-\varphi}}{2}, \sinh\sigma = \frac{e^{\varphi} - e^{-\varphi}}{2}\right)$$

that parametrize the unit hyperbola

$$z^2 - x^2 = 1$$

in the same way that $(\cos\varphi, \sin\varphi)$ parametrize the unit circle. That is,

$$\cosh^2\varphi - \sinh^2\varphi \equiv 1,$$

and if

$$z_0^2 - x_0^2 = 1,$$

then there is a φ such that $z_0 = \cosh\varphi$ and $x_0 = \sinh\varphi$.

Exercise 144 (HG). Show that the matrix

$$\begin{pmatrix} \cosh\varphi & 0 & |K|^{1/2} \cdot \sinh\varphi \\ 0 & 1 & 0 \\ |K|^{-1/2} \cdot \sinh\varphi & 0 & \cosh\varphi \end{pmatrix}$$

is K-orthogonal. Describe geometrically what this transformation is doing to the the K-geometry.

11.2.2. Moving a point and vector to the North Pole by a rigid motion. So, first of all, the North Pole is the point

$$N = (0, 0, 1)\,.$$

Suppose we start with a point

$$X_0 = (x_0, y_0, z_0)$$

in the geometry, that is, satisfying equation (11.2.1).

Exercise 145 (HG). Write an explicit K-rigid motion

$$M_1 = \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes the point X_0 to a point $X_1 = (x_1, 0, z_0)$.

Exercise 146 (HG). Write an explicit K-rigid motion

$$M_2 = \begin{pmatrix} \cosh\varphi & 0 & |K|^{1/2} \cdot \sinh\varphi \\ 0 & 1 & 0 \\ |K|^{-1/2} \cdot \sinh\varphi & 0 & \cosh\varphi \end{pmatrix}$$

that takes the point $X_1 = (x_1, 0, z_0)$ to $N = (0, 0, 1)$.
Hint: Notice that

$$Kx_1^2 + z_0^2 = 1 = -\left(-|K|^{1/2} \cdot x_1\right)^2 + z_0^2.$$

So there is a φ with

$$\cosh\varphi = z_0$$

and

$$\sinh\varphi = -|K|^{1/2} \cdot x_1.$$

Try that φ in M_2.

Using these last two exercises, we conclude that the transformation

$$\left(\underline{x}, \underline{y}, \underline{z}\right) = (x, y, z) \cdot (M_1 \cdot M_2)$$

is a K-rigid motion (why?) and that

$$N = (x_0, y_0, z_0) \cdot (M_1 \cdot M_2)$$

(why?).

Let

$$V_2 = (a_2, b_2, 0)$$

be a tangent vector to K-geometry at the North Pole N.

Exercise 147 (HG). Write an explicit K-rigid motion

$$M_3 = \begin{pmatrix} \cos\vartheta' & \sin\vartheta' & 0 \\ -\sin\vartheta' & \cos\vartheta' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that takes V_2 to the vector

$$\left(\sqrt{a_2^2+b_2^2},0,0\right)=\left(\sqrt{V_2 \bullet_K V_2},0,0\right).$$

Why does the transformation given by M_3 leave the North Pole N fixed?

11.2.3. Moving a (point, direction) to any other (point, direction) by a rigid motion. Now suppose we have any point

$$X_0=(x_0,y_0,z_0)$$

in K-geometry and any K-tangent vector

$$V_0=(a_0,b_0,c_0)$$

at that point.

Exercise 148 (HG). Explain why the K-rigid motion

$$(\underline{x},\underline{y},\underline{z})=(x,y,z)\cdot(M_1\cdot M_2\cdot M_3)$$

constructed over the last couple of sections takes the point X_0 to N and the tangent vector V_0 to $\left(\sqrt{V_0 \bullet_K V_0},0,0\right)$.

Now suppose that (X_0,V_0) gives a point X_0 in K-geometry and a tangent direction V_0 to K-geometry at X_0. Suppose that (X_0',V_0') gives another point in K-geometry and a tangent direction to K-geometry at X_0'. Finally suppose that

$$V_0 \bullet_K V_0 = V_0' \bullet_K V_0'.$$

As above, find a K-rigid motion given by

$$M=(M_1\cdot M_2\cdot M_3)$$

taking X_0 to the North Pole and V_0 to $\left(\sqrt{V_0 \bullet_K V_0},0,0\right)$. Similarly find a K-rigid motion given by

$$M'=(M_1'\cdot M_2'\cdot M_3')$$

taking X_0' to the North Pole and V_0' to $\left(\sqrt{V_0' \bullet_K V_0'},0,0\right)$.

Exercise 149 (HG). Explain why the K-rigid motion given by

$$M\cdot(M')^{-1}$$

takes (X_0,V_0) to (X_0',V_0').

By completing this exercise we have shown that **HG** looks the same at each point and in each direction at that point. That is, we have shown that **HG** is homogeneous.

11.3. Lines in hyperbolic geometry

11.3.1. Hyperbolic coordinates, the shortest path from the North Pole. Next, we will figure out the shortest path you can take between two points in **HG**. Again we will do our calculation using only (x, y, z)-coordinates (since, as we have seen in (9.1.8), we don't have $(\hat{x}, \hat{y}, \hat{z})$-coordinates). The (x, y, z)-coordinates for **SG**, namely

$$\begin{aligned} x(\sigma, \tau) &= R \cdot \sin\sigma \cdot \cos\tau, \\ y(\sigma, \tau) &= R \cdot \sin\sigma \cdot \sin\tau, \\ z(\sigma, \tau) &= \cos\sigma, \end{aligned}$$

won't work this time because they involve R which has gone off to infinity. Fortunately there are hyperbolic coordinates

$$\left(\cosh\sigma = \frac{e^{\sigma} + e^{-\sigma}}{2}, \sinh\sigma = \frac{e^{\sigma} - e^{-\sigma}}{2}\right)$$

that parametrize the "unit" hyperbola just like $(\cos\sigma, \sin\sigma)$ parametrize the unit circle. So we define

$$\begin{aligned} x(\sigma, \tau) &= |K|^{-1/2} \cdot \sinh\sigma \cdot \cos\tau, \\ y(\sigma, \tau) &= |K|^{-1/2} \cdot \sinh\sigma \cdot \sin\tau, \\ z(\sigma, \tau) &= \cosh\sigma. \end{aligned}$$

Exercise 150 (HG). Show that these hyperbolic coordinates do actually parametrize the K-geometry, that is, that

$$K\left(x(\sigma, \tau)^2 + y(\sigma, \tau)^2\right) + z(\sigma, \tau)^2 \equiv 1$$

for all (σ, τ).

Again notice that you can write a path on the R-sphere by giving a path $(\sigma(t), \tau(t))$ in the (σ, τ)-plane. In fact, you can use σ as the parameter t and just write

$$(\sigma, \tau(\sigma)),$$

where τ is a function of σ. To write a path that starts at the North Pole, just write

$$(\sigma, \tau(\sigma)), \ 0 \leq \sigma \leq \varepsilon$$

and demand that

$$\tau(0) = 0.$$

If you want the path to end on the plane $y = 0$, demand additionally that

$$\tau(\varepsilon) = 0.$$

But if we are going to describe paths on **HG** by paths in the (σ, τ)-plane, we are going to need to figure out the K-dot-product in (σ, τ)-coordinates so that we can compute the lengths of paths in these coordinates.

Exercise 151 (HG). a) Compute the 2×3 matrix D_{hyp} such that

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(\frac{d\sigma}{dt}, \frac{d\tau}{dt}\right) \cdot D_{hyp}$$

when a path in K-geometry is given by a path in the (σ, τ)-plane.
Hint: By the Chain Rule from several variable calculus

$$D_{hyp} = \begin{pmatrix} \frac{dx}{d\sigma} & \frac{dy}{d\sigma} & \frac{dz}{d\sigma} \\ \frac{dx}{d\tau} & \frac{dy}{d\tau} & \frac{dz}{d\tau} \end{pmatrix}.$$

b) Use a) to compute the K-dot-product in (σ, τ)-coordinates, namely

$$\begin{aligned} &\left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \bullet_{hyp} \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right) \\ &= \left(\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}\right) \bullet_K \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}, \frac{dz_2}{dt}\right) \\ &= \left(\frac{dx_1}{dt}, \frac{dy_1}{dt}, \frac{dz_1}{dt}\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot \left(\frac{dx_2}{dt}, \frac{dy_2}{dt}, \frac{dz_2}{dt}\right)^t \\ &= \left(\frac{d\sigma_1}{dt}, \frac{d\tau_1}{dt}\right) \cdot D_{hyp} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K^{-1} \end{pmatrix} \cdot D_{hyp}^t \cdot \left(\frac{d\sigma_2}{dt}, \frac{d\tau_2}{dt}\right)^t. \end{aligned}$$

Exercise 152 (HG). Show that the length L of any path in our K-geometry given by

$$(\sigma, \tau(\sigma)), \ 0 \leq \sigma \leq \varepsilon$$

with

$$\tau(0) = 0$$

and

$$\tau(\varepsilon) = 0$$

is given by the formula

$$L = |K|^{-1/2} \int_0^\varepsilon \sqrt{\left(1, \frac{d\tau}{d\sigma}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 \sigma \end{pmatrix} \cdot \left(1, \frac{d\tau}{d\sigma}\right)^t} d\sigma.$$

This last formula for L lets us figure out the shortest path from

$$N = (\sinh 0 \cdot \cos 0, R \cdot \sinh 0 \cdot \sin 0, \cosh 0)$$

to

$$\left(|K|^{-1/2} \cdot \sinh \varepsilon, 0, \cosh \varepsilon\right) = \left(|K|^{-1/2} \cdot \sin \varepsilon \cdot \cos 0, |K|^{-1/2} \cdot \sin 0 \cdot \sin 0, \cosh \varepsilon\right).$$

Since

$$L = |K|^{-1/2} \cdot \int_0^\varepsilon \sqrt{1 + \sinh^2 \sigma \cdot \left(\tfrac{d\tau}{d\sigma}\right)^2} d\sigma$$

and $\sinh^2 \sigma$ is positive for almost all $\sigma \in [0, \varepsilon]$, L is minimal only when $\frac{d\tau}{d\sigma}$ is identically zero. But this means that $\tau(\sigma)$ is a constant function. Since $\tau(0) = 0$, this means that $\tau(\sigma)$ is identically zero. So we have shown the following result.

Theorem 16 (HG). *The shortest path in K-geometry from the North Pole to a point $(x, y, z) = \left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$ is the path lying in the plane $y = 0$. The K-length of that shortest path is*

$$|K|^{-1/2} \cdot \varepsilon.$$

11.3.2. Shortest path between any two points.

Theorem 17 (HG). *Given any two points $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ in K-geometry, the shortest path between the two points is the path cut out by*

$$K\left(x^2 + y^2\right) + z^2 = 1$$

and the plane

$$\left|\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}\right| = 0, \tag{11.3.1}$$

that is, the plane containing $(0, 0, 0)$, X_1, and X_2.

Proof. Let $V_1 = (a_1, b_1, c_1)$ be the unit tangent vector at X_1 to the curve cut out by the plane (11.3.1) of K-length one. Then $(x, y, z) = (a_1, b_1, c_1)$ also satisfies equation (11.3.1), and so the equation for that plane can also be written

$$\left|\begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ a_1 & b_1 & c_1 \end{pmatrix}\right| = 0. \tag{11.3.2}$$

By Exercise 131 there is a K-rigid motion M that takes X_1 to the North Pole N and V_1 to $(1, 0, 0)$. So M takes the plane (11.3.2) to the plane given by the equation

$$\left|\begin{pmatrix} x & y & z \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right| = 0,$$

namely the plane

$$y = 0.$$

So $X_2 \cdot M$ must also lie in the plane $y = 0$ since X_2 lies in the plane (11.3.2). So

$$X_2 \cdot M = \left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$$

for some ε since all points in K-geometry with $y = 0$ can be written as $\left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$ for some ε. Since M is a K-rigid motion, it must take the shortest path from X_1 to X_2 to the shortest path from $X_1 \cdot M = N$ to $X_2 \cdot M = \left(|K|^{-1/2} \cdot \sinh\varepsilon, 0, \cosh\varepsilon\right)$. And we already know that the shortest path from $X_1 \cdot M$ to $X_2 \cdot M$ is the one that lies in the plane $y = 0$. But that path comes from the path cut out by the plane given by equation (11.3.2), that is, the plane given by equation (11.3.1). □

The path given by equation (11.3.1) is called the *great hyperbolic arc* between X_1 and X_2.

Exercise 153 (HG). Explain why lines in **HG** extend infinitely in each direction. Hint: There is a K-rigid motion that takes any two points to $(0,0,1)$ and $\left(|K|^{-1/2} \cdot \sinh \varepsilon, 0, \cosh \varepsilon\right)$ for some $\varepsilon > 0$. Why does that mean that the K-distance between the two points is equal to $|K|^{-1/2} \cdot \varepsilon$?

Definition 23. A *line* in **HG** will be a curve that extends infinitely in each direction and has the property that given any two points X_1 and X_2 on the path, the shortest path between X_1 and X_2 lies along that curve. Lines in **HG** are the intersections of the K-geometry with planes through $(0,0,0)$. The length of the shortest path between two points in K-geometry will be called the K-distance.

11.4. Central projection in HG

11.4.1. The edge of the universe. Again **HG** is a K-geometry in the sense of Part V since, in (x,y,z)-coordinates, the equation for the K-geometry is

$$K\left(x^2+y^2\right)+z^2=1 \tag{11.4.1}$$

with

$$K<0,$$

and the K-dot-product is a valid dot-product. So all the calculations in Part V hold, in particular (9.2.3). So the (x_c, y_c)-coordinates that parametrize the entire K-geometry are such that

$$x_c^2+y_c^2<\frac{1}{|K|}.$$

We call the circle

$$x_c^2+y_c^2=\frac{1}{|K|}$$

the *edge of the universe*. (The (x_c, y_c)-coordinates are called Klein coordinates and the disk of radius $|K|^{-1/2}$ is called the *Klein model* for **HG** in honor of the famous German geometer, Felix Klein.)

Exercise 154. a) Use formula (9.2.4) for K-rigid motions to check that for K-rigid motions,

$$\begin{pmatrix} \underline{x} & \underline{y} & \underline{z} \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & K^{-1} \end{pmatrix}\begin{pmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{pmatrix}=0$$

whenever

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & K^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$$

and vice versa.

b) Recall that $m_{13}x+m_{23}y+m_{33}z=\underline{z}$ for the K-rigid motion given by the matrix M in (x,y,z)-coordinates. If $K<0$, use a) to explain why (x,y,z) lies on the light cone if and only if $\left(\underline{x}, \underline{y}, \underline{z}\right)$ lies on the light cone.

c) Use

$$\underline{z}=m_{13}x+m_{23}y+m_{33}z$$

to conclude that the plane given by

$$m_{13}x + m_{23}y + m_{33}z = 0$$

intersects the light cone only at $(0, 0, 0)$.

d) Show that the line given by

$$m_{13}x_c + m_{23}y_c + m_{33} = 0$$

never intersects the edge of the universe in central projection coordinates (nor in stereographic projection coordinates), no matter the (negative) value of K.

We will use this last exercise in what follows since it implies that a K-rigid motion in (x_c, y_c)-coordinates takes any tangent line to the edge of the universe circle to another tangent line to the edge of the universe circle.

11.4.2. Lines go to chords. Again all the calculations in Part V hold, in particular Exercise 118a). We conclude that lines in **HG** correspond to chords on the Klein (x_c, y_c)-disk that connect two points on the edge of the universe.

Exercise 155 (HG). a) Explain why the K-line $y = 0$ is given by the x_c-axis and the North Pole N is given by $(x_c, y_c) = (0, 0)$.

b) Explain why the point $\left(|K|^{-1/2} \cdot \sinh \varepsilon, 0, \cosh \varepsilon\right)$ in the K-geometry is given by the point

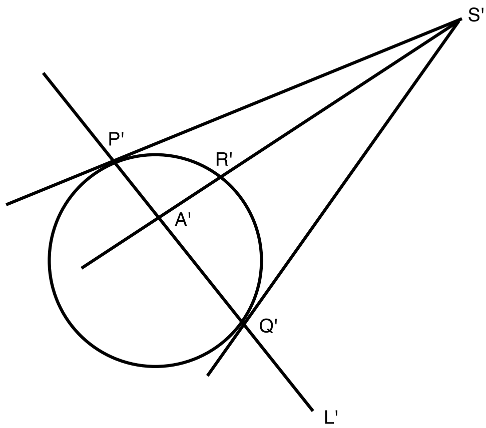
$$(x_c, y_c) = \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}}, 0\right).$$

c) Explain why the K-distance between $(x_c, y_c) = (0, 0)$ and $(x_c, y_c) = \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon} - e^{-\varepsilon}}{e^{\varepsilon} + e^{-\varepsilon}}, 0\right)$ is $|K|^{-1/2} \cdot \varepsilon$.

Hint: Convert the statement to a statement about the distance between two points in (x, y, z)-coordinates, a distance that we have computed previously.

Exercise 156 (HG). Use (x_c, y_c)-coordinates to show that **HG** satisfies the four Euclidean postulates E1, E2, E3, and E4. Thus hyperbolic geometry is a neutral geometry (**NG**).

11.4.3. K-perpendicularity in the Klein model for HG. Suppose we are given any three distinct points P', R', and Q' on the edge of the universe of the Klein K-disk. We construct the line L' through P' and Q' and mark a point A' on it as shown in the following figure.



We know that there is a K-rigid motion M_c that takes A' to $(0,0)$ and L' to the x_c-axis. (Why?) Viewed as a transformation of the entire (x_c, y_c)-plane, this transformation takes the tangent line to the edge of the universe at P' to the tangent line to the edge of the universe at $\left(-|K|^{-1/2}, 0\right)$ and the tangent line to the edge of the universe at Q' to the tangent line to the edge of the universe at $\left(|K|^{-1/2}, 0\right)$. Since the tangent lines at $\left(-|K|^{-1/2}, 0\right)$ and $\left(|K|^{-1/2}, 0\right)$ are vertical, the point S' must have gone to infinity under the K-rigid motion, and so the line through A' and R' must go to the y_c-axis.

Exercise 157. Explain why there is a K-rigid motion M_c that takes any three points P', R', and Q' in order along the edge of the universe to any other three points P'', R'', and Q'' in order along the edge of the universe.

Hint: Use the fact that the set of K-rigid motions forms a group under the composition operation.

Exercise 158 (HG). Explain why the above discussion implies that the angles $\angle P'A'R'$ and $\angle Q'A'R'$ must both be K-right angles; that is, their K-measures must each be 90°. So the line segments $\overline{P'Q'}$ and $\overline{A'R'}$ are K-perpendicular [**MJG**, 238–239].

Hint: You may need to use the fact that since there is a K-rigid motion that interchanges $\left(-|K|^{-1/2}, 0\right)$ and $\left(|K|^{-1/2}, 0\right)$ and leaves $(0,0)$ fixed, the x_c-axis and the y_c-axis are K-perpendicular.

Exercise 159 (HG). Use the previous exercise and the fact that A' can be any point along the chord $\overline{P'Q'}$ in the figure above to explain why the Klein model is not conformal; that is, it does not faithfully represent the measure of angles in **HG**.

Exercise 160. Use (x_c, y_c)-coordinates to show that **HG** does not satisfy Euclid's postulate E5. That is, through a point not on a line, it is not true that there passes a unique parallel (i.e., nonintersecting) line.

11.4.4. Computing K-distances using cross-ratio in Klein coordinates. In fact, the tool that will let us compute all K-distances in (x_c, y_c)-coordinates is the cross-ratio from Definition 17. Let $d_K(A_c, B_c)$ denote the K-distance between two points A_c and B_c in the Klein K-disk. Now we know that

$$d_K\left((0,0), \left(|K|^{-1/2} \cdot \frac{e^\varepsilon - e^{-\varepsilon}}{e^\varepsilon + e^{-\varepsilon}}, 0\right)\right) = |K|^{-1/2} \cdot \varepsilon.$$

To see what this has to do with cross-ratio, we begin by computing the cross-ratio

$$\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^\varepsilon - e^{-\varepsilon}}{e^\varepsilon + e^{-\varepsilon}} : |K|^{-1/2}\right)$$

given by the two points $(0,0)$, $\left(|K|^{-1/2} \cdot \frac{e^\varepsilon - e^{-\varepsilon}}{e^\varepsilon + e^{-\varepsilon}}, 0\right)$ and the two points $\left(-|K|^{-1/2}, 0\right)$ and $\left(|K|^{-1/2}, 0\right)$ where the x_c-axis intersects the edge of the universe.

Exercise 161 (HG). a) Draw a picture of the Klein K-disk, the edge of the universe, and the four points on the x_c-axis.

b) Show that

$$\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^\varepsilon - e^{-\varepsilon}}{e^\varepsilon + e^{-\varepsilon}} : |K|^{-1/2}\right) = \left(0 : -1 : \frac{e^\varepsilon - e^{-\varepsilon}}{e^\varepsilon + e^{-\varepsilon}} : 1\right).$$

In particular, notice that the computation doesn't depend on K.

c) Show that

$$\left(0 : -1 : \frac{e^\varepsilon - e^{-\varepsilon}}{e^\varepsilon + e^{-\varepsilon}} : 1\right) = e^{-2\varepsilon}.$$

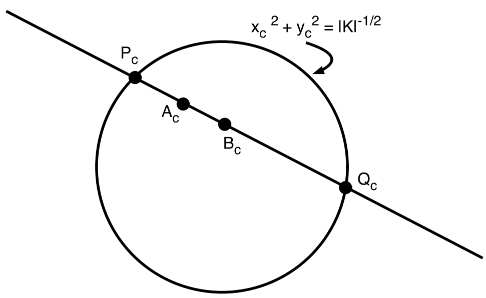
From this exercise we conclude that

$$(11.4.2)\quad d_K\left((0,0), \left(|K|^{-1/2} \cdot \frac{e^\varepsilon - e^{-\varepsilon}}{e^\varepsilon + e^{-\varepsilon}}, 0\right)\right) = \frac{|K|^{-1/2}}{2} \cdot \left|\ln\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^\varepsilon - e^{-\varepsilon}}{e^\varepsilon + e^{-\varepsilon}} : |K|^{-1/2}\right)\right|.$$

Now suppose we are given any two points A_c and B_c in the Klein K-disk. They determine a line

$$(11.4.3)\qquad \alpha x_c + \beta y_c + \gamma = 0$$

and points P_c and Q_c where that line meets the edge of the universe as shown in the figure below.



We are now ready to prove the following theorem.

Theorem 18 (HG). *For any two points A_c and B_c on the Klein K-disk, the K-distance $d_K(A_c, B_c)$ between them is given by the formula*

$$d_K(A_c, B_c) = \frac{|K|^{-1/2}}{2} \cdot |\ln(x_c(A_c) : x_c(P_c) : x_c(B_c) : x_c(Q_c))|,$$

where P_c and Q_c are the endpoints of the chord through A_c and B_c. (*Compare with* [**MJG**, 268].)

Proof. We know that there is a K-rigid motion $(\underline{x_c}, \underline{y_c}) = M_c(x_c, y_c)$ of the Klein disk that takes A_c to $(0,0)$ and B_c to some point $\left(|K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}}, 0\right)$ on the positive x_c-axis. From (9.2.4) we know that

$$\underline{x_c} = \frac{m_{11}x_c + m_{21}y_c + m_{31}}{m_{13}x_c + m_{23}y_c + m_{33}}.$$

But from (11.4.3) we know that for our four points A_c, B_c, P_c, and Q_c,

$$\alpha x_c + \beta y_c + \gamma = 0$$
$$y_c = \frac{-\alpha x_c - \gamma}{\beta}.$$

So if we calculate M_c only for these four points, we have

$$\underline{x_c} = \frac{m_{11}x_c + m_{21}\left(\frac{-\alpha x_c - \gamma}{\beta}\right) + m_{31}}{m_{13}x_c + m_{23}\left(\frac{-\alpha x_c - \gamma}{\beta}\right) + m_{33}} = \frac{\left(m_{11} - \frac{m_{21}\alpha}{\beta}\right)x_c + \left(m_{31} - \frac{m_{21}\gamma}{\beta}\right)}{\left(m_{13} - \frac{m_{23}\alpha}{\beta}\right)x_c + \left(m_{33} - \frac{m_{23}\gamma}{\beta}\right)}.$$

That is, the function $x_c \mapsto \underline{x_c}$ is a linear fractional transformation! So by Theorem 13,

$$\begin{aligned}(x_c(A_c) : x_c(P_c) : x_c(B_c) : x_c(Q_c)) &= (\underline{x_c}(A_c) : \underline{x_c}(P_c) : \underline{x_c}(B_c) : \underline{x_c}(Q_c)) \\ &= \left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}} : |K|^{-1/2}\right) \\ &= e^{-2\varepsilon}.\end{aligned}$$

Therefore

$$\begin{aligned}d_K(A_c, B_c) &= d_K\left((0,0), \left(|K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}}, 0\right)\right) \\ &= \frac{|K|^{-1/2}}{2} \cdot \left|\ln\left(0 : -|K|^{-1/2} : |K|^{-1/2} \cdot \frac{e^{\varepsilon}-e^{-\varepsilon}}{e^{\varepsilon}+e^{-\varepsilon}} : |K|^{-1/2}\right)\right| \\ &= \frac{|K|^{-1/2}}{2} \cdot |\ln(x_c(A_c) : x_c(P_c) : x_c(B_c) : x_c(Q_c))|.\end{aligned}$$

□

Exercise 162. For $K = -1$, calculate the K-distance between the two points given in (x_c, y_c)-coordinates by $(0,0)$ and $(1/2, 0)$.

11.4.5. Areas of hyperbolic lunes. Finally, there is one K-area computation that is convenient to do in Klein coordinates. Namely, suppose that we take the ordinary triangle T_c in the (x_c, y_c)-plane with vertices $(0,0)$, $\left(|K|^{-1/2}\cos\beta, |K|^{-1/2}\sin\beta\right)$, and $\left(|K|^{-1/2}\cos\beta, -|K|^{-1/2}\sin\beta\right)$. Notice that two of the three vertices lie on the edge of the universe of the Klein K-disk and that the K-angle at the third vertex is

$$\alpha = 2\beta.$$

We will call the interior of this triangle, or any K-rigid motion of it, an α-*lune*. So we wish to compute the K-area of an α-lune. Since **HG** is a K-geometry, we know from Exercise 113 that this area $A_K(\alpha)$ is given by the formula

$$A_K(\alpha) = \int_{T_c} \frac{1}{\left(1 - |K|\left(x_c^2 + y_c^2\right)\right)^{3/2}} dx_c dy_c.$$

Exercise 163. Show that

$$A_K(\alpha) = |K|^{-1}(\pi - \alpha).$$

Hint: Use the substitution

$$\begin{aligned} \underline{x_c} &= |K|^{1/2} x_c, \\ \underline{y_c} &= |K|^{1/2} y_c \end{aligned}$$

to reduce the computation to the computation in the case that $|K| = 1$. Then use polar coordinates to get

$$A_{-1}(\alpha) = \int_{\vartheta=-\beta}^{\vartheta=\beta} \left(\int_{r=0}^{r=\frac{\cos\beta}{\cos\vartheta}} \frac{1}{\left(1 - r^2\right)^{3/2}} r \cdot dr \right) d\vartheta.$$

Then do the substitution

$$\begin{aligned} u &= 1 - r^2, \\ du &= -2r dr \end{aligned}$$

to compute $\int_{r=0}^{r=\frac{\cos\beta}{\cos\vartheta}} \frac{1}{(1-r^2)^{3/2}} r \cdot dr$. In the final step use the substitution

$$t = \sin\vartheta$$

to reduce to an integral of the form

$$\int \frac{a^{-1}}{\sqrt{1 - \left(\frac{t}{a}\right)^2}} dt.$$

Definition 24. For $K < 0$, let T_K denote the region in K-geometry given by the triangle T_c whose area is calculated in Exercise 163. An α-*lune* in K-geometry is any region in K-geometry that is obtained from the region T_K by a K-rigid motion. So the K-area of any α-lune is $|K|^{-1}(\pi - \alpha)$.

11.5. Stereographic projection in HG

11.5.1. The Poincaré K-disk and the edge of the universe. Under stereographic projection, the center of projection is the South Pole $(0,0,-1)$. So if $K<0$ and a point $(x,0,z)$ on the K-geometry goes out the hyperboloid to infinity while continuing to lie on the (x,z)-plane, the line joining $(0,0,-1)$ to that point becomes parallel to an asymptote of the hyperbola

$$Kx^2+z^2=1.$$

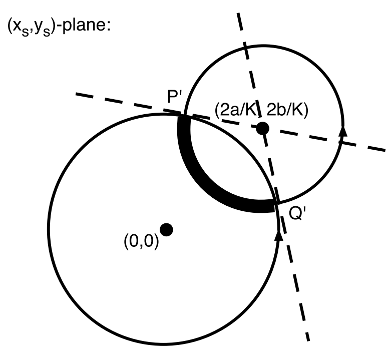
So the line approaches a line of slope $\pm|K|^{1/2}$ in the (x,z)-plane, and therefore that limit line is the line $z=\pm|K|^{1/2}x-1$. Therefore the intersection of that line with the line $z=1$ in the (x,z)-plane approaches the point with $x=\frac{\pm 2}{|K|^{1/2}}$. Therefore under stereographic projection, the edge of the universe is given by the circle

$$x_s^2+y_s^2=\frac{4}{|K|}.$$

The interior of this circle, that is, the image of K-geometry under stereographic projection, is called the Poincaré model of hyperbolic geometry, of course after the famous geometer, Henrí Poincaré. Again, since **HG** is a K-geometry, all the rules of Part V apply. So by Exercise 118b), lines in the K-geometry are given by circles of the form

$$\left(x_s-\frac{2a}{K}\right)^2+\left(y_s-\frac{2b}{K}\right)^2=\frac{4\left(K+a^2+b^2\right)}{K^2} \qquad (11.5.1)$$

in the Poincaré K-disk. The darker arc below



represents the K-line

$$ax+by+1=0$$

in the Poincaré K-disk.

11.5.2. Stereographic projection preserves angles.

Exercise 164 (HG). a) Show that stereographic projection is conformal, that is, that the measure of K-angles between K-lines on K-geometry is just the ordinary Euclidean measure of angles formed by their (usually circular) stereographic projections.

Hint: See subsection 9.3.3.

b) For $K = -1$, construct the K-line in (x_s, y_s)-coordinates that meets the K-line

$$(x_s - 2)^2 + (y_s - 2)^2 = 4$$

perpendicularly at the point $\left(2 - \sqrt{2}, 2 - \sqrt{2}\right)$.

Exercise 165. Show that in K-geometry for any K, the angle between two tangent vectors at the North Pole is the same as the ordinary Euclidean angle between the two corresponding tangent vectors in the (x_c, y_c)-plane, and that angle is also the same as the ordinary Euclidean angle between the two corresponding tangent vectors in the (x_s, y_s)-plane.

To get a more precise idea of what K-lines look like under stereographic projection, again consider equation (11.5.1) and the picture below it. The equations of the circles in the picture are

$$x_s^2 + y_s^2 = \frac{4}{|K|} \tag{11.5.2}$$

and

$$\left(x_s - \frac{2a}{K}\right)^2 + \left(y_s - \frac{2b}{K}\right)^2 = \frac{4\left(K + a^2 + b^2\right)}{K^2}. \tag{11.5.3}$$

Construct a third circle whose diameter is the line segment from $(0,0)$ to $\left(\frac{2a}{K}, \frac{2b}{K}\right)$, namely the circle $\left(x_s - \frac{a}{K}\right)^2 + \left(y_s - \frac{b}{K}\right)^2 = \left(\frac{a}{K}\right)^2 + \left(\frac{b}{K}\right)^2$ which can be rewritten

$$x_s^2 + y_s^2 - \frac{2a}{K}x_s - \frac{2b}{K}y_s = 0. \tag{11.5.4}$$

Lemma 8. *The circles* (11.5.2), (11.5.3), *and* (11.5.4) *all pass through two common points.*

Proof. From (11.5.2) and (11.5.3) we get by addition that

$$x_s^2 + y_s^2 + \left(x_s - \frac{2a}{K}\right)^2 + \left(y_s - \frac{2b}{K}\right)^2 = \frac{4}{|K|} + \frac{4\left(K + a^2 + b^2\right)}{K^2}.$$

Simplifying this last equation and dividing both sides by two, we obtain equation (11.5.4). So the two points P' and Q' in picture (11.5.1) that satisfy both equations (11.5.2) and (11.5.3) also satisfy equation (11.5.4). □

The lemma tells us that the angle formed by the segments $\overline{(0,0)\,P'}$ and $\overline{P'\left(\frac{2a}{K}, \frac{2b}{K}\right)}$ is a right angle since it is an inscribed angle in the circle (11.5.4) whose associated central angle is a diameter of that circle. But $\overline{(0,0)\,P'}$ is a radius of circle (11.5.2) and so $\overline{P'\left(\frac{2a}{K}, \frac{2b}{K}\right)}$ is tangent to circle (11.5.2). Similarly $\overline{P'\left(\frac{2a}{K}, \frac{2b}{K}\right)}$ is a radius of circle (11.5.3) and so $\overline{(0,0)\,P'}$ is tangent to circle (11.5.3). So we conclude the following theorem.

Theorem 19 (HG). *In the Poincaré model for K-geometry, the K-lines are represesented by circular arcs that meet the edge of the universe perpendicularly.*

11.5.3. Infinite triangles in the Poincaré K-disk. By Exercise 118a) and 118b), lines in **HG** become circles under stereographic projection unless the line in **HG** passes through the North Pole (in which case it corresponds to a line through $(x_s, y_s) = (0,0)$ in the (x_s, y_s)-plane). Suppose a hyperbolic triangle T corresponds to a region T_s in (x_s, y_s)-coordinates and the vertices of T correspond to $(x_s, y_s) = (-2, 0)$, $(x_s, y_s) = (2, 0)$, and $(x_s, y_s) = (0, 2)$. So one side of T_s lies on the line $y_s = 0$.

Exercise 166 (HG). a) Use Exercise 118b) to compute the equations for the other two sides of T_s.

b) In the (x_s, y_s)-plane, draw T_s as accurately as you can when $K = -\frac{1}{4}$, then when $K = -1$.

The area of a hyperbolic triangle T is given by the formula

$$\int_{T_s} \frac{1}{\left(1 + \frac{K}{4}\left(x_s^2 + y_s^2\right)\right)^2} dx_s dy_s.$$

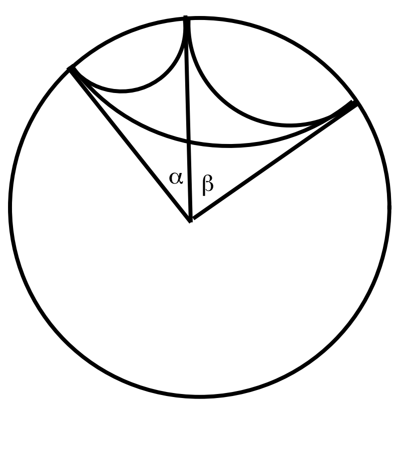
However, we do not yet have a way to calculate the area numerically for any given triangle T. The last topic in this book will remedy that situation. Analogous to the case of spherical triangles, we start from the fact that we do know the area of α-lunes. From Exercise 163 the K-area of an α-lune with one vertex at $(0,0)$ in (x_c, y_c)-coordinates is

$$|K|^{-1}(\pi - \alpha).$$

Exercise 167. Draw the α-lune in Exercise 163 in (x_s, y_s)-coordinates.

Since rigid motions preserve K-lengths and angles, and since stereographic projection is conformal, the ordinary Euclidean measure of the angle of the figure in Exercise 167 is α.

Since rotation of the (x_c, y_c)-plane around $(0,0)$ is a K-rigid motion, this formula holds for any K-lune with vertex at $(0,0)$. Now by Exercise 165 we can represent the *same* lune using the same angle in the (x_s, y_s)-plane. Below is a picture in the (x_s, y_s)-plane of some of these K-lunes.



Exercise 168 (HG). Use Exercise 163 to show that in the above picture the K-area that lies in the union of the α-lune and the β-lune but does not lie in the $(\alpha+\beta)$-lune is $|K|^{-1}\pi$.

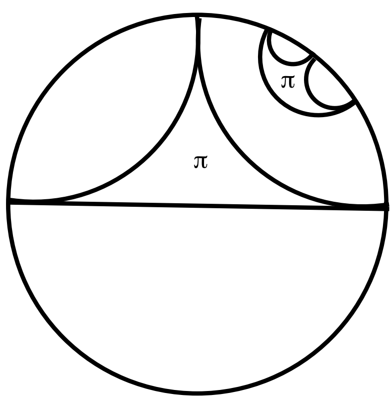
Exercise 169. Move the vertex of the α-lune in Exercise 167 to any other point of K-geometry by a rigid motion in (x_s, y_s)-coordinates. Draw the resulting figure (that we will continue to call an α-lune).

Definition 25 (HG). An infinite K-triangle is the figure given in stereographic projection coordinates by the stereographic projection of three K-lines such that any two meet the edge of the universe at a common point.

Exercise 170. a) (**HG**) Use Exercise 157 to show that the area of (the interior of) any infinite triangle has K-area

$$|K|^{-1} \cdot \pi.$$

For example, if $K = -1$, we have

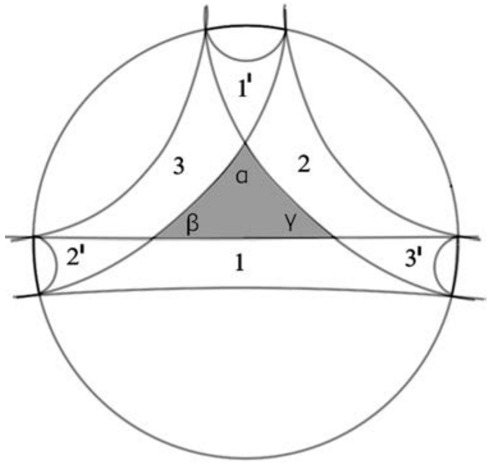


b) Use a) to give a formula for the K-area of any infinite n-gon in **HG**, that is, a figure described by a set of n disjoint K-lines that is the limit of a family of finite n-gons, all of whose vertices have gone to infinity. In particular, what is the area of any infinite hexagon?

Hint: Divide the infinite n-gon into infinite triangles.

11.5.4. Areas of polygons in HG. Consider the picture below in the Poincaré model for **HG**. Locate the infinite hexagon whose vertices are the points at infinity on the lines expending the sides of the hyperbolic triangle with interior angles α, β, and γ.

Exercise 171 (HG). a) Use the picture



and remarks just above to explain why the K-area of the hyperbolic triangle is

$$|K|^{-1} \cdot (\pi - (\alpha + \beta + \gamma)) .$$

Hint: Locate α-lunes, two β-lunes, and two γ-lunes in the picture and notice that they cover the hyperbolic triangle three times.

b) Use a) to give a formula for the K-area of a hyperbolic n-gon.