Preface

Real analysis is a beautiful subject that brings together several areas of mathematics. Its origins go back to calculus, a subject that is certainly at the center of our scientific understanding of the world. When trying to justify and give a firm foundation to the fundamental concepts and theorems of calculus, one is faced with having to understand the real numbers, functions, and their properties. The properties of functions include continuity, differentiation, and integration, which lead us to study the convergence of sequences and of series (of numbers and of functions). To understand the real numbers, one has to study properties such as compactness. Compactness also plays a crucial role when studying functions and sequences of functions. This in turn is expressed in the language of set theory, thus one needs a basic acquaintance with properties of sets.

One of the challenges in teaching and learning real analysis is that the subject was developed about 150 years after calculus. We know that calculus is largely the result of the work of Newton and Leibniz in the second half of the seventeenth century, though some of its ideas, such as the approximation methods of Archimedes, go back much earlier. Analysis as we know it today is the result of the work of mathematicians at the start of the nineteenth century. Students are confronted with concepts such as continuity, differentiation, and integration, and wonder why they have to relearn them when they have already seen them in calculus. A challenge for the instructor is to make these ideas come alive in new ways that shed new and important light on an older subject.

Audience

This book is addressed to students who have completed calculus and linear algebra courses as they are normally taught at colleges and universities in the United States. At the same time, all the notions that are used are defined, and linear algebra is not used explicitly. The book covers the basic theorems of calculus and also introduces students to more modern concepts such as open and closed sets, compactness, and uniform convergence. Metric spaces are introduced early in this book, but the instructor who wishes to avoid them or cover very little of them may do so.

Chapter 0 covers basic logic, sets, an introduction to proof techniques, functions, and mathematical induction. This chapter can be used as an introduction to proof-writing techniques, for self-study, or it can be omitted by students who have had a course covering sets, proof-writing, and induction. Readers who plan to omit this chapter should review Section 0.1, where all the notation used in the book regarding sets and functions is covered. On the other hand, students who are not familiar with these concepts or who have no or little experience with proof-writing and logic should cover Chapter 0 starting with Section 0.2. Depending on the class, the instructor may decide to spend one or more weeks with Chapter 0. The chapter ends with an optional section, which could be assigned for independent reading or covered later in the course, on the axioms of set theory and the construction of the natural numbers, the integers, and the rational numbers from these axioms.

Chapter 1 introduces the real numbers, including the notion of (order) completeness. It covers the field and order properties of the real numbers in detail, with an emphasis on understanding the various implications among these properties. The notion of supremum is introduced and used to define the completeness property. There is also an optional section on the construction of the real numbers that may be omitted as it is not needed later in the book.

Chapter 2 covers sequences of numbers, including the Cauchy property. This chapter introduces the formal definition of the limit of a sequence and emphasizes proof techniques dealing with limits.

Chapter 3 is probably the chapter with the largest number of concepts that may be new to the reader as they are not typically covered in a calculus course. These include open and closed sets, accumulation points, compactness, and the Cantor set. This chapter also covers those concepts in the context of metric spaces, but those sections may be omitted. The n-dimensional Euclidean space is treated as an example of a metric space, and the metric space sections are where the notions of open and closed sets are discussed in $\mathbb{R}^n$. However, the chapter is written so that the reader may omit the sections dealing with metric spaces and study these notions only in the context of the set of real numbers $\mathbb{R}$, or come back to them later. For example, it is possible to start with Section 3.2 instead of starting with Section 3.1.

Chapter 4 treats continuity, including uniform continuity, and it has a discussion of proof techniques for continuity. This chapter also introduces limits of functions and ends with continuity in the context of metric spaces.

Differentiation and integration and treated in Chapters 5 and 6, respectively. While they cover concepts from calculus, the approach is different and probably new to the reader; these should be covered in detail.

Series are the subject of Chapter 7. Chapter 8 introduces sequences and series of functions, and it brings together all the notions that have been studied earlier.

The book's emphasis is on understanding these ideas and making connections between the different concepts. Sometimes we give more than one proof of a result when the second proof brings out new ideas. Students are expected to write complete and careful proofs for the exercises.

This book contains two types of exercises. Regular exercises at different levels of difficulty are at the end of each section: those that may be particularly hard are marked with a star ($\star$); all exercises are numbered within their chapters. The second type, called "Questions", are interspersed throughout the exposition so that readers can verify their understanding of the material; solutions to all of the Questions are supplied in Appendix A.

The end of a definition is marked by the symbol "$\Diamond$".

Convention for Numbering Theorems and Exercises

Chapters are divided into sections. Corollaries, definitions, lemmas, propositions, questions, remarks, and theorems are numbered consecutively within each section. Figures are numbered consecutively within each chapter. Some sections have subsections, but they don't change the numbering. Exercises are at the end of each section and are numbered within that section.

Building a Course

The book has been designed and used for a one-semester course in real analysis, but there is more material than would typically be covered in a semester. At Williams College, in our regular course we start with Chapter 0 and cover logic, sets, functions, and induction (Sections 0.2–0.5) in about one to one and a half weeks. Then we cover the real numbers and completeness in detail, and the construction of the real numbers quickly and with some independent reading. The chapter on sequences (Chapter 2) is covered in detail; there is also the opportunity here for additional exercises and reading on various characterizations of completeness. Chapter 3 contains several new ideas, and we cover most of it, including the Cantor set and several characterizations of compactness; often metric spaces are not covered at the start and only later and as additional reading topics. Chapter 4 on continuous functions is important for developing familiarity with epsilon-delta arguments; the part dealing with metric spaces is usually omitted. Students' familiarity with differentiation is used to cover the first couple of sections in Chapter 5 rather quickly, with emphasis paid to Taylor's theorem. Integration in Chapter 6 is covered in detail, but not improper integrals. Series are also covered rather quickly. Chapter 8 brings together many of the topics that have been studied; emphasis is put on the exchange of limit theorems with applications to power series.

A course with more emphasis on applications could start by reviewing the section on sets and induction. In such a case it is possible to start with Chapter 2 on sequences; one can use the monotone sequence property as the characterization of order completeness. For the reader interested in applications, it is useful to cover metric spaces as there are several applications that can be based on them. The remaining topics are as in the previous course, but there will be more time to spend on Chapter 8.

The book can also be used for a two-semester course that spends a significant amount of time in Chapter 0, where students are introduced to proof-writing techniques. The first part can cover up to limits and continuity, and the second semester could start with differentiation, cover integration, including improper integrals, then series, and finally sequences and series of functions.

For the instructor who wants to include an independent reading component, there are several exercises that complement the material. For example, there is a section on the construction of the integers and rational numbers using equivalence relations, where the student is asked to complete significant parts of the material. The sections on metric spaces could be assigned for independent reading as some parts parallel the theory that has been developed for the real line.

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`www.ams.org/bookpages/amstext-36`.

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