Agenda

This is a book about mathematical modelling of strategic behavior, defined as behavior that arises whenever the outcome of an individual's actions depends on actions to be taken by other individuals. The individuals may be either human or nonhuman beings, the actions either premeditated or instinctive. Thus models of strategic behavior are applicable in both the social and the natural sciences.

Examples of humans interacting strategically include store managers setting prices (the number of customers who buy at one store's price will depend on the price at other stores in the neighborhood) and drivers negotiating a 4-way junction (whether it is advantageous for a driver to assume right of way depends on whether the driver facing her concedes right of way).[1] Examples of nonhumans interacting strategically include spiders disputing a territory (the risk of injury to one animal from being an aggressor will depend on whether the other is prepared to fight), insects foraging for sites at which to lay eggs (the number of one insect's eggs that mature into adults at a given site, where food for growth is limited, will depend on the number of eggs laid there by other insects), and mammals in a trio deciding between sharing a resource three ways or attempting to exclude one or both of their partners from it (whether it is advantageous to go it alone depends on whether the other two form an opposing coalition, as well as on the animals' strengths, and on the extent to which those strengths are amplified by coalition membership), and on a host of other factors, including chance.[2] These and other strategic interactions will be modelled in detail, beginning in Chapter 1.[3]

[1] We assume throughout that a human of indeterminate gender is female in odd-numbered chapters or appendices and male in even-numbered ones. This convention is simply the reverse of that which I adopted in *A Concrete Approach to Mathematical Modelling* [**210**] and obviates the continual use of "his or her" and "he or she" in place of epicene pronouns. For nonhumans, however, I adhere to the biologist's convention of using neuter pronouns—e.g., for the damselflies in §6.2 and the wasps in §6.7, even though their gender is known (male and female, respectively).

[2] As whimsically suggested by the picture on the front cover, where the two smallest meerkats of the trio roll a backgammon die together, in coalition, while the largest meerkat rolls a die by itself.

[3] Specifically, store managers setting prices will be modelled in §1.4 and §1.5; drivers negotiating a 4-way junction in §1.1, §1.3, §2.1, and §2.3; spiders disputing a territory in §7.3; insects foraging for oviposition sites in §5.1 and §5.7; and coalitions of two versus one in §4.9 and §8.3.

Table 0.1. Student achievement (number of satisfactory solutions) in mathematics as a function of effort

	Very hard ($E = 5$)	Quite hard ($E = 3$)	Hardly at all ($E = 1$)
Student 1	10	8	6
Student 2	8	7	5
Student 3	7	5	3
Student 4	7	4	3
Student 5	5	4	2
Student 6	4	2	1

To fix ideas, however, it will be helpful first to consider an example that, although somewhat fanciful, will serve to delineate the important distinction between strategic and nonstrategic decision making. Let us therefore suppose that the enrollment for some mathematics course is a mere six humans, and that grades for this course are based exclusively on answers to ten questions. Answers are judged to be either satisfactory or unsatisfactory, and the number of satisfactory solutions determines the final letter grade for the course—A, B, C, D, or F. In the usual way, A corresponds to 4 units of merit, and B, C, D, and F correspond to 3, 2, 1, and 0 units of merit, respectively. The students vary in motivation and intellectual ability, and all are capable of working very hard, or only quite hard, or hardly at all; but there is nothing random about student achievement as a function of effort, which (in these fanciful circumstances) is precisely defined as in Table 0.1. Thus, for example, Student 5 will produce five satisfactory solutions if she works very hard, but only four if she works quite hard; whereas Student 4 will produce seven satisfactory solutions if he works very hard, but only three if he works hardly at all. The students have complete control over how much effort they apply, and so we refer to effort as a decision variable. Furthermore, for the sake of definiteness, we assume that working very hard corresponds to 5 units of effort, quite hard to 3, and hardly at all to 1. Thus, if we denote effort by E and merit by M, then working very hard corresponds to $E = 5$; obtaining the letter grade A corresponds to $M = 4$; and similarly for the other values of E and M.

Let us now suppose that academic standards are absolute, i.e., the number of satisfactory solutions required for each letter grade is prescribed in advance. Then no strategic behavior is possible. This doesn't eliminate scope for decision making—quite the contrary. If, for example, 9 or 10 satisfactory solutions were required for an A, 7 or 8 for a B, 5 or 6 for a C, and 3 or 4 for a D, then Student 3 would earn 3 units of merit for $E = 5$, 2 for $E = 3$, and only 1 for $E = 1$. If she wished to maximize merit per unit of effort, or M/E, then she would still have to solve a simple optimization problem, namely, to determine the maximum of $3 \div 5 = 0.6$, $2 \div 3 = 0.67$, and $1 \div 1 = 1$. The answer, of course, is 1, corresponding to $E = 1$: to maximize M/E, Student 3 should hardly work at all. Nevertheless, such a decision would not be strategic because its outcome would depend solely on the individual concerned; it would not depend in any way on the behavior of other students.

The story is very different, however, if academic standards are relative, i.e., if letter grades depend on collective student achievement. To illustrate, let s denote

Table 0.2. The grading scheme

A:	$\frac{1}{5}(4b+w)$	$\leq s \leq$	b
B:	$\frac{1}{5}(3b+2w)$	$\leq s <$	$\frac{1}{5}(4b+w)$
C:	$\frac{1}{5}(2b+3w)$	$\leq s <$	$\frac{1}{5}(3b+2w)$
D:	$\frac{1}{5}(b+4w)$	$\leq s <$	$\frac{1}{5}(2b+3w)$
F:	w	$\leq s <$	$\frac{1}{5}(b+4w)$

the number of satisfactory solutions; let b denote the number of satisfactory solutions obtained by the best student, and let w denote the number obtained by the worst. Additionally, let grades be assigned according to the scheme in Table 0.2, which awards A to any student in the top fifth of the range, B to any student in the next fifth of the range, and so on. Thus, if all students chose $E = 5$, then Student 1 would get A, Student 2 would get B, Students 3 and 4 would get C, and Students 5 and 6 would both fail; whereas if all students chose $E = 1$, then Students 1 and 2 would get A, Students 3 and 4 would get C, Student 5 would get D, and only Student 6 would fail.

Students who wish to maximize M or M/E must now anticipate (and perhaps seek to influence) how hard the others will work, and choose E accordingly. In other words, students who wish to attain their goals must behave strategically. For example, Student 1 is now guaranteed an A if she works at least quite hard; and $E = 3$ yields a higher value of M/E than $E = 5$. But $4/3 = 1.33$ is not the highest value of M/E that Student 1 can obtain. If she knew that Student 2 would work at most quite hard, and if she also knew that either Student 6 would choose $E \leq 3$ or Student 5 would choose $E = 1$, then the grading scheme would award her an A no matter how hard she worked—in particular, if she chose $E = 1$. But Students 5 and 6 can avoid failing only if they obtain at least 4 satisfactory solutions and either Student 3 or Student 4 chooses $E = 1$; in which case, $E = 1$ and its 6 satisfactory solutions could earn Student 1 only B (although it would still maximize M/E). And so on. We have made our point: changing from absolute to relative standards brings ample scope for strategic behavior to an interaction among individuals that otherwise has none.

If an interaction among individuals gives rise to strategic behavior, and if the interaction can be described mathematically, then we refer to this description as a *game* and to each individual in the game as a *player*. Thus a game in the mathematician's sense is a model of strategic interaction, and game-theoretic modelling is the process by which such games are constructed. Correspondingly, game theory is a diverse assemblage of ideas, theorems, analytical methods, and computational tools for the study of strategic interaction, perhaps even more appropriately viewed as a language than as a theory [**292**, p. 4]. You won't often find "game" so defined in a dictionary.[4] But new meanings take time to diffuse into dictionaries; and in any event, "game" is the word, and "game theory" the phrase, we shall use.

[4]For example, you won't find "game" so defined in the Oxford English Dictionary, though you will find "game theory" defined as "Mathematical treatment of the processes involved in making decisions and developing strategies, esp. in situations of competition or conflict, applied originally to games of skill and to economics, and later to many other fields including biology and psychology."

As an acknowledged field of study in its own right, game theory began with the publication in 1944 of a treatise on games and economic behavior by von Neumann and Morgenstern.[5] Yet some game-theoretic concepts have been traced to earlier work by Cournot [**72**], Edgeworth [**87**], Böhm-Bawerk [**34**], Borel [**35**], and Zeuthen [**370**] in the context of economics, and to Fisher [**98**] in the context of evolutionary biology. By tradition, games are classified as either cooperative or noncooperative, although this dichotomy is universally acknowledged to be imperfect: almost every conflict has an element of cooperation, and almost all cooperation has an element of conflict. In this regard we abide, more or less, by tradition: Chapters 1, 2, and 6–8 are mostly about noncooperative solution concepts, whereas Chapters 3 and 4 are about cooperative ones. But the distinction is especially blurred in Chapter 5, where we study cooperation within the context of a noncooperative game. Games are also classified as having either strategic or nonstrategic form.[6] For present purposes, it will suffice to say that a game is in strategic form when the strategies to be used are explicitly specified within the model, and that a game is in nonstrategic form when we focus on a benefit distribution or a coalition structure and strategies are merely implicit. Only cooperative games appear in nonstrategic form. In particular, when the focus is on a benefit distribution, the conflict is analogous to sharing a pie among players who would each like all of it, and who collectively can obtain all of it, but who as individuals cannot obtain any of it; then the game is said to be in characteristic function form. Such characteristic function games or CFGs are studied in Chapter 4, where we discuss, for example, how to split the costs of a car pool fairly. Games are studied in strategic form in Chapters 1–3 and 5–8, where we study, among other things, vaccination behavior, conflict among crustaceans or insects, the use of landmarks as territorial boundaries, victory displays, and food sharing among ravens. A further useful distinction is between games whose players are specific actors and games whose players are individuals drawn randomly from a large population: drivers at a 4-way junction may be either neighbors or strangers. We refer to the first kind of game as a community game (Chapter 1) and to the second kind as a population game (Chapter 2).[7] The overall classification that results for this book is depicted in Figure 0.1.

To interpret games in the wider context of optimization theory, it will be helpful now to return for a while to our six mathematics students. Let us suppose that Student 2 will work quite hard ($E = 3$), whereas Students 3 to 6 will work very hard ($E = 5$); and that Student 1 already knows this. If Student 1 is rewarded *either* by high achievement *or* by high achievement per unit effort, then she has a single reward—either M or M/E—and a single decision variable E with which to maximize it. If, on the other hand, Student 1 is rewarded *both* by high achievement

[5] Nowadays, however, it is customary to consult the third edition [**350**]. The field has since diversified to embrace a multitude of disciplines, in each of which it forms a largely separate subfield with its own traditions and priorities. These many subfields include algorithmic [**289**], biological [**49**], combinatorial [**309**], economic [**241**, **343**], mathematical [**110**], and political [**167**] game theory.

[6] Game theorists further distinguish between strategic games in extensive form and strategic games in normal form; see, e.g., [**257**, pp. 1–5]. We have little use for this distinction, although we discuss the extensive form in §1.7.

[7] To a large extent, community games are a concern of economic game theory, whereas population games are the concern of biological game theory. So if you are a biologist, then you may be tempted to proceed directly to Chapter 2; however, I would encourage you not to skip Chapter 1, because a prior acquaintance with community games can greatly facilitate an understanding of population games.

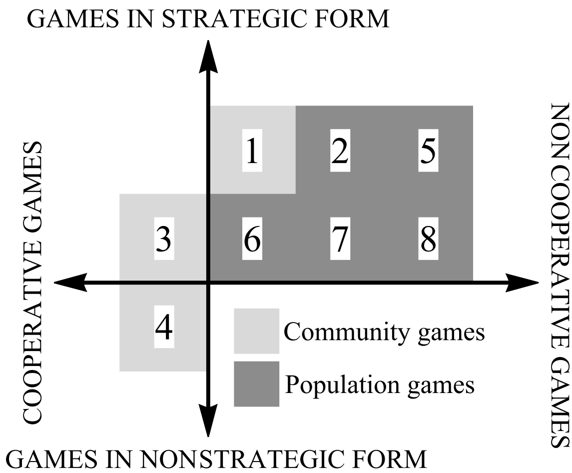

Figure 0.1. Classification of games that appear in this book. Numbers refer to chapters: above the horizontal axis corresponds to games in strategic form; below that axis to games in nonstrategic form; leftward of the vertical axis to cooperative games; rightward of that axis to noncooperative games; light shading to community games; and dark shading to population games.

and by high achievement per unit effort, then she has two rewards—M *and* M/E—but still only a single decision variable E with which to maximize them. In the first case, with a single reward, we say that Student 1 faces a *scalar* optimization problem (whose solution is clearly $E = 5$ or $E = 3$ if rewarded by M, but $E = 1$ if rewarded by M/E). In the second case, with two rewards, we say that Student 1 faces a *vector* optimization problem (whose solution is far from clear, because the value of E that maximizes M fails also to maximize M/E, and vice versa). More generally, an optimization problem requires a single decision maker to select a single decision variable (over which this decision maker has complete control) to optimize r rewards; and if $r = 1$, then the problem is a scalar optimization problem, whereas if $r \geq 2$, then the problem is a vector optimization problem.[8]

By contrast, the games that we are about to study require each of n decision makers, or players, to select a single decision variable—called a strategy—to optimize a single reward. But each player's reward depends on the other players' strategies, that is, on decision variables that the other players completely control. That players lack control over all decision variables affecting their rewards is what makes a game a game, and what distinguishes it from an optimization problem. In this book, we do not discuss optimization problems in their own right, nor do we allow players to have more than one reward—in other words, we do not consider *vector games*. Thus our games lie in the shaded region of Figure 0.2.[9] We will discover that they all have four key ingredients, as summarized in Table 0.3.

[8] As we shall see in due course, a decision variable can itself be a vector. What distinguishes a scalar from a vector optimization problem (or a game from a vector game), however, is the number of rewards per decision maker.

[9] For further examples of optimization problems see, e.g., [**210**, Chapters 3, 7, and 12]; for vector games, see [**246**, **371**]. In biological contexts, optimization models are also called "games against nature" (see [**188**, **277**, **356**]), an example of which appears in §2.10.

Table 0.3. The key ingredients of a game

INGREDIENT	DESCRIPTION
Player set	Set of interacting individuals
Strategy set	Set of feasible choices or plans of action through which the players interact
Reward function	Defines the reward to each player from every feasible strategy combination
Solution concept	Selects a strategy or strategy combination that is stable in an appropriate sense. Suitable concepts are Nash equilibrium for community games and evolutionarily stable strategy for population games

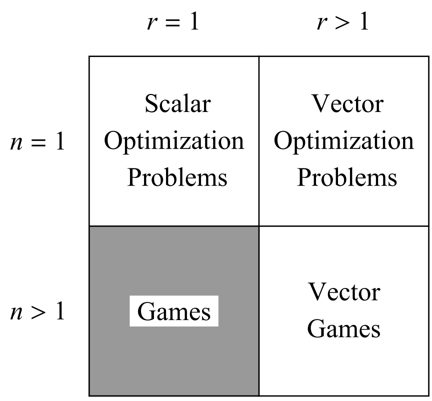

Figure 0.2. Games in the context of optimization theory: r is the number of rewards per decision maker, and n is the number of decision makers.

Now, a strategic interaction can be very complicated, and a game in our sense does not exist unless the interaction can be described mathematically. But this step is often exceedingly difficult, especially if players are many, and especially if we insist—and as modellers we should—that players' rewards be explicitly defined. Therefore, we confine our agenda to strategic interactions that lend themselves readily to a concrete mathematical description—and hence, for the most part, to games with few distinct players. Indeed six is many in this regard: Students 1 to 6, having served us so well, must now depart the scene.

Throughout the book we introduce concepts by means of specific models, later indicating how to generalize them; however, we avoid rigorous statements and proofs of theorems, referring instead to the standard texts. Our approach to games is thus largely the opposite of the classical approach, but has the clear advantage in an introductory text that it fosters substantial progress. We can downplay the issue of what—in most general terms—constitutes a decision maker's reward, because the reward is self-evident in the particular examples we choose.[10] We can demonstrate

[10] For community games, books that discuss this issue thoroughly include [**174**], [**306**], [**307**] and [**350**]. The last of these books defines the classical approach to the theory of games; the first is an excellent later text covering the same ground and more, but with much less technical mathematical detail; and the other two constitute one of the most comprehensive works on game theory ever published.

the usefulness and richness of games while avoiding unnecessary distractions; even two-player games in strategic form still have enormous potential, which is decades from being fully realized. Moreover, we can rely on intuitions about everyday conflicts to strengthen our grasp of game theory's key ideas, and we can be flexible and creative in applying those ideas.

Our agenda is thus defined. We carry it out in Chapters 1–8, with summaries and suggestions for further reading and research in commentaries at the ends of the chapters; and in Chapter 9 we reflect upon our accomplishments, and look forward to the future of games.

For population games, although the reward appears in different guises in different models, it can always be regarded as a measure of fitness (p. 71), to which benefits or costs make positive or negative contributions.

Chapter 1

Community Games

A community game is a model of strategic interaction among specific actors, as opposed to among individuals drawn randomly from a large population (as will be discussed in Chapter 2). Motoring behavior provides our first example of such a community game,[1] which will pave the way for several solution concepts, both in this chapter and later in Chapters 2 and 3. Here the example will introduce what is probably game theory's most enduring concept, Nash's concept of noncooperative equilibrium [**243**].

1.1. Crossroads: a motorist's dilemma

Consider a pair of motorists who are driving in opposite directions along a 2-lane road when they arrive simultaneously at a 4-way junction, where each would like to cross the path of the other. For the sake of definiteness, let us suppose that the first motorist, say Nan, is travelling north but would like to go west; whereas the second motorist, say San, is travelling south but would like to go east. Nan and San cannot proceed simultaneously: one must proceed before the other. Then who should it be? How should the motorists behave? Here is potential for conflict that's fit for a game. We call the game Crossroads.

To keep matters simple, let's suppose that a motorist has but two choices: she[2] can either wait for the other motorist to turn first, in which case we shall say that she selects pure strategy W (for Wait), or she can proceed to turn and hope that the other motorist will refrain from doing so, in which case we shall say that she selects pure strategy G (for Go). We use the word "strategy" because we wish to think of Nan as the first player or decision maker and of San as the second player in a two-player game, and we call G and W pure strategies to distinguish them from mixed strategies, which we shall introduce in §1.2. If Nan selects pure strategy X

[1] As indicated by Figure 0.1, in this chapter we consider only noncooperative community games; cooperative community games will be considered in Chapters 3 and 4.

[2] Recall from p. 1 that, in place of epicene pronouns, female pronouns will be used for humans of indeterminate gender in Chapter 1, male pronouns in Chapter 2, and so on.

and if San selects pure strategy Y, then we shall say that the players have jointly selected the pure strategy combination XY. Thus, our game has precisely four pure strategy combinations, namely, GG, GW, WG, and WW.

In the case where each player decides to defer to the other (WW), let ϵ denote the time they spend dithering and frantically waving to each other, before one of them eventually moves. Likewise, in the case where each decides not to defer (GG), let δ denote the time they spend intimidating each other in the middle of the junction until one of them eventually backs down. It seems reasonable to suppose that the time they waste if both are aggressive (GG) exceeds that which they waste if both are accommodating (WW), even if not by much. Therefore we shall assume throughout that

$$0 < \epsilon < \delta < \infty, \tag{1.1}$$

even if ϵ/δ is close to 1. Let τ_1 denote the time it takes Nan to negotiate the turn without interruption, i.e., the time that elapses (if San lets her go) between her front bumper crossing the northbound stop bar in Figure 1.1 and her back bumper crossing the junction exit line; let τ_2 denote the corresponding time for San. We are now in a position to analyze the confrontation from Nan's point of view.

Suppose, first, that pure strategy combination GW is selected: Nan decides to go, San decides to wait. Then Nan's delay is zero. Now suppose that WG is selected: San decides to go, Nan decides to wait. Then Nan's delay is τ_2, the time it takes San to cross the junction. Suppose, next, that WW is selected: both decide to wait. There follows a bout of rapid gesticulation, after which it is still the case that either Nan or San is first to proceed, as they can't just sit there all day. Just how is it decided who, given WW, should go first is a game within the game—or a *subgame*—of Crossroads. But we shall not attempt to model it explicitly,[3] rather, we shall simply assume that the two motorists are then equally likely to be first to turn. Accordingly, let F denote the motorist who (given WW) turns first. Then F is a random variable, whose sample space[4] is {Nan, San}, and

$$\text{Prob}(F = \text{Nan}) \;=\; \tfrac{1}{2}, \quad \text{Prob}(F = \text{San}) \;=\; \tfrac{1}{2}. \tag{1.2}$$

Note that F could easily be converted to an integer-valued random variable by labelling Nan as 1 and San as 2, but it is more convenient not to do so.

If Nan turns first (F = Nan), then she suffers a delay of only ϵ; whereas if San turns first (F = San), then Nan—from whose viewpoint we are analyzing the confrontation—suffers a delay of $\epsilon + \tau_2$. Thus the expected value of Nan's delay (given WW) is

$$\epsilon \cdot \text{Prob}(F = \text{Nan}) + (\epsilon + \tau_2) \cdot \text{Prob}(F = \text{San}) = \epsilon + \tfrac{1}{2}\tau_2. \tag{1.3}$$

Suppose, finally, that GG is selected: both decide to go. Then there follows a minor skirmish, of duration δ, which one of the players must eventually win. Let random variable V, with sample space {Nan, San}, denote the victor (given GG). Next suppose that if Player k is the victor, then the time she takes to negotiate the junction (given GG) is simply δ greater than she would have taken anyway, i.e.,

[3] For an example of an explicitly modelled subgame, see §1.7.

[4] Regardless of whether a random variable is discrete (as here) or continuous (as in §1.4), its sample space is the set of all possible values that the random variable could take.

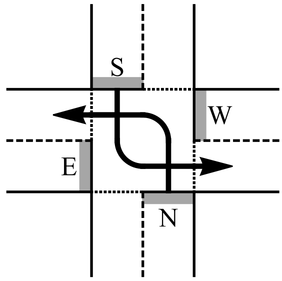

Figure 1.1. The scene of the action: a crossroad

Table 1.1. Payoff matrix for Nan

		San	
		G	W
Nan	G	$-\delta - \frac{1}{2}\tau_2$	0
	W	$-\tau_2$	$-\epsilon - \frac{1}{2}\tau_2$

Table 1.2. Payoff matrix for San

		San	
		G	W
Nan	G	$-\delta - \frac{1}{2}\tau_1$	$-\tau_1$
	W	0	$-\epsilon - \frac{1}{2}\tau_1$

$\delta + \tau_k$. If neither player is especially aggressive, then it seems reasonable to suppose that each is as likely as the other to find her path cleared, in which case, (1.2) holds with V in place of F. If, given GG, Nan is the victor (V = Nan), then her delay is only δ; whereas if San is the victor (V = San), then Nan's delay is $\delta + \tau_2$. So the expected value of her delay, given GG, is $\delta \cdot \text{Prob}(V = \text{Nan}) + (\delta + \tau_2) \cdot \text{Prob}(V = \text{San}) = \delta + \frac{1}{2}\tau_2$, by analogy with (1.3).

We are tacitly assuming that δ and ϵ are both independent of τ_1 and τ_2. You may be tempted to criticize this assumption—but tread daintily if you do so. In the real world, it is more than likely that δ and ϵ would depend upon various aspects of the personalities of the drivers in conflict. But our model—by virtue of being an abstraction, a deliberate simplification of reality (as noted on p. xi)—ignores them. It differentiates between Nan and San solely by virtue of their transit times τ_1 and τ_2, and any dependence of δ and ϵ on these may well be weak.

Everyone likes delays to be as short as possible, but in noncooperative game theory it is traditional to like payoffs that are as large as possible. So, because making a delay small is the same thing as making its negative large, we agree that the payoff to Nan is the negative of her delay. Thus the payoffs to Nan associated with pure strategy combinations GG, GW, WG, and WW are $-\delta - \frac{1}{2}\tau_2, 0, -\tau_2$, and $-\epsilon - \frac{1}{2}\tau_2$, respectively. It is customary to store these payoffs in a matrix, as in Table 1.1, where the rows correspond to strategies of Player 1 (Nan) and the columns correspond to strategies of Player 2 (San).

Now, in any particular confrontation, the actual payoff to Nan from pure strategy combination GG or WW is a random variable. If the game is played repeatedly, however—perhaps because Nan and San meet every morning at the same time, and at the same place, as they travel to work in opposite directions—then Nan's average payoff from GG or WW over an extended period should be well approximated by the random variable's expected value, and this is how we justify using expected values as payoffs.

To obtain the matrix in Table 1.1, we analyzed the game from Nan's point of view. A similar analysis, but from San's point of view, yields the payoff matrix in Table 1.2. Indeed it is hardly necessary to repeat the analysis, because the only difference between Nan and San that is incorporated into our model of their conflict—or, as game theorists prefer to say, the only *asymmetry* between the players—is that Nan's transit time may be different from San's ($\tau_1 \neq \tau_2$). Thus San's payoff matrix is the transpose of Nan's with suffix 2 replaced by suffix 1. Transposition is necessary because rows correspond to strategies of Player 1, and columns to those of Player 2, in both tables.

In terms of game theory, the payoff matrices in Tables 1.1 and 1.2 define a two-player game in which each player has two pure strategies, G and W. If we assume that Nan and San act out of rational self-interest, *and that they cannot communicate prior to the game*, then the game becomes a noncooperative one. (As we shall discover in Chapter 3, what really distinguishes a noncooperative game from a cooperative game is the inability to make commitments; but the players cannot possibly make binding agreements if they cannot even communicate prior to the game.) Not being especially selfish is not necessarily or even usually a violation of rational self-interest on the part of Nan or San; and waving at one another does not constitute prior communication. It is therefore legitimate to regard Tables 1.1 and 1.2 as the payoff matrices for a noncooperative, two-player game.

Specifically, Crossroads is an example of a *bimatrix* game, or simply a matrix game, especially if (1.4) below is satisfied. A more general bimatrix game in which Player 1 has m_1 pure strategies and Player 2 has m_2 pure strategies is defined by a pair of $m_1 \times m_2$ matrices A, B in which a_{ij} denotes the payoff to Player 1 and b_{ij} the payoff to Player 2 from the strategy combination (i, j). If $m_1 = m_2 = m$ and

$$b_{ij} = a_{ji}, \qquad 0 \leq i, j \leq m, \tag{1.4}$$

i.e., if B is the transpose of A, then the game is *symmetric*. If

$$a_{ij} + b_{ij} = c, \qquad 0 \leq i \leq m_1,\ 0 \leq j \leq m_2, \tag{1.5}$$

where c is a constant, then the game is *constant-sum*; and if in addition $c = 0$, then the game is *zero-sum*.[5] Thus Crossroads is symmetric (with $m = 2$) if, and only if, $\tau_1 = \tau_2$. Even if $\tau_1 = \tau_2$, however, the game is not constant-sum, because $\delta > \epsilon > 0$ by (1.1).

For further discussion of bimatrix games, see Appendix A.

1.2. Optimal reaction sets and Nash equilibria

To determine which strategy Nan should adopt in Crossroads, let's begin by supposing that San is so slow that

$$\tau_2 > 2\delta > 2\epsilon. \tag{1.6}$$

[5]In either case, a gain to one player means a loss to the other; the games are equivalent because subtracting the same constant from (or adding the same constant to) every payoff in a player's matrix has no strategic effect—it cannot affect the conditions for Nash equilibrium, which we derive in §1.2.

Then whether San chooses W or G is quite irrelevant because

$$-\delta - \tau_2/2 > -\tau_2, \tag{1.7}$$

and so it follows immediately from Table 1.1 that Nan's best strategy is to hit the gas: every element in the first row of her payoff matrix is greater than the corresponding element in the second row of her payoff matrix. We say that strategy G dominates strategy W for Nan, and that G is a dominant strategy for Nan. More generally, if A is Player 1's payoff matrix (defined at the end of §1.1), pure strategy i is said to *dominate* pure strategy k for Player 1 if $a_{iq} \geq a_{kq}$ for *all* $q = 1, \ldots, m_2$ and $a_{iq} > a_{kq}$ for *some* q; if i dominates k for all $k \neq i$, then i is called a *dominant strategy*. Dominance is *strong* if the above inequalities are all strictly satisfied, and otherwise (i.e., if even one inequality is not strictly satisfied) *weak*. Thus, in particular, (1.7) implies that G is strongly dominant for Player 1. Similarly, if B is Player 2's payoff matrix, then pure strategy j dominates pure strategy l for Player 2 if $b_{pj} \geq b_{pl}$ for all $p = 1, \ldots, m_1$ with $b_{pj} > b_{pl}$ for some p; j is a dominant strategy if j dominates l for all $l \neq j$. Again, dominance is weak unless all inequalities are strictly satisfied. Note that we speak of *a* strongly dominant strategy, rather than *the* strongly dominant strategy, because both players may have one. Clearly, if one strategy (weakly or strongly) dominates another, then the other is (weakly or strongly) dominated by the one.

In practice, if (1.6) is used to define a slow San, then we could interpret our model as yielding the advice: "If you think the driver across the road is a slowpoke, then put down your foot and go." Furthermore, if (1.6) is used to define a slow San, then it might well be appropriate to define an intermediate San by

$$2\delta > \tau_2 > 2\epsilon \tag{1.8}$$

and a fast San by

$$2\delta > 2\epsilon > \tau_2. \tag{1.9}$$

Nan would be slow for $\tau_1 > 2\delta > 2\epsilon$, intermediate for $2\delta > \tau_1 > 2\epsilon$, and fast for $2\delta > 2\epsilon > \tau_1$. G is a dominant strategy for San if Nan is slow, because then $-\delta - \tau_1/2 > -\tau_1$.[6] So if both drivers are slow, then we can regard the strategy combination GG as the solution of our game: it contains a dominant strategy for each player. In terms of Table 0.3, dominant strategy is an adequate solution concept—in this very special case.

Suppose, however, that either (1.8) or (1.9) holds. Then (1.7) is false, and what's best for Nan is no longer independent of what San chooses because the second element in row 1 of Nan's payoff matrix is still greater than the second element in row 2, whereas the first element in row 1 is now smaller than the first in row 2. No pure strategy is obviously better for Nan. Then what should Nan choose?

If sometimes G is better and sometimes W is better (depending on what San chooses), then shouldn't Nan's choice be in some sense a mixture of strategies

[6] We do not consider that, say, τ_2 and 2δ might actually be equal. In general, we ignore as fanciful the possibility that strict equality between parameters aggregating behavior—population parameters—can ever be meaningful unless justified by a prior assumption of symmetry. This assumption is discussed at length elsewhere under the guise of "genericity" [**292**, p. 30] or the "generic payoffs assumption" [**49**, pp. 21–23].

G and W? One way to mix strategies would be to play G with probability u, and hence W with probability $1-u$. Accordingly, let N denote Nan's choice of pure strategy. Then N is a random variable, with sample space $\{G, W\}$; $\text{Prob}(N = G) = u$ and $\text{Prob}(N = W) = 1-u$. If Nan plays G with probability u, then we will say that she selects *mixed strategy* u, where $u \in [0,1]$.[7] Similarly, if San plays G with probability v and hence W with probability $1-v$, that is, if $\text{Prob}(S = G) = v$ and $\text{Prob}(S = W) = 1-v$, where the random variable S, again with sample space $\{G, W\}$, is San's pure strategy, then we shall say that San selects mixed strategy v. Thus, for either player, the set of all feasible strategies or *strategy set* is the unit interval $\Delta = [0,1]$. When Player 1 selects strategy u and Player 2 selects strategy v, we refer to the row vector (u, v) as their mixed strategy combination; and we refer to the set of all feasible strategy combinations, which we denote by D, as the *decision set* (because this phrase is less cumbersome than "strategy combination set"). In this particular case,

$$D = \{(u,v) \mid 0 \le u, v \le 1\} = \{(u,v) \mid u, v \in \Delta\} = \Delta \times \Delta, \tag{1.10}$$

the unit square of the Cartesian coordinate plane.[8] Note that $(1,1)$, $(1,0)$, $(0,1)$, and $(0,0)$ are equivalent to GG, GW, WG, and WW, respectively, in §1.1.

But how could Nan and San arrange all this? Let's suppose that the spinning arrow depicted in Figure 1.2 is mounted on Nan's dashboard. When confronted by San, Nan gives the arrow a quick spin. If it comes to rest in the shaded sector of the disk, then she plays G; if it comes to rest in the unshaded sector, then she plays W. Thus selecting strategy u means having a disk with shaded sectoral angle $2\pi u$, and changing one's strategy means changing the disk. What about the time required to spin the arrow—does it matter? Not if San also has a spinning arrow mounted on her dashboard and takes about as long to spin it (of course, San's shaded sector would subtend angle $2\pi v$ at the center). And if you think it's a bit far-fetched that motorists would drive around with spinning arrows on their dashboards, then you can think of Nan's spinning arrow as merely the analogue of a mental process through which she decides whether to go or wait at random—but in such a way that she goes, on average, fraction u of the time. Similarly for San. Note, incidentally, the important point that strategies are selected prior to interaction: the players arrive at the junction with their disks already shaded.

Let F_1 denote the payoff to Nan. Then F_1 is a random variable with sample space

$$\left\{-\delta - \tfrac{1}{2}\tau_2, 0, -\tau_2, -\epsilon - \tfrac{1}{2}\tau_2\right\}. \tag{1.11}$$

If strategies are chosen independently, then

$$\begin{aligned}\text{Prob}\left(F_1 = -\delta - \tfrac{1}{2}\tau_2\right) &= \text{Prob}(N = G \text{ and } S = G) \\ &= \text{Prob}(N = G) \cdot \text{Prob}(S = G) = uv.\end{aligned}$$

---

[7]Here "$\in$" means "belongs to" (and in other contexts, e.g., (1.14), "belonging to"), and $[a, b]$ denotes the set of all real numbers between a and b, inclusive.

[8]In (1.10), the vertical bar means "such that" and "$\times$" denotes Cartesian product. In general, $\{x \mid P\}$ denotes the set of all x such that P is satisfied; for example, $[a,b] = \{x \mid a \le x \le b\}$. The Cartesian product of sets U and V is the set $U \times V = \{(u,v) | u \in U, v \in V\}$, and more generally, the Cartesian product of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n = \{(a_1, \ldots, a_n) | a_i \in \mathcal{A}_i, 1 \le i \le n\}$.

Although the decision set is usually a Cartesian product, it does not have to be; for an exception, see §1.4, in particular (1.44).

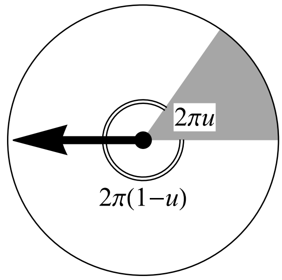

Figure 1.2. Nan's spinning arrow and disk

Similarly,

$$\text{Prob}(F_1 = 0) = u(1 - v),$$
$$\text{Prob}(F_1 = -\tau_2) = (1 - u)v,$$

and

$$\text{Prob}\big(F_1 = -\epsilon - \tfrac{1}{2}\tau_2\big) = (1 - u)(1 - v).$$

Let E denote expected value, and let $f_1(u, v)$ denote the expected value of Nan's payoff from strategy combination (u, v). We will refer to the expected value of a payoff as a *reward*. Thus Nan's reward is

$$\begin{aligned} f_1(u, v) = \text{E}[F_1] = &-(\delta + \tfrac{1}{2}\tau_2) \cdot \text{Prob}(F_1 = -\delta - \tfrac{1}{2}\tau_2) \\ &+ 0 \cdot \text{Prob}(F_1 = 0) - \tau_2 \cdot \text{Prob}(F_1 = -\tau_2) \\ &- (\epsilon + \tfrac{1}{2}\tau_2) \cdot \text{Prob}(F_1 = -\epsilon - \tfrac{1}{2}\tau_2) \end{aligned}$$

or

$$f_1(u, v) = \big(\epsilon + \tfrac{1}{2}\tau_2 - \{\delta + \epsilon\}v\big)\, u + \big(\epsilon - \tfrac{1}{2}\tau_2\big)\, v - \epsilon - \tfrac{1}{2}\tau_2 \tag{1.12a}$$

after simplification. San's reward from the same combination is

$$f_2(u, v) = \big(\epsilon + \tfrac{1}{2}\tau_1 - \{\delta + \epsilon\}u\big)\, v + \big(\epsilon - \tfrac{1}{2}\tau_1\big)\, u - \epsilon - \tfrac{1}{2}\tau_1 \tag{1.12b}$$

after a similar calculation—or simply exchanging Nan's strategy and San's transit time in (1.12a) for San's strategy and Nan's transit time. Note that (1.12) can be written more compactly as

$$\begin{aligned} f_1(u, v) &= (u, 1 - u)A(v, 1 - v)^T, \\ f_2(u, v) &= (u, 1 - u)B(v, 1 - v)^T, \end{aligned} \tag{1.13}$$

where A and B are the matrices in Tables 1.1 and 1.2, respectively.[9] Equations (1.13) define a vector-valued function $f = (f_1, f_2)$ from the decision set D into the f_1-f_2 plane. In terms of Table 0.3, this reward function is the third key ingredient of our game of Crossroads.

Both Nan and San would like their reward to be as large as possible. Unfortunately, Nan does not know what San will do (and San does not know what Nan

[9] In turn, (1.13) is a special case of (A.2).

will do) because this is a noncooperative game. Therefore, Nan should reason as follows: "I do not know which v San will pick—but for every v, I will pick u to make $f_1(u,v)$ as large as possible." In this way, Nan obtains a set of points in the unit square. Each of these points corresponds to a strategy combination (u,v) that is optimal for Nan, in the sense that for each v (over which Nan has no control) a corresponding u is a strategy that makes Nan's reward as large as possible. We will refer to this set of strategy combinations as Nan's *optimal reaction set* and denote it by R_1.[10] In mathematical terms, we have

$$R_1 = \Big\{(u,v) \in D \,\Big|\, f_1(u,v) = \max_{\overline{u}\in\Delta} f_1(\overline{u},v)\Big\} = \big\{(u,v) \in D \mid u = \mathcal{B}_1(v)\big\}, \tag{1.14a}$$

where D is the decision set defined by (1.10) and $\mathcal{B}_1(v)$ denotes a best reply to v for Player 1—*a* best reply, rather than *the* best reply, because there may be more than one. Note that R_1 is obtained in practice by holding v constant and maximizing $f_1(\overline{u},v)$ as a function of a single variable $\overline{u}$. For given v, a maximizing $\overline{u}$ is denoted by u and will in general depend upon v.

Likewise, San should reason as follows: "I do not know which u Nan will pick, but for every u I will pick v to make $f_2(u,v)$ as large as possible." In this way, San obtains a set of points in the unit square. Each of these points corresponds to a strategy combination that is optimal for San in the sense that for each u (over which San has no control) a corresponding v is a strategy that makes San's reward as large as possible. We will refer to this set of strategy combinations as San's optimal reaction set, and denote it by R_2. That is,

$$R_2 = \Big\{(u,v) \in D \,\Big|\, f_2(u,v) = \max_{\overline{v}\in[0,1]} f_2(u,\overline{v})\Big\} = \big\{(u,v) \in D \mid v = \mathcal{B}_2(u)\big\}, \tag{1.14b}$$

where $\mathcal{B}_2(u)$ denotes a best reply to u for Player 2. Again, in practice, R_2 is obtained by holding u constant and maximizing $f_2(u,\overline{v})$ as a function of a single variable $\overline{v}$, and for given u, a maximizing $\overline{v}$ is denoted by v and will in general depend upon u. Note the important point that each player can determine her optimal reaction set without any knowledge of the other player's reward.

Suppose, for example, that $\delta < \frac{1}{2}\min(\tau_1,\tau_2)$; i.e., both drivers are slow. Then, because $0 \le v \le 1$ implies that the coefficient of $\overline{u}$ in

$$f_1(\overline{u},v) = \big(\epsilon\{1-v\} + \tfrac{1}{2}\tau_2 - \delta v\big)\overline{u} + \big(\epsilon - \tfrac{1}{2}\tau_2\big)v - \epsilon - \tfrac{1}{2}\tau_2 \tag{1.15}$$

is always positive, $f_1(\overline{u},v)$ is maximized for $0 \le \overline{u} \le 1$ by choosing $\overline{u} = 1$. Therefore, Nan's optimal reaction set is

$$R_1 = \big\{(u,v) \mid u = 1, 0 \le v \le 1\big\}, \tag{1.16}$$

the edge of the unit square that runs between $(1,0)$ and $(1,1)$; see Figure 1.3(a) where R_1 is represented by a thick solid line. Similarly, because $0 \le u \le 1$ implies

[10] "Optimal" for Nan is also "rational" for Nan. Correspondingly, a more common term for R_1, in both classical [**313**] and evolutionary [**347**] game theory, is *rational reaction set*, and I followed the prevailing custom by using it in the second edition of this book. It can be argued, however, that even though rationality "does not necessarily imply the use of reason, when the term is used as part of the study of animal economics" [**194**, p. 167], "rational" is an unnecessary qualifier in the evolutionary context of Chapters 2 and 5–8 and, hence, is best avoided.

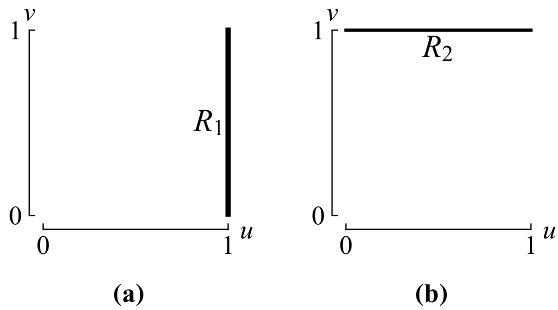

Figure 1.3. Optimal reaction sets when $\tau_1 > 2\delta, \tau_2 > 2\delta$

that the coefficient of $\overline{v}$ in

$$f_2(u, \overline{v}) = \big(\epsilon\{1-u\} + \tfrac{1}{2}\tau_1 - \delta u\big)\overline{v} + \big(\epsilon - \tfrac{1}{2}\tau_1\big)u - \epsilon - \tfrac{1}{2}\tau_1 \tag{1.17}$$

is always positive, $f_2(u, \overline{v})$ is maximized for $0 \le \overline{v} \le 1$ by choosing $\overline{v} = 1$. Therefore, San's optimal reaction set is

$$R_2 = \big\{(u, v) \mid 0 \le u \le 1, v = 1\big\}, \tag{1.18}$$

the edge of the unit square that runs between $(0, 1)$ and $(1, 1)$; see Figure 1.3(b) where R_2 is represented by a thin solid line. Of course, all that Figure 1.3 tells us is that the best strategy against a slow driver is G, which we knew long before we began to talk about mixed strategies. But something new will shortly emerge.

Notice that the reaction sets R_1 and R_2 have a nonempty intersection $R_1 \cap R_2 = \{(1, 1)\}$. The strategy combination $(1, 1)$ that lies in both sets has the following property: if either player selects strategy 1, then the other cannot obtain a greater reward by selecting a strategy other than 1. In other words, neither player can increase her reward by a *unilateral* departure from the strategy combination $(1, 1)$. By virtue of having this property, $(1, 1)$ is said to be a *Nash-equilibrium strategy combination*. More generally, $(u^*, v^*) \in D$ is said to be a Nash-equilibrium strategy combination, or simply a *Nash equilibrium*, of a noncooperative, two-player game whenever

$$(u^*, v^*) \in R_1 \cap R_2 \tag{1.19}$$

because if either player sticks rigidly to her Nash-equilibrium strategy (u^* for Player 1, v^* for Player 2), then the other player cannot increase her reward by selecting a strategy other than her Nash-equilibrium strategy. That is,

$$f_1(u^*, v^*) \;\ge\; f_1(u, v^*) \tag{1.20a}$$

for all $(u, v^*) \in D$ and

$$f_2(u^*, v^*) \;\ge\; f_2(u^*, v) \tag{1.20b}$$

for all $(u^*, v) \in D$, so that u^* is a best reply to v^* *and* v^* is a best reply to u^*. If (1.20a) is satisfied with strict inequality for all $u \ne u^*$ and (1.20b) is satisfied with strict inequality for all $v \ne v^*$ (that is, if not only are u^* and v^* mutual best replies, but also each is uniquely the best reply to the other), then the Nash equilibrium is said to be *strong*; otherwise it is said to be *weak*.

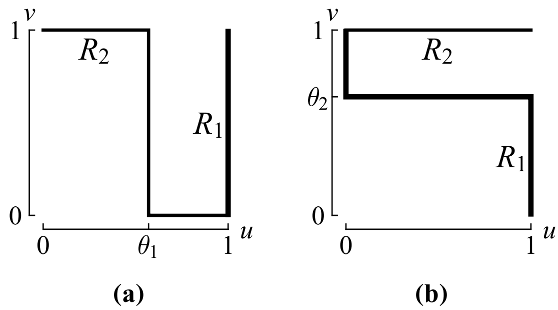

Figure 1.4. R_1 and R_2 for **(a)** $\tau_1 < 2\delta < \tau_2$, **(b)** $\tau_2 < 2\delta < \tau_1$

The analogue of (1.20) for the discrete bimatrix game with matrices A, B (p. 12) is that the strategy combination (i, j) is a Nash equilibrium if

$$a_{ij} \geq a_{kj} \tag{1.21a}$$

for all $k = 1, \dots, m_1$ and

$$b_{ij} \geq b_{il} \tag{1.21b}$$

for all $l = 1, \dots, m_2$. Again, the Nash equilibrium is strong if (1.21a) and (1.21b) hold with strict inequality for all $k \neq i$ and all $l \neq j$, respectively; otherwise, it is weak.

In terms of Table 0.3, the concept of Nash equilibrium is our solution concept for noncooperative community games—their fourth and last key ingredient. If we were interested solely in finding the best pair of strategies for two slow drivers in Crossroads, however, then introducing the concept of Nash equilibrium would be like using a sledgehammer to burst a soap bubble. It is obvious from Figure 1.3 that $(1, 1)$ is the only pair of strategies that two rational players would select, because (u, v) will be selected only if u lies in R_1 and v in R_2. But things get a bit more complicated when either driver is either fast or intermediate.

To determine R_1 and R_2 in these circumstances, it will be convenient first to define parameters θ_1 and θ_2 by

$$(\delta + \epsilon)\theta_k = \epsilon + \tfrac{1}{2}\tau_k, \quad k = 1, 2. \tag{1.22}$$

Then from (1.12),

$$f_1(\overline{u}, v) = (\delta + \epsilon)(\theta_2 - v)\overline{u} + \left(\epsilon - \tfrac{1}{2}\tau_2\right)v - \epsilon - \tfrac{1}{2}\tau_2, \tag{1.23a}$$

$$f_2(u, \overline{v}) = (\delta + \epsilon)(\theta_1 - u)\overline{v} + \left(\epsilon - \tfrac{1}{2}\tau_1\right)u - \epsilon - \tfrac{1}{2}\tau_1. \tag{1.23b}$$

If $\frac{1}{2}\tau_1 < \delta < \frac{1}{2}\tau_2$ (San slow, Nan fast or intermediate), then $\theta_1 < 1 < \theta_2$. So the $\overline{u}$ that maximizes $f_1(\overline{u}, v)$ for $0 \leq \overline{u} \leq 1$ is still $\overline{u} = 1$ because $\frac{\partial f_1}{\partial \overline{u}} > 0$ for all $\overline{u} \in [0, 1]$. But the $\overline{v}$ that maximizes $f_2(u, \overline{v})$—that is, Player 2's best reply to u—is now

$$v = \mathcal{B}_2(u) = \begin{cases} 1 & \text{if } \; 0 \leq u < \theta_1, \\ \text{any } \overline{v} \in [0, 1] & \text{if } \; u = \theta_1, \\ 0 & \text{if } \; \theta_1 < u \leq 1, \end{cases} \tag{1.24}$$

because $\frac{\partial f_2}{\partial \overline{v}}$ is positive, zero, or negative according to whether $u < \theta_1$, $u = \theta_1$, or $u > \theta_1$. Thus R_1 is the same as before; whereas $(u, v) \in R_2$ is equivalent to $v = \mathcal{B}_2(u)$. So R_2 consists of three straight-line segments as shown in Figure 1.4(a).[11] We see that, if San has no knowledge of Nan's reward function f_1, then any $v \in [0, 1]$ could be optimal for San because, for all she knows, Nan could select the strategy $u = \theta_1$, to which any $v \in [0, 1]$ is a best reply. If, on the other hand, San knows Nan's reward function, then the only optimal choice for San is $v = 0$ because only $v = 0$ is a best reply to $u = 1$. Of course, $u = 1$ is also a best reply to $v = 0$ because it's the best reply to anything. Thus $(1, 0)$, the only point in the intersection of R_1 and R_2, is a Nash equilibrium. In terms of pure strategies, the Nash equilibrium is GW: G is a best reply to W (regardless), and W is a slow driver's best reply to an intermediate or fast driver's G.

Similarly, if $\frac{1}{2}\tau_2 < \delta < \frac{1}{2}\tau_1$ or $\theta_2 < 1 < \theta_1$ (Nan slow, San fast or intermediate) then, either from symmetry or proceeding as above (Exercise 1.3), we find that $\overline{v} = 1$ maximizes $f_2(u, \overline{v})$ because $\frac{\partial f_2}{\partial \overline{v}}$ is strictly positive; whereas the $\overline{u}$ that maximizes $f_1(\overline{u}, v)$ for $0 \le \overline{u} \le 1$ (that is, Player 1's best reply to v) is

$$u = \mathcal{B}_1(v) = \begin{cases} 1 & \text{if } 0 \le v < \theta_2, \\ \text{any } \overline{u} \in [0,1] & \text{if } v = \theta_2, \\ 0 & \text{if } \theta_2 < v \le 1, \end{cases} \tag{1.25}$$

because $\frac{\partial f_1}{\partial \overline{u}}$ is positive, zero, or negative according to whether $v < \theta_2$, $v = \theta_2$, or $v > \theta_2$. Thus R_1 and R_2 are as shown in Figure 1.4(b). If Nan has no knowledge of San's reward function f_2, then any $u \in [0, 1]$ could be optimal for Nan because, for all she knows, San could select the strategy $v = \theta_2$, to which any $u \in [0, 1]$ is a best reply. If, on the other hand, Nan knows San's reward function, then the only optimal choice for Nan is $u = 0$ because only $u = 0$ is a best reply to $v = 1$. Because $v = 1$ is also a best reply to $u = 0$, then $(0, 1)$, the only point in the intersection of R_1 and R_2, is a Nash equilibrium. In terms of pure strategies, this time the Nash equilibrium is WG, but the interpretation is otherwise the same: G is a best reply to W, and W is a slow driver's best reply to an intermediate or fast driver's G. We see that the concept of Nash equilibrium depends crucially on each player knowing the other player's reward (whereas the concept of optimal reaction set does not). If such is the case, then it is customary to say that the players have *complete information*.

Provided each player has knowledge of the other's reward, and despite a lack of explicit cooperation between the players, a Nash equilibrium is self-enforcing in the sense that neither player has a unilateral incentive to depart from it. Consider, however, the case in which neither driver is slow, so that $\max(\tau_1, \tau_2) < 2\delta$, or $\max(\theta_1, \theta_2) < 1$. The $\overline{u}$ that maximizes $f_1(\overline{u}, v)$ for $0 \le \overline{u} \le 1$ (that is, Player 1's best reply to v) is now still $\mathcal{B}_1(v)$ defined by (1.25), and the $\overline{v}$ that maximizes $f_2(u, \overline{v})$ (that is, Player 2's best reply to u) is again $\mathcal{B}_2(u)$ defined by (1.24). That is, $(u, v) \in R_1$ is equivalent to $u = \mathcal{B}_1(v)$ and $(u, v) \in R_2$ is equivalent to $v = \mathcal{B}_2(u)$,

[11] Note that $\mathcal{B}_2$ defined by (1.24) is not a function because $\mathcal{B}_2(u)$ takes more than one value where $u = \theta_1$. In general, a function is a rule that assigns a unique element of a set V (called the range), to each element of a set U (called the domain), and a rule that assigns a subset of V to each element of U is called a multivalued function, or *correspondence*. Thus (1.24) defines only a correspondence with $U = V = [0, 1]$. Often, however, a best-reply correspondence is also a function; see, e.g., (2.30).

Table 1.3. Nash-equilibrium rewards in Crossroads

(u,v)	$f_1(u,v)$	$f_2(u,v)$
$(1,0)$	0	$-\tau_1$
(θ_1,θ_2)	$-\big(\delta+\frac{1}{2}\tau_2\big)\theta_2$	$-\big(\delta+\frac{1}{2}\tau_1\big)\theta_1$
$(0,1)$	$-\tau_2$	0

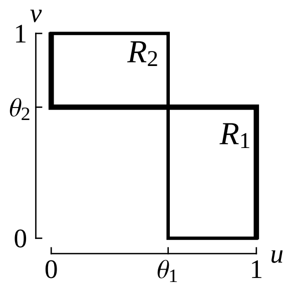

Figure 1.5. Optimal reaction sets when $\max(\tau_1,\tau_2) < 2\delta$

so that R_1 and R_2 are as shown in Figure 1.5. We observe at once that $R_1 \cap R_2 = \{(1,0),(\theta_1,\theta_2),(0,1)\}$: there are three Nash equilibria. Then which do we regard as the solution?

The rewards associated with the three Nash equilibria are given in Table 1.3. You can readily show that

$$-\big(\delta+\tfrac{1}{2}\tau_k\big)\theta_k - \big(-\tau_k\big) = \tfrac{(2\delta-\tau_k)(\tau_k-2\epsilon)}{4(\delta+\epsilon)}, \qquad k=1,2. \tag{1.26}$$

Thus $(1,0)$ is always the best Nash equilibrium for Nan, and (θ_1,θ_2) is second or third best, according to whether $2\delta > \tau_2 > 2\epsilon$ (intermediate San) or $2\delta > 2\epsilon > \tau_2$ (fast San). Likewise, $(0,1)$ is always the best Nash equilibrium for San, and (θ_1,θ_2) is second or third best according to whether $2\delta > \tau_1 > 2\epsilon$ (intermediate Nan) or $2\delta > 2\epsilon > \tau_1$ (fast Nan). Even though θ_1 is a best reply to θ_2 and θ_2 is a best reply to θ_1, there is no reason to expect the players to select these strategies because for each player there is another combination of mutual best replies that yields a higher reward. But if Nan selects her best Nash-equilibrium strategy $u=1$, and if San selects her best Nash-equilibrium strategy $v=1$, then the resulting strategy combination $(1,1)$ belongs to neither player's optimal reaction set! Then which—if any—of the Nash equilibria should we regard as the solution of the game? We will return to this matter in Chapter 2.

1.3. Four Ways: a motorist's trilemma

Nan and San's dilemma becomes even more intriguing if we allow a third pure strategy, denoted by C, in which each player's action is contingent upon that of the other.[12] A player who adopts C will select G if the other player selects W, but she will select W if the other player selects G. Let us suppose that, if Nan is

[12]Such a strategy is sometimes called a *conditional strategy*, although use of this term should perhaps be discouraged, essentially on the grounds of redundancy; see, e.g., [**139**, p. 77].

Table 1.4. Nan's payoff matrix in Four Ways

	G	W	C
G	$-\delta - \frac{1}{2}\tau$	0	0
W	$-\tau$	$-\epsilon - \frac{1}{2}\tau$	$-\tau$
C	$-\tau$	0	$-\delta - \frac{1}{2}\tau$

Table 1.5. San's payoff matrix in Four Ways

	G	W	C
G	$-\delta - \frac{1}{2}\tau$	$-\tau$	$-\tau$
W	0	$-\epsilon - \frac{1}{2}\tau$	0
C	0	$-\tau$	$-\delta - \frac{1}{2}\tau$

a C-strategist, then the first thing she does when she arrives at the junction is to wave San on; but if San replies by waving Nan on, then immediately Nan puts down her foot and drives away. If, on the other hand, San replies by hitting the gas, then Nan waits until San has traversed the junction. But what happens if San is also a C-strategist? As soon as they reach the junction, Nan and San both wave at one another. Nan interprets San's wave to mean that San wants to wait, so Nan drives forward; San interprets Nan's wave to mean that Nan wants to wait, so San also drives forward, and the result is the same as if both had selected strategy G. Thus if a G-strategist can be described as aggressive and a W-strategist as cooperative, then a C-strategist could perhaps be described as an impatient cooperator.

For the sake of simplicity, let us assume that the game is symmetric, i.e., $\tau_1 = \tau_2$, and denote the common value of these two parameters by τ. Then Nan and San's payoff matrices, A and B, respectively, are as shown in Tables 1.4 and 1.5 (assuming the time for which two C-strategists wave at each other to be negligibly small, that is, very small compared to ϵ). As always, the rows correspond to strategies of Player 1 (Nan), and the columns correspond to strategies of Player 2 (San). Thus, the entry in row i and column j is the payoff to the player whose payoffs are stored in the matrix if Player 1 selects strategy i and Player 2 selects strategy j. Because the game is symmetric, B is just the transpose of A. To distinguish this game from Crossroads, we will refer to it as Four Ways.

If the drivers are so slow that $\tau > 2\delta$ or $\sigma > 1$, where

$$\sigma \;=\; \tau/2\delta, \tag{1.27}$$

then their best strategy is to hit the gas because G dominates C and strictly dominates W for Nan from Table 1.4; similarly for San from Table 1.5. Thus G is a (weakly) dominant strategy for both players: neither has an incentive to depart from it, which makes strategy combination GG a Nash equilibrium. Furthermore, GG is the only Nash equilibrium when $\sigma > 1$ (see Exercise 1.2), and so we do not hesitate to regard it as the solution of the game: when there is only one Nash equilibrium, there is no indeterminacy to resolve.[13]

The game becomes interesting, however, when $\tau < 2\delta$ or $\sigma < 1$, which we assume for the rest of this section. As in Crossroads, no pure strategy is now dominant. We therefore consider mixed strategies. If Nan selects pure strategy G with probability u_1 and pure strategy W with probability u_2, then we shall say that Nan selects strategy u, where $u = (u_1, u_2)$ is a two-dimensional row vector. Thus Nan selects pure strategy C with probability $1 - u_1 - u_2$, where

$$0 \le u_1 \le 1, \qquad 0 \le u_2 \le 1, \qquad 0 \le u_1 + u_2 \le 1. \tag{1.28a}$$

So Nan's strategies correspond to points of a closed triangle in two-dimensional space. Similarly, if San selects G with probability v_1 and W with probability v_2, then we shall say that San selects strategy v, where $v = (v_1, v_2)$. Because San selects C with probability $1 - v_1 - v_2$, we have

$$0 \le v_1 \le 1, \qquad 0 \le v_2 \le 1, \qquad 0 \le v_1 + v_2 \le 1. \tag{1.28b}$$

Subsequently, we shall use Δ to denote the closed triangle in two-dimensional space defined *either* as the set of all points u satisfying (1.28a) *or* as the set of all points v satisfying (1.28b): Δ is the same strategy set, regardless of whether we use u or v to label a point in it. If Nan selects $u \in \Delta$ and San selects $v \in \Delta$, then they jointly select strategy combination (u, v), where $(u, v) = (u_1, u_2, v_1, v_2)$ is a four-dimensional vector.[14]

The sample space of N, Nan's choice of pure strategy, is now $\{G, W, C\}$ instead of $\{G, W\}$; $\text{Prob}(N = G) = u_1$, $\text{Prob}(N = W) = u_2$, and $\text{Prob}(N = C) = 1 - u_1 - u_2$. San's choice of pure strategy, S, has the same sample space, but with

$$\text{Prob}(S = G) = v_1, \quad \text{Prob}(S = W) = v_2, \quad \text{and } \text{Prob}(S = C) = 1 - v_1 - v_2.$$

The payoff to Nan, F_1, now has sample space $\left\{-\delta - \frac{1}{2}\tau, 0, -\tau, -\epsilon - \frac{1}{2}\tau\right\}$, and if strategies are still chosen independently, then

$$\begin{aligned}
\text{Prob}(F_1 = -\delta - \tau/2) &= \text{Prob}(N = G, S = G \text{ or } N = C, S = C) \\
&= \text{Prob}(N = G, S = G) + \text{Prob}(N = C, S = C) \\
&= \text{Prob}(N = G) \cdot \text{Prob}(S = G) + \text{Prob}(N = C) \cdot \text{Prob}(S = C) \\
&= u_1 v_1 + (1 - u_1 - u_2)(1 - v_1 - v_2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\text{Prob}(F_1 = 0) &= u_1 v_2 + u_1(1 - v_1 - v_2) + (1 - u_1 - u_2)v_2, \\
\text{Prob}(F_1 = -\tau) &= u_2 v_1 + u_2(1 - v_1 - v_2) + (1 - u_1 - u_2)v_1,
\end{aligned}$$

[13] Even if there were more than one Nash equilibrium, there would be no indeterminacy if all combinations of Nash-equilibrium strategies yielded the same payoffs. This equivalence holds in general only for zero-sum games; see, for example, Owen [**257**] or Wang [**354**]. For an example of a zero-sum game, see Exercise 1.30.

[14] If $u = (u_1, u_2) \in \Delta$ and $v = (v_1, v_2) \in \Delta$ are both two-dimensional row vectors, then strictly $(u, v) \in D = \Delta \times \Delta$ is a two-dimensional row vector of two-dimensional row vectors. But we prefer to think of strategy combination (u, v) as a single four-dimensional row vector (u_1, u_2, v_1, v_2), whose first two components are Player 1's strategy and whose last two are Player 2's, because we can then write Player i's reward as $f_i(u, v) = f_i(u_1, u_2, v_1, v_2)$ instead of $f_i(u, v) = f_i((u_1, u_2), (v_1, v_2))$. Preferring to avoid such a cumbersome notation, we ignore whatever claims it may have to being more technically correct, whenever it is convenient to do so, e.g., in §2.4 (p. 76) and in §7.3 (p. 308).

and

$$\mathrm{Prob}(F_1 = -\epsilon - \tau/2) = u_2 v_2.$$

Thus Nan's reward from the mixed strategy combination (u, v) is

$$\begin{aligned} f_1(u,v) = \mathrm{E}[F_1] = &-\big(\delta + \tfrac{1}{2}\tau\big)\cdot \mathrm{Prob}\big(F_1 = -\delta - \tfrac{1}{2}\tau\big) \\ &+ 0 \cdot \mathrm{Prob}\big(F_1 = 0\big) - \tau \cdot \mathrm{Prob}\big(F_1 = -\tau\big) \\ &- \big(\epsilon + \tfrac{1}{2}\tau\big)\cdot \mathrm{Prob}\big(F_1 = -\epsilon - \tfrac{1}{2}\tau\big) \end{aligned}$$

or, after simplification,

$$\begin{aligned} f_1(u,v) \;=\; &-\big(2\delta v_1 + \{\delta + \tfrac{1}{2}\tau\}\{v_2 - 1\}\big)u_1 \\ &-\big(\{\delta - \tfrac{1}{2}\tau\}\{v_1 - 1\} + \{\delta + \epsilon\}v_2\big)u_2 \\ &+\big(\delta - \tfrac{1}{2}\tau\big)v_1 + \big(\delta + \tfrac{1}{2}\tau\big)(v_2 - 1). \end{aligned} \tag{1.29a}$$

Similarly, San's reward from the strategy combination (u, v) is

$$\begin{aligned} f_2(u,v) = &-\big(2\delta u_1 + \{\delta + \tfrac{1}{2}\tau\}\{u_2 - 1\}\big)v_1 \\ &-\big(\{\delta - \tfrac{1}{2}\tau\}\{u_1 - 1\} + \{\delta + \epsilon\}u_2\big)v_2 \\ &+\big(\delta - \tfrac{1}{2}\tau\big)u_1 + \big(\delta + \tfrac{1}{2}\tau\big)(u_2 - 1). \end{aligned} \tag{1.29b}$$

By virtue of symmetry,

$$f_2(u,v) = f_1(v,u) \tag{1.30}$$

for all u and v satisfying (1.28). Note that (1.29) can be written more compactly as

$$\begin{aligned} f_1(u,v) &= (u_1, u_2, 1 - u_1 - u_2)A(v_1, v_2, 1 - v_1 - v_2)^T, \\ f_2(u,v) &= (u_1, u_2, 1 - u_1 - u_2)B(v_1, v_2, 1 - v_1 - v_2)^T, \end{aligned} \tag{1.31}$$

where A and $B = A^T$ are defined by Tables 1.4 and 1.5.[15]

Although u and v are now vectors, as opposed to scalars, everything we have said about optimal reaction sets and Nash equilibria with respect to Crossroads remains true for Four Ways, provided only that we replace $\Delta = [0, 1]$ in (1.10) and (1.14) by Δ defined as the set of points satisfying (1.28). Thus the players' optimal reaction sets in Four Ways are still defined by

$$R_1 = \big\{(u,v) \in D \mid f_1(u,v) = \max_{\overline{u} \in \Delta} f_1(\overline{u}, v)\big\}, \tag{1.32a}$$

$$R_2 = \big\{(u,v) \in D \mid f_2(u,v) = \max_{\overline{v} \in \Delta} f_2(u, \overline{v})\big\}, \tag{1.32b}$$

and the set of all Nash equilibria is still $R_1 \cap R_2$. On the other hand, because the optimal reaction sets now lie in a four-dimensional space, as opposed to a two-dimensional space, we cannot locate the Nash equilibria by drawing diagrams equivalent to Figures 1.3–1.5. Instead, we proceed as follows.

We first define dimensionless parameters

$$\gamma = \frac{\epsilon}{\delta}, \quad \alpha = \frac{(\sigma + \gamma)(\sigma + 1)}{1 + 2\gamma + \sigma^2}, \quad \beta = \frac{(1 - \sigma)^2}{1 + 2\gamma + \sigma^2}, \quad \omega = \frac{2\sigma}{1 + \sigma}, \tag{1.33a}$$

[15] Moreover, (1.31) is another special case of (A.2), just like (1.13).

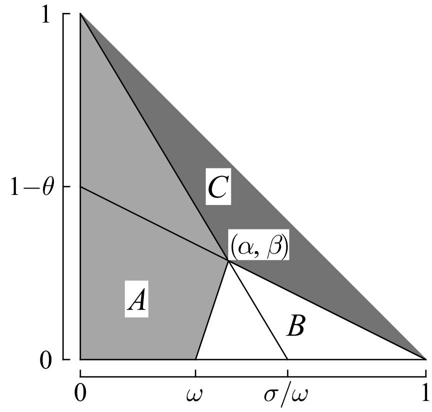

Figure 1.6. Subsets of the strategy set Δ defined by (1.28)

and

$$\theta \;=\; \frac{\epsilon + \tau/2}{\epsilon + \delta} \;=\; \frac{\sigma + \gamma}{1 + \gamma}, \tag{1.33b}$$

where σ is defined by (1.27). In view of (1.1), α, β, γ, σ, θ, and ω all lie between 0 and 1. If the coefficients of u_1 and u_2 in (1.29a) are both negative, then $f_1(u, v)$ is maximized by selecting $u_1 = 0$ and $u_2 = 0$, or $u = (0, 0)$; moreover, $(0, 0)$ is the only maximizing strategy for Player 1. If these coefficients are merely nonpositive, then there will be more than one maximizing strategy; nevertheless, $u = (0, 0)$ will continue to be one of them. But the coefficient of u_1 in (1.29a) is nonpositive when the point (v_1, v_2) lies on or above the line in two-dimensional space that joins the point $(\sigma/\omega, 0)$ to the point $(0, 1)$; whereas the coefficient of u_2 in (1.29a) is nonpositive when (v_1, v_2) lies on or above the line that joins $(1, 0)$ to $(0, 1 - \theta)$. Thus, the coefficients of u_1 and u_2 in (1.29a) are both nonpositive when the point (v_1, v_2) lies in that part of Δ which corresponds to (the interior or boundary of) the triangle marked C in Figure 1.6. Let us denote by $v^C = (v_1^C, v_2^C)$ any strategy for San that corresponds to a point in C. Then what we have shown is that all four-dimensional vectors of the form $\big(0, 0, v_1^C, v_2^C\big)$ must lie in R_1.

Extending our notation in an obvious way, let $v^A = (v_1^A, v_2^A)$ denote any strategy for San that corresponds to a point in A, let $v^{AC} = (v_1^{AC}, v_2^{AC})$ denote any strategy for San that corresponds to a point lying in both A and C, and so on. Then, by considering the various cases in which the coefficient of u_1 or the coefficient of u_2 or both in (1.29a) are nonpositive, nonnegative, or zero, it can be shown that all strategy combinations in Table 1.6 must lie in Nan's optimal reaction set, R_1; see Exercise 1.6. Furthermore, if we repeat the analysis for f_2 and San (as opposed to f_1 and Nan), and if we let $u^A = (u_1^A, u_2^A)$ denote any strategy for Nan that corresponds to a point in A, let $u^{AC} = (u_1^{AC}, u_2^{AC})$ denote any strategy for Nan that corresponds to a point in both A and C, and so on, then we find that all strategy combinations in Table 1.7 must lie in San's optimal reaction set R_2. Indeed, in view of symmetry condition (1.30), it is hardly necessary to repeat the analysis.

Table 1.6. R_1 for Four Ways

u_1	u_2	v_1	v_2	CONSTRAINTS
1	0	v_1^A	v_2^A	
0	1	v_1^B	v_2^B	
0	0	v_1^C	v_2^C	
u_1	0	v_1^{AC}	v_2^{AC}	$0 \le u_1 \le 1$
0	u_2	v_1^{BC}	v_2^{BC}	$0 \le u_2 \le 1$
u_1	u_2	v_1^{AB}	v_2^{AB}	$u \in \Delta,\ u_1 + u_2 = 1$
u_1	u_2	α	β	$u \in \Delta$

Table 1.7. R_2 for Four Ways

u_1	u_2	v_1	v_2	CONSTRAINTS
u_1^A	u_2^A	1	0	
u_1^B	u_2^B	0	1	
u_1^C	u_2^C	0	0	
u_1^{AC}	u_2^{AC}	v_1	0	$0 \le v_1 \le 1$
u_1^{BC}	u_2^{BC}	0	v_2	$0 \le v_2 \le 1$
u_1^{AB}	u_2^{AB}	v_1	v_2	$v \in \Delta,\ v_1 + v_2 = 1$
α	β	v_1	v_2	$v \in \Delta$

Table 1.8. Nash equilibria for Four Ways

u_1	u_2	v_1	v_2	CONSTRAINTS
1	0	0	1	
0	1	1	0	
1	0	0	0	
0	0	1	0	
1	0	0	v_2	$0 < v_2 < 1$
0	u_2	1	0	$0 < u_2 < 1$
0	1	v_1	0	$\omega \le v_1 < 1$
u_1	0	0	1	$\omega \le u_1 < 1$
α	β	α	β	

A strategy combination is a Nash equilibrium if, and only if, it appears both in Table 1.6 and in Table 1.7. Therefore, to find all Nash equilibria, we must match strategy combinations from Table 1.6 with strategy combinations from Table 1.7 in every possible way. For example, consider the first row of Table 1.6. It does not match the first, fourth, or sixth row of Table 1.7 because $(1, 0)$ does not lie in A. It does not match the last row of Table 1.7, even for $(v_1, v_2) \in A$, because $\alpha < 1$ (or $\beta > 0$). Because $(1, 0)$ lies in B and $(0, 1)$ lies in A, however, we can match the first row of Table 1.6 with the second row of Table 1.7, and so $(1, 0, 0, 1)$ is a Nash equilibrium. Likewise, because $(1, 0)$ lies in C and $(0, 0)$ in A, we can match the first row of Table 1.6 with the third row of Table 1.7, so that $(1, 0, 0, 0)$ is a Nash equilibrium too. Finally, we can match the first row of Table 1.6 with the fifth row of Table 1.7 to deduce that $(1, 0, 0, v_2)$ is a Nash equilibrium not only for $v_2 = 1$ and $v_2 = 0$, but also for $0 < v_2 < 1$, because then $(0, v_2)$ lies in A. The Nash equilibria we have found in this way are recorded in rows 1, 3, and 5 of Table 1.8.

Repeating the analysis for the remaining six rows of Table 1.8, we obtain (see Exercise 1.7) an exhaustive list of Nash-equilibrium strategy combinations. They are recorded in Table 1.8. The first four rows of this table correspond to equilibria in pure strategies: rows 1 and 2 correspond to equilibria in which one player selects G and the other W; rows 3 and 4, to equilibria in which one player selects G and the other C. The remaining five rows correspond to equilibria in mixed strategies.[16] We see that, although rows 1–4 and 9 of the table correspond to isolated equilibria, there are infinitely many equilibria of the other types. If you thought that having three equilibria to choose from in Crossroads was bad enough, then I wonder what you are thinking now. Which, if any, of all these infinitely many equilibria do we regard as the solution of Four Ways? We will return to this question in Chapter 2.

1.4. Store Wars: a continuous game of prices

Although it is always reasonable to suppose that decision makers have only a finite number of pure strategies, when the number is large it is often convenient to imagine instead that the strategies form a continuum. Suppose, for example, that the price of some item could reasonably lie anywhere between \$5 and \$10. Then if a cent is the smallest unit of currency and if selecting a strategy corresponds to setting the price of the item, then the decision maker has a finite total of 501 pure strategies. Because this number is large, however, it may be preferable to suppose that the price in dollars can take any value between 5 and 10 (and round to two decimal places). Then rewards are calculated directly, i.e., without the intermediate step of calculating payoff matrices, and the game is said to be *continuous*, in order to distinguish it from matrix games like Crossroads and Four Ways.[17] The definition of Nash equilibrium is not in the least affected, but whereas matrix games are guaranteed to have at least one Nash equilibrium, continuous games may have none at all.[18] These ideas are illustrated by the following example.

A district or subdivision of an area of 50 square miles consists of two rectangles of land, as shaded in Figure 1.7: the smaller rectangle measures 15 square miles; the larger rectangle, 35. If we take the southwest corner of the subdivision to be the origin of a Cartesian coordinate system Oxy, with x increasing to the east and y to the north, then the subdivision contains all points (x, y) such that either $0 \le x \le 7, 0 \le y \le 5$ or $7 \le x \le 10, 5 \le y \le 10$. All roads through the subdivision

[16] If one has no wish to distinguish pure strategies from mixed ones, then Table 1.8 can be reduced to just five rows by weakening the inequalities in rows 5 and 6.

[17] To be sure, these games have a continuum of mixed strategies; however, we reserve the phrase "continuous game" for a game that has a continuum of strategies but is not also a bimatrix game—or, to be more precise, is not also the mixed extension of a discrete bimatrix game; see Appendix A, p. 364. Thus, on the one hand, there are games with a continuous reward function that we do not regard as continuous games, because they are also bimatrix games, and, on the other hand, there exist games that we think of as continuous games, even though their reward functions have isolated discontinuities on the decision set D—e.g., across a line in the unit square, as in Exercise 1.28.

[18] For a proof that matrix games have at least one Nash equilibrium, see, for example, [**354**].

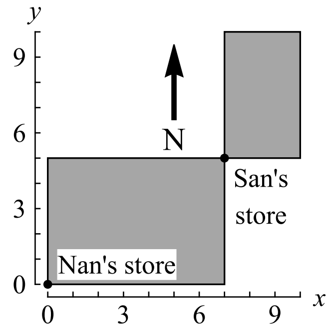

Figure 1.7. Battleground for Store Wars

run either from east to west or from north to south. There are two stores, one at $(0, 0)$, the other at $(7, 5)$. Each sells a product for which the daily demand is uniformly distributed over the 50 square miles, in the sense that customers are equally likely to live anywhere in the subdivision; the product might, for example, be bags of ice. If buyers select a store solely by weighing the price of the product against the cost of getting there (bags of ice at the first store are identical to those at the second, etc.), and if each store wishes to maximize revenue from the product in question, then how should prices be set? Because the best price for each store depends on the other store's price, their decisions are interdependent; and if they do not communicate with one another before setting prices, then we have all the necessary ingredients for a noncooperative game. We call this game Store Wars.[19]

Let Player 1 be Nan, who is manager of the store at $(0, 0)$, and let Player 2 be San, who is manager of the store at $(7, 5)$.[20] Let p_1 be Nan's price for the product, let p_2 be San's price, and let c be the cost per mile of travel to the store, assumed the same for all customers. Thus the round-trip cost of travel from Nan's store to San's store would be $24c$—no matter how you went, because all roads through the subdivision run from east to west or from north to south. Clearly, if Nan's price were to exceed this round-trip travel cost plus San's price for the item in question, then Nan could never expect anyone to buy from her. Accordingly, we can safely assume that

$$p_1 \leq p_2 + 24c. \tag{1.34a}$$

Similarly, because nobody in the larger rectangle can be expected to buy from San if her price exceeds Nan's by the round-trip travel cost between the stores, and assuming that San would like to attract at least some customers from the larger rectangle, we have

$$p_2 \leq p_1 + 24c. \tag{1.34b}$$

[19]Store Wars was suggested by the Hotelling model described in Phlips [**271**, pp. 42–45]. Phlips assumes that prospective customers are uniformly distributed along a line, whereas Store Wars assumes, in effect, that they are nonuniformly distributed along a line.

[20]If Nan were to live near San's store and San were to live near Nan's store, then we could easily explain why they keep meeting each other in Crossroads!

Furthermore, there are upper and lower limits to the price that a store can charge for a product, and it will be convenient to write these as

$$p_1 \leq 4c\alpha, \quad p_2 \leq 4c\alpha, \tag{1.34c}$$

$$p_1 \geq 4c\beta, \quad p_2 \geq 4c\beta, \tag{1.34d}$$

where α and β are dimensionless parameters.[21] For the sake of simplicity, however, we assume throughout that $\beta = 0$; provided that β is sufficiently small, this assumption will not affect the principal results of our analysis.[22] Also, we assume that $\alpha > 6$, as in Figure 1.8. Note that (1.34) can be written more compactly as $|p_1 - p_2| \leq 24c$: the difference in prices does not exceed the cost of round-trip travel between the stores.

Now, let (X, Y) be the residential coordinates of the next customer for the product in question. Because all roads run either north and south or east and west, her distance from Nan's store is $|X| + |Y|$ and her distance from San's store is $|7 - X| + |5 - Y|$. Thus, assuming that she selects a store *solely* by weighing the price of the product against the cost of travel from her residence (she doesn't, for example, buy the product on her way home from work), this customer will buy from Nan if

$$p_1 + 2c(|X| + |Y|) < p_2 + 2c(|7 - X| + |5 - Y|); \tag{1.35a}$$

whereas she will buy from San if

$$p_1 + 2c(|X| + |Y|) > p_2 + 2c(|7 - X| + |5 - Y|). \tag{1.35b}$$

But $X \geq 0, Y \geq 0$; thus $|X| + |Y|$ is the same thing as $X + Y$. Furthermore, the shape of the subdivision precludes either $X > 7, Y < 5$ or $X < 7, Y > 5$; therefore, $|7 - X| + |5 - Y|$ is the same thing as $|12 - X - Y|$. So, if we had $X + Y > 12$ in (1.35a), then it would now imply $p_1 + 24c < p_2$, which violates (1.34b). Accordingly, we can both assume that $X + Y \leq 12$ in (1.35a) and rewrite it as $p_1 + 2c(X + Y) < p_2 + 2c(12 - X - Y)$. Hence the next customer will buy from Nan if

$$X + Y < \frac{p_2 - p_1}{4c} + 6. \tag{1.36a}$$

Similarly, if $X + Y \leq 12$, then the next customer will buy from San if

$$X + Y > \frac{p_2 - p_1}{4c} + 6. \tag{1.36b}$$

If, on the other hand, $X + Y > 12$, then the customer will certainly buy from San (and (1.35b) reduces to $p_1 + 24c > p_2$). So the next customer will buy from San either if $X + Y > 12$ or if $X + Y \leq 12$ and (1.36b) is satisfied. But $X + Y > 12$ implies (1.36b) because the right-hand side of (1.36b) is less than or equal to 12, by virtue of (1.34b). Thus, in any event, the next customer will buy from San if

[21] Note that $4c$ is the cost of driving around a square-mile block. Because c is a cost per unit length, we must multiply it by a distance (here 4) to make the right-hand side of each inequality a quantity with the dimensions of price.

[22] In terms of the economist's inverse demand curve, with quantity measured along the horizontal axis and price along the vertical axis, $4\alpha c$ is the price at which the demand curve meets the vertical axis, whereas $4\beta c$ is simply the cost price of the item. Strictly, however, we ignore questions of supply and demand, or, if you prefer, we assume that demand is infinitely elastic at $4\alpha c$ but infinitely inelastic at greater or lower prices.

(1.36b) is satisfied. Of course, San's monopoly over the smaller rectangle was built into the model when we assumed (1.34b).

Because the next customer could live anywhere in the subdivision, X and Y are (continuous) random variables; hence so is $X+Y$. Let G denote its cumulative distribution function, i.e., define

$$G(s) = \text{Prob}(X+Y \le s), \quad 0 \le s \le 20, \tag{1.37}$$

and let F_1 denote Nan's payoff from the next customer. Then F_1 is also a random variable, which in view of (1.36) is defined by

$$F_1 = \begin{cases} p_1 & \text{if } X+Y < \frac{p_2-p_1}{4c}+6, \\ 0 & \text{if } X+Y > \frac{p_2-p_1}{4c}+6. \end{cases} \tag{1.38}$$

Because F_1 is a random variable, it cannot itself be maximized; instead we can maximize its expected value, which we shall denote by f_1, and define to be Nan's reward.

It will be convenient to make prices dimensionless, by scaling them with respect to $4c$. Let us therefore define u and v by

$$u = \frac{p_1}{4c}, \qquad v = \frac{p_2}{4c}, \tag{1.39}$$

where u is Nan's strategy and v is San's. Then, from (1.37)–(1.39),

$$\begin{aligned} f_1(u,v) = \text{E}[F_1] &= p_1 \cdot \text{Prob}(X+Y < v-u+6) \\ &\quad + 0 \cdot \text{Prob}(X+Y > v-u+6) \\ &= 4cuG(v-u+6); \end{aligned} \tag{1.40}$$

of course, $\text{Prob}(X+Y = v-u+6) = 0$, because $X+Y$ is a continuous random variable.[23] Similarly, San's payoff is the random variable

$$F_2 = \begin{cases} 0 & \text{if } X+Y < v-u+6, \\ 4cv & \text{if } X+Y > v-u+6, \end{cases} \tag{1.41}$$

and her reward is

$$\begin{aligned} f_2(u,v) = \text{E}[F_2] &= 4cv \cdot \text{Prob}(X+Y > v-u+6) \\ &= 4cv\,\{1-G(v-u+6)\}. \end{aligned} \tag{1.42}$$

Note that, in view of (1.39), (1.34a) requires $u \le v+6$, whereas (1.34b) requires $v \le u+6$. Thus, in view of (1.34c), the set of all feasible strategy combinations—the decision set—is

$$D = \big\{(u,v) \mid 0 \le u, v \le \alpha,\ |u-v| \le 6\big\}. \tag{1.43}$$

It will be convenient to define three subsets of D by

$$D_A = \big\{(u,v) \mid u \le \alpha,\ v \ge 0,\ 1 \le u-v \le 6\big\}, \tag{1.44a}$$

$$D_B = \big\{(u,v) \mid 0 \le u, v \le \alpha,\ |u-v| \le 1\big\}, \tag{1.44b}$$

$$D_C = \big\{(u,v) \mid u \ge 0,\ v \le \alpha,\ 1 \le v-u \le 6\big\}, \tag{1.44c}$$

so that $D = D_A \cup D_B \cup D_C$. For $\alpha = 10$, D is depicted in Figure 1.8. The lighter shaded region is D_B; the darker region lies outside D.

[23] See, for example, [**210**, pp. 523–524].

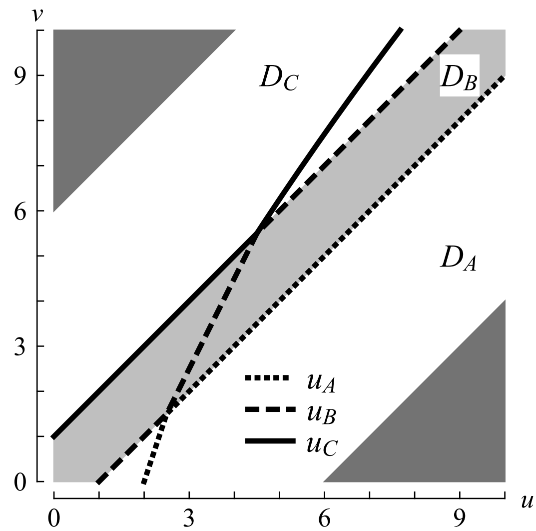

Figure 1.8. The decision set D and a set of points containing R_1 for $\alpha = 10$; the dark shaded triangles lie outside D.

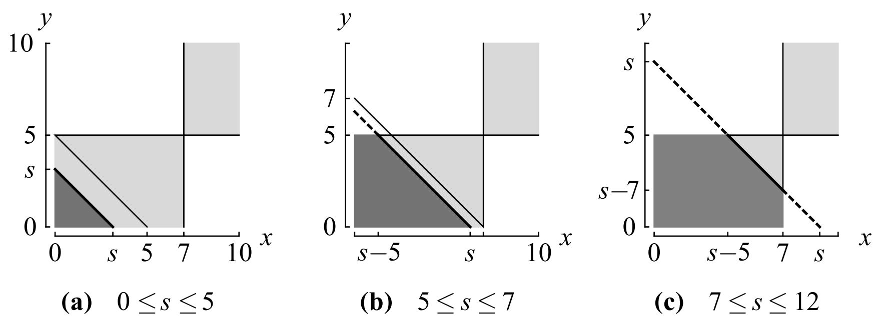

Figure 1.9. Calculation of G defined by (1.37); see text for discussion.

If we assume that customers are uniformly distributed throughout the subdivision, then $G(s)$ is readily calculated with the help of Figure 1.9, because $\text{Prob}(X + Y \leq s)$ is just the fraction of the total area of the subdivision that lies below the line $x + y = s$. Suppose, for example, that $0 \leq s \leq 5$. Then the dark shaded area in Figure 1.9(a) is $\frac{1}{2}s^2$; so the fraction of total area below the line $x+y = s$ is $\frac{1}{100}s^2$ (because the populated area is 50 square miles). Or suppose that $5 \leq s \leq 7$. Then, similarly, the fraction of total area shaded dark in Figure 1.9(b) is $\frac{1}{10}s - \frac{1}{4}$. Continuing in this manner, we readily find that

$$G(s) = \begin{cases} \frac{1}{100}s^2 & \text{if } 0 \leq s \leq 5, \\ \frac{2s-5}{20} & \text{if } 5 \leq s \leq 7, \\ \frac{7}{10} - \frac{1}{100}(12 - s)^2 & \text{if } 7 \leq s \leq 12. \end{cases} \tag{1.45}$$

Because San has a monopoly over the upper rectangle in Figure 1.9, $G(s)$ is not needed for $s \geq 12$.[24]

We can now obtain the optimal reaction sets. We have

$$\begin{aligned} R_1 &= \Big\{(u,v) \in D \mid f_1(u,v) = \max_{\overline{u}\,:\,(\overline{u},v)\in D} f_1(\overline{u},v)\Big\} \\ &= \{(u,v) \in D \mid u = \mathcal{B}_1(v)\}, \end{aligned} \tag{1.46a}$$

where ":" means "such that", and

$$\begin{aligned} R_2 &= \Big\{(u,v) \in D \mid f_2(u,v) = \max_{\overline{v}\,:\,(u,\overline{v})\in D} f_2(u,\overline{v})\Big\} \\ &= \{(u,v) \in D \mid v = \mathcal{B}_2(u)\}, \end{aligned} \tag{1.46b}$$

where D is defined by (1.43) and $\mathcal{B}_i$ denotes Player i's best reply to the other player. First we find R_1. From (1.40), (1.42), and (1.45),

$$f_1(u,v) = \frac{cu}{25}\begin{cases} (v-u+6)^2 & \text{if } (u,v) \in D_A, \\ 5(2v-2u+7) & \text{if } (u,v) \in D_B, \\ 70-(u-v+6)^2 & \text{if } (u,v) \in D_C \end{cases} \tag{1.47a}$$

and

$$f_2(u,v) = \frac{cv}{25}\begin{cases} (u-v+4)(v-u+16) & \text{if } (u,v) \in D_A, \\ 5(2u-2v+13) & \text{if } (u,v) \in D_B, \\ 30+(u-v+6)^2 & \text{if } (u,v) \in D_C. \end{cases} \tag{1.47b}$$

From (1.47), if (u,v) lies inside D_A, implying $v+1 < u < \min(v+6,\alpha)$ and hence $v+1 < u < v+6$), then

$$\partial f_1/\partial u = \tfrac{1}{25}c(v-u+6)(v-3u+6) \tag{1.48}$$

is positive for $u < \frac{1}{3}(v+6)$ and negative for $u > \frac{1}{3}(v+6)$. Thus if $v+1 \geq \frac{1}{3}(v+6)$, then f_1 has its maximum on D_A at $u = v+1$. If, on the other hand, $\frac{1}{3}(v+6) \geq v+1$, then f_1 has its maximum on D_A at $u = \frac{1}{3}(v+6)$ because $\frac{1}{3}(v+6) \leq \min(v+6,\alpha)$.[25] In other words, the maximum of f_1 over the region D_A occurs at $u = u_A(v)$, where

$$u_A(v) = \begin{cases} \frac{1}{3}v+2 & \text{if } 0 \leq v \leq \frac{3}{2}, \\ v+1 & \text{if } \frac{3}{2} \leq v \leq \alpha-1. \end{cases} \tag{1.49a}$$

The curve $u = u_A$ is represented in Figure 1.8 by dotted lines. For any $v \in [0,\alpha-1]$, $u = u_A(v)$ is Nan's best reply to v with $(u,v) \in D_A$.

If $(u,v) \in D_B$ or $\max(0,v-1) \leq u \leq \min(v+1,\alpha)$, then $\partial f_1/\partial u$ is positive for $u < \frac{1}{4}(2v+7)$ but negative for $u > \frac{1}{4}(2v+7)$. So the maximum of f_1 over D_B occurs at $u = u_B(v)$, where

$$u_B(v) = \begin{cases} v+1 & \text{if } 0 \leq v \leq \frac{3}{2}, \\ \frac{1}{2}v+\frac{7}{4} & \text{if } \frac{3}{2} \leq v \leq \frac{11}{2}, \\ v-1 & \text{if } \frac{11}{2} \leq v \leq \alpha. \end{cases} \tag{1.49b}$$

[24] It will shortly transpire that, in effect, $0 \leq s \leq 5$ corresponds to D_A, $5 \leq s \leq 7$ to D_B and $7 \leq s \leq 12$ to D_C in Figure 1.8.

[25] This inequality clearly holds if $v+6 \leq \alpha$ or $v \leq \alpha-6$; whereas if $\alpha-1 \geq v > \alpha-6$ (> 0, as assumed on p. 28), then the inequality reduces to $v+6 \leq 3\alpha$, which again must hold because $v \leq \alpha$ and α exceeds 6 (hence also 3).

The curve $u = u_B$ is represented in Figure 1.8 by dashed lines. For any $v \in [0, \alpha]$, $u = u_B(v)$ is Nan's best reply to v with $(u, v) \in D_B$.

Similarly (see Exercise 1.8), the maximum of f_1 over D_C occurs at $u = u_C(v)$, where

$$(1.49c) \qquad u_C(v) \;=\; \begin{cases} v - 1 & \text{if } 1 \le v \le \frac{11}{2}, \\ \frac{1}{3}\left(2v + \sqrt{(v-6)^2 + 210}\right) - 4 & \text{if } \frac{11}{2} \le v \le \alpha, \end{cases}$$

and it is assumed (on p. 28) that $\alpha > 6$. The curve $u = u_C$ is shown solid in Figure 1.8; although it appears to consist of two straight line segments, for $v \ge \frac{11}{2}$ it has a slight downward curvature. For any $v \in [1, \alpha]$, $u = u_C(v)$ is Nan's best reply to v with $(u, v) \in D_C$.

Now, for any $v \in [0, \alpha]$, a comparison of the conditional best replies in (1.49) yields Nan's unconditional best reply to v with $(u, v) \in D_A \cup D_B \cup D_C = D$. From (1.46a) and Figure 1.8, we obtain

$$(1.50a) \qquad \mathcal{B}_1(v) \;=\; \begin{cases} \frac{1}{3}v + 2 & \text{if } 0 \le v \le \frac{3}{2}, \\ \frac{1}{2}v + \frac{7}{4} & \text{if } \frac{3}{2} \le v \le \frac{11}{2}, \\ \frac{1}{3}\left(2v + \sqrt{(v-6)^2 + 210}\right) - 4 & \text{if } \frac{11}{2} \le v \le \alpha. \end{cases}$$

Equivalently, Nan's optimal reaction set is

$$(1.50b) \qquad \begin{aligned} R_1 \;=\; & \left\{(u_A(v), v) \mid 0 \le v \le \tfrac{3}{2}\right\} \\ & \cup \left\{(u_B(v), v) \mid \tfrac{3}{2} \le v \le \tfrac{11}{2}\right\} \\ & \cup \left\{(u_C(v), v) \mid \tfrac{11}{2} \le v \le \alpha\right\}. \end{aligned}$$

To verify (1.50), suppose, for example, that $0 \le v \le \frac{3}{2}$. Then f_1 is larger along $u = u_A(v)$ than elsewhere in D_A, including the boundary with D_B; but because this boundary is where f_1 is maximized on D_B (for $0 \le v \le \frac{3}{2}$), f_1 must be larger along $u = u_A(v)$ than elsewhere in both D_A and D_B, including its boundary with D_C (when $1 \le v \le \frac{3}{2}$). But because this boundary is where f_1 is maximized on D_C (for $1 \le v \le \frac{3}{2}$), f_1 must be larger along $u = u_A(v)$ than anywhere else in $D_A \cup D_B \cup D_C = D$ (for $0 \le v \le \frac{3}{2}$). Similarly for $\frac{3}{2} \le v \le \alpha$. The result for $\alpha = 10$ is sketched in Figure 1.11(a) as a thick solid curve. Note in particular that Nan's optimal reaction to $v = 0$ would be $u = 2$. Thus, even if San were to give away the product ($p_2 = 0$), Nan should still charge $p_1 = 8c$ for it because she would still attract customers who reside south or west of the line $x + y = 4$.

Although R_1 is *connected*—it's all in one piece—connectedness is not a general property of optimal reaction sets. To see this, note that if $\alpha = 10$ and if the maxima of f_2 over subsets D_A, D_B, and D_C of D occur where $v = v_A(u)$, $v = v_B(u)$, and $v = v_C(u)$, respectively, then from Exercise 1.9 we have

$$(1.51a) \qquad v_A(u) \;=\; \begin{cases} u - 1 & \text{if } 1 \le u \le \frac{17}{2}, \\ \frac{1}{3}\left(2u + \sqrt{(u-6)^2 + 300}\right) - 4 & \text{if } \frac{17}{2} \le u \le 10 \end{cases}$$

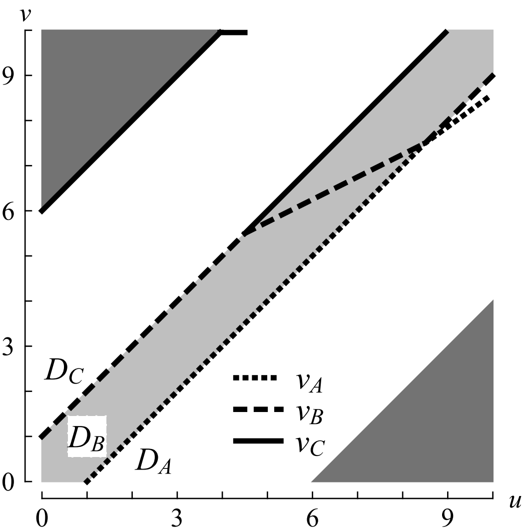

Figure 1.10. The decision set D and a set of points containing R_2 for $\alpha = 10$; the dark shaded triangles lie outside D.

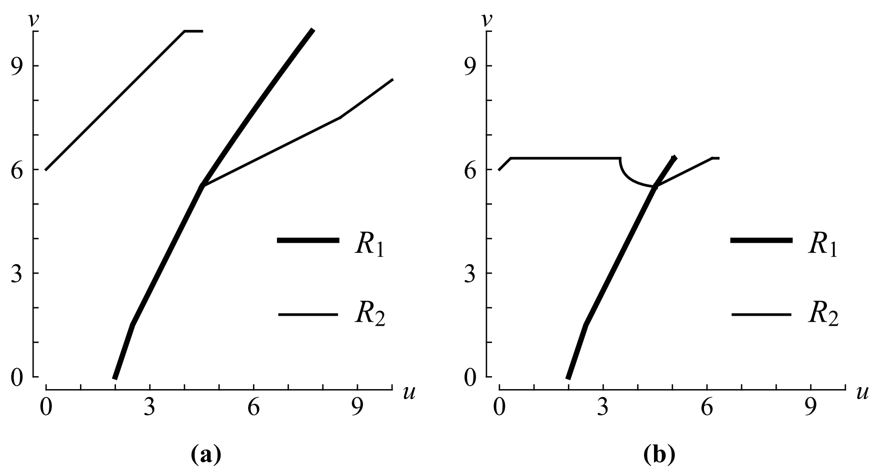

Figure 1.11. R_1 and R_2 for **(a)** $\alpha = 10$ and **(b)** $\alpha = 2\sqrt{10}$

because $\max(0, u-6) \leq v \leq u-1$ for $(u, v) \in D_A$;

$$v_B(u) \;=\; \begin{cases} u+1 & \text{if } 0 \leq u \leq \frac{9}{2}, \\ \frac{1}{2}u + \frac{13}{4} & \text{if } \frac{9}{2} \leq u \leq \frac{17}{2}, \\ u-1 & \text{if } \frac{17}{2} \leq u \leq 10 \end{cases} \tag{1.51b}$$

because $\max(0, u-1) \le v \le \min(u+1, 10)$ for $(u,v) \in D_B$; and

$$\text{(1.51c)} \qquad v_C(u) \;=\; \begin{cases} u+6 & \text{if } 0 \le u \le 4, \\ 10 & \text{if } 4 \le u \le \frac{9}{2}, \\ u+1 & \text{if } \frac{9}{2} \le u \le 9 \end{cases}$$

because $u+1 \le v \le \min(u+6, 10)$ for $(u,v) \in D_C$. The graphs of $v = v_A(u), v = v_B(u)$, and $v = v_C(u)$ are depicted in Figure 1.10. Note that $v = v_C(u)$ is not strictly a function but rather a multivalued function or correspondence; it is double valued at $u = \frac{9}{2}$ because $v = \frac{11}{2}$ and $v = 10$ both maximize $f_2(9/2, v)$.

By analogy with (1.50), it follows that San's best reply to u for $(u,v) \in D$ is

$$\text{(1.52a)} \qquad \mathcal{B}_2(u) \;=\; \begin{cases} \min(u+6, 10) & \text{if } 0 \le u \le \frac{9}{2}, \\ \frac{1}{2}u + \frac{13}{4} & \text{if } \frac{9}{2} \le u \le \frac{17}{2}, \\ \frac{1}{3}\left(2u + \sqrt{(u-6)^2 + 300}\right) - 4 & \text{if } \frac{17}{2} \le u \le 10. \end{cases}$$

Equivalently, San's optimal reaction set is

$$\text{(1.52b)} \qquad \begin{aligned} R_2 \;=\;& \left\{(u, v_C(u)) \mid 0 \le u \le \tfrac{9}{2}\right\} \\ &\cup \left\{(u, v_B(u)) \mid \tfrac{9}{2} \le u \le \tfrac{17}{2}\right\} \\ &\cup \left\{(u, v_A(u)) \mid \tfrac{17}{2} \le u \le 10\right\}. \end{aligned}$$

R_2 is sketched in Figure 1.11(a) as a thin solid curve. Note that the maximum of

$$f_2\left(\tfrac{9}{2}, v\right) \;=\; \tfrac{1}{25}c\left\{30 + \left(v - \tfrac{21}{2}\right)^2\right\}$$

for $\frac{11}{2} \le v \le 10$, namely, $\frac{121c}{10}$, occurs at both ends of the interval, and because $f_2\left(\frac{9}{2}, v\right)$ is less than $\frac{121c}{10}$ at every intermediate point, R_2 is disconnected along $u = \frac{9}{2}$; it contains both $\left(\frac{9}{2}, \frac{11}{2}\right)$ and $\left(\frac{9}{2}, 10\right)$ but no points that lie in between. Nevertheless, R_1 and R_2 still intersect one another at the (only) Nash equilibrium $(u^*, v^*) = \left(\frac{9}{2}, \frac{11}{2}\right)$. If this is accepted as the solution of the noncooperative game, then Nan's price is $p_1 = 18c$ and San's price is $p_2 = 22c$ from (1.39).

This result is strongly dependent on the value we chose for α. Indeed $\alpha = 10$ has a critical property: it is the largest value of α for which a Nash equilibrium exists. As α increases beyond 10, the left endpoint of the right-hand segment of R_2 moves away from $D_B \cap D_C$ into the interior of D_B, so that $R_1 \cap R_2 = \varnothing$, the empty set. As α moves below 10, on the other hand, the same endpoint moves into the interior of D_C, and there is a second critical value, namely, $\alpha = 2\sqrt{10}$, at which R_2 becomes connected; for this value of α, R_1 and R_2 are sketched in Figure 1.11(b). These results are best left as exercises, however; see Exercises 1.10–1.12.[26]

1.5. Store Wars II: a three-player game

We could easily turn Store Wars into a three-player continuous noncooperative game by placing a third store, say Van's, somewhere else in the subdivision, perhaps at the northeast corner; however, we prefer to devise an example of a three-player game by supposing instead that the interior of some circular island is uninhabitable (perhaps because of a volcano), so that all prospective customers for a certain product must

[26]And for further illustration of the point that connectedness is not a general property of optimal reaction sets, see Figure 6.2.

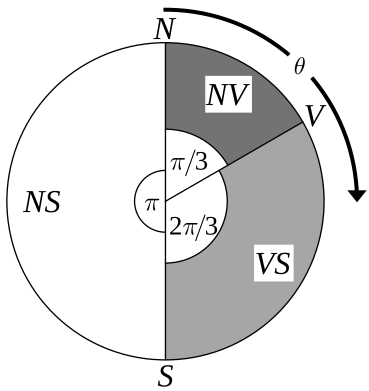

Figure 1.12. Map of battleground for Store Wars II

reside on the island's circumference. To be specific, let us suppose that Nan's store is at the most northerly point of the island and that Van's store is east of Nan's and one third of the way from Nan's store to the most southerly point of the island, which is also the location of the third store, San's; Nan is Player 1, Van is Player 2, and San is now Player 3. Let the radius of the island be a miles, and let $a\theta$ denote distance along the circumference, measured clockwise from the most northerly point. Then $0 \le \theta < 2\pi$, and the location of a customer's residence is determined by her θ-coordinate, with Nan's store at $\theta = 0$, Van's at $\theta = \pi/3$, and San's at $\theta = \pi$; see Figure 1.12. We call this game Store Wars II.

We will suppose that customers are uniformly distributed along the circumference. Thus if Θ denotes the θ-coordinate of a randomly chosen customer, then

$$\text{Prob}(0 \le \theta_1 < \Theta < \theta_2 < 2\pi) \;=\; \tfrac{1}{2\pi}(\theta_2 - \theta_1). \tag{1.53}$$

For example, if NV denotes the event that Θ lies between 0 and $\pi/3$, VS the event that Θ lies between $\pi/3$ and π, and NS the event that Θ lies between π and 2π (see Figure 1.12), then (1.53) implies

$$\text{Prob}(NV) = \tfrac{1}{6}, \quad \text{Prob}(VS) = \tfrac{1}{3}, \quad \text{Prob}(NS) = \tfrac{1}{2}. \tag{1.54}$$

Let p_i denote Player i's price for the product in question, for $i = 1, 2, 3$. Then we shall assume, as in §1.4, that the difference in prices between adjacent stores does not exceed the round-trip cost of travel between them. Thus if travel costs c dollars per mile, then

$$|p_1 - p_2| \le \tfrac{2}{3}\pi ac, \quad |p_2 - p_3| \le \tfrac{4}{3}\pi ac, \tag{1.55}$$

and $|p_1 - p_3| \le 2\pi ac$, which (1.55) implies.[27] As in §1.4, there are lower and upper bounds on the prices:

$$8\pi ac\beta \le p_i \le 8\pi ac\alpha, \quad i = 1, 2, 3. \tag{1.56}$$

But again as in §1.4, we shall assume throughout that $\beta = 0$.

[27] By the triangle inequality $|\delta_1 + \delta_2| \le |\delta_1| + |\delta_2|$ with $\delta_1 = p_1 - p_2$, $\delta_2 = p_2 - p_3$.

Now, let Θ be the residential coordinate of the next customer (hence $0 \le \Theta < 2\pi$); and suppose, as in §1.4, that this customer selects a store solely by weighing the price of the product against the cost of travel from her residence. Then, in view of (1.55), she will always buy from one of the two stores between which she lives. For example, the customer will buy from Nan if she resides in the dark shaded sector denoted by NV in Figure 1.12 and the total cost of buying from Nan is less than the total cost of buying from Van, i.e., if $0 < \Theta < \pi/3$ and $p_1 + 2ac\Theta < p_2 + 2ac(\pi/3 - \Theta)$ or, equivalently, $0 < \Theta < \pi/6 + (p_2 - p_1)/4ac$.[28] The customer will also buy from Nan, however, if she resides in the unshaded sector denoted by NS in Figure 1.12 and the total cost of buying from Nan is less than the total cost of buying from San, i.e., if $\pi < \Theta < 2\pi$ and $p_1 + 2ac(2\pi - \Theta) < p_3 + 2ac(\Theta - \pi)$ or, equivalently, $3\pi/2 + (p_1 - p_3)/4ac < \Theta < 2\pi$. As usual, we need not worry about the event that, for example, $p_1 + 2ac\Theta$ equals $p_2 + 2ac(\pi/3 - \Theta)$ precisely, because the event is associated with probability zero. Thus the next customer will buy from Nan if

$$0 < \Theta < \tfrac{\pi}{6} + \tfrac{p_2 - p_1}{4ac} \quad \text{or} \quad \tfrac{3\pi}{2} + \tfrac{p_1 - p_3}{4ac} < \Theta < 2\pi. \tag{1.57}$$

From (1.53), the probability of this event is

$$\tfrac{1}{2\pi}\left(\tfrac{\pi}{6} + \tfrac{p_2 - p_1}{4ac}\right) + \tfrac{1}{2\pi}\left(2\pi - \left\{\tfrac{3\pi}{2} + \tfrac{p_1 - p_3}{4ac}\right\}\right) = \tfrac{1}{3} + \tfrac{p_2 - 2p_1 + p_3}{8ac\pi}. \tag{1.58a}$$

Similarly, the next customer will buy from Van if either $0 < \Theta < \pi/3$ and $p_2 + 2ac(\pi/3 - \Theta) < p_1 + 2ac\Theta$ or $\pi/3 < \Theta < \pi$ and $p_2 + 2ac(\Theta - \pi/3) < p_3 + 2ac(\pi - \Theta)$, i.e., if $\frac{\pi}{6} + \frac{p_2 - p_1}{4ac} < \Theta < \frac{2\pi}{3} + \frac{p_3 - p_2}{4ac}$. From (1.53), the probability of this event is

$$\tfrac{1}{4} + \tfrac{p_1 - 2p_2 + p_3}{8ac\pi}. \tag{1.58b}$$

A similar calculation (Exercise 1.13) shows that the next customer will buy from San if $2\pi/3 + (p_3 - p_2)/4ac < \Theta < 3\pi/2 + (p_1 - p_3)/4ac$, and that the probability of this event is

$$\tfrac{5}{12} + \tfrac{p_1 - 2p_3 + p_2}{8ac\pi}. \tag{1.58c}$$

Of course, the three probabilities in (1.58) must sum to 1.

For $i = 1, 2, 3$, let the random variable F_i denote Player i's payoff from the next customer; its expected value, $f_i = \mathrm{E}[F_i]$, is Player i's reward. By analogy with (1.38), F_1 is p_1 if (1.57) is satisfied and zero otherwise, so that Nan's reward is p_1 times (1.58a). Likewise, Van's reward is p_2 times (1.58b), and San's reward is p_3 times (1.58c). It will be convenient, however, to make prices dimensionless by scaling them with respect to $8\pi ac$. Accordingly, we define strategies u, v, and z for Nan, Van, and San, respectively, by

$$u = \frac{p_1}{8\pi ac}, \quad v = \frac{p_2}{8\pi ac}, \quad z = \frac{p_3}{8\pi ac}. \tag{1.59}$$

Then the players' rewards are as follows:

$$f_1(u, v, z) = 8\pi acu\left(\tfrac{1}{3} + v - 2u + z\right), \tag{1.60a}$$

$$f_2(u, v, z) = 8\pi acv\left(\tfrac{1}{4} + u - 2v + z\right), \tag{1.60b}$$

$$f_3(u, v, z) = 8\pi acz\left(\tfrac{5}{12} + u - 2z + v\right). \tag{1.60c}$$

Moreover, from (1.55) and (1.56) with $\beta = 0$, their decision set is

$$D = \left\{(u, v, z)\,\middle|\,|u - v| \le \tfrac{1}{12},\ |v - z| \le \tfrac{1}{6},\ 0 \le u, v, z \le \alpha\right\}. \tag{1.61}$$

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[28] Note that (1.55) ensures $\pi/6 + (p_2 - p_1)/4ac \le \pi/3$.

Extending (1.46) in the obvious way, optimal reaction sets R_1, R_2, and R_3 for this three-player, noncooperative game are defined by

$$R_1 = \Big\{(u,v,z) \in D \Big| f_1(u,v,z) = \max_{\overline{u}\,:\,(\overline{u},v,z)\in D} f_1(\overline{u},v,z)\Big\}, \tag{1.62a}$$

$$R_2 = \Big\{(u,v,z) \in D \Big| f_2(u,v,z) = \max_{\overline{v}\,:\,(u,\overline{v},z)\in D} f_2(u,\overline{v},z)\Big\}, \tag{1.62b}$$

$$R_3 = \Big\{(u,v,z) \in D \Big| f_3(u,v,z) = \max_{\overline{z}\,:\,(u,v,\overline{z})\in D} f_3(u,v,\overline{z})\Big\}. \tag{1.62c}$$

Furthermore, the strategy combination (u^*, v^*, z^*) is a Nash equilibrium if u^* is a best reply to (v^*, z^*), v^* a best reply to (u^*, z^*), and z^* a best reply to (u^*, v^*); i.e., if Player 1 has nothing to gain by selecting $u \neq u^*$ when Players 2 and 3 have already selected (v^*, z^*), Player 2 has nothing to gain by selecting $v \neq v^*$ when Players 1 and 3 have already selected (u^*, z^*), and Player 3 has nothing to gain by selecting $z \neq z^*$ when Players 1 and 2 have already selected (u^*, v^*). Thus (u^*, v^*, z^*) is a Nash equilibrium if

$$\begin{aligned} f_1(u^*,v^*,z^*) &\geq f_1(u,v^*,z^*) \text{ for all } (u,v^*,z^*) \in D,\\ f_2(u^*,v^*,z^*) &\geq f_2(u^*,v,z^*) \text{ for all } (u^*,v,z^*) \in D,\\ f_3(u^*,v^*,z^*) &\geq f_3(u^*,v^*,z) \text{ for all } (u^*,v^*,z) \in D \end{aligned} \tag{1.63a}$$

or, equivalently, if

$$(u^*,v^*,z^*) \in R_1 \cap R_2 \cap R_3. \tag{1.63b}$$

As usual, if u^*, v^*, and z^* are all unique best replies, that is, if inequalities (1.63a) are strictly satisfied for, respectively, all $u \neq u^*$, all $v \neq v^*$, and all $z \neq z^*$, then the Nash equilibrium is said to be strong; otherwise, it is weak.

To obtain R_1, we must maximize f_1 as a function of u for all v and z, subject to the constraint that $(u,v,z) \in D$; to obtain R_2, we must maximize f_2 as a function of v for all u and z, subject to the same constraint; and similarly for R_3. Typically, the optimal reaction sets of a three-player game are much more difficult to calculate and visualize than those of a two-player game; but in the particular case of Store Wars II, they are all readily calculated—at least when α is sufficiently large, which we assume henceforward to simplify the analysis. It will also help to simplify matters if we define quantities $\hat{u}, \hat{v}$, and $\hat{z}$ by

$$\hat{u} = \tfrac{1}{4}\big(\tfrac{1}{3} + v + z\big), \tag{1.64a}$$

$$\hat{v} = \tfrac{1}{4}\big(\tfrac{1}{4} + u + z\big), \tag{1.64b}$$

$$\hat{z} = \tfrac{1}{4}\big(\tfrac{5}{12} + u + v\big). \tag{1.64c}$$

First we calculate R_3. From (1.61) in the limit as $\alpha \to \infty$, for all u, v satisfying $|u - v| \leq \frac{1}{12}$, we must maximize f_3 as a function of z subject to $-\frac{1}{6} \leq z - v \leq \frac{1}{6}$ and $z \geq 0$ or

$$\max\big(0, v - \tfrac{1}{6}\big) \leq z \leq v + \tfrac{1}{6}. \tag{1.65}$$

It is straightforward (see Exercise 1.14) to show that the maximum of f_3 on the interval $[0,\infty)$ occurs at $z = \hat{z}$, where $\hat{z}$ is defined by (1.64c), and that $|u - v| \leq \frac{1}{12}$ implies $\hat{z} \leq \frac{1}{2}v + \frac{1}{8}$, which in turn implies $\hat{z} < v + \frac{1}{6}$. Thus, because f_3 is increasing on $[0, \hat{z}]$ and decreasing on $[\hat{z}, \infty)$, the maximum of f_3 subject to (1.65) must occur

Table 1.9. R_1 for Store Wars II

u	v	z
$v^A + \frac{1}{12}$	v^A	z^A
$\frac{1}{4}\left(\frac{1}{3} + v^B + z^B\right)$	v^B	z^B
$v^C - \frac{1}{12}$	v^C	z^C

Table 1.10. R_2 for Store Wars II

u	v	z
u^D	$u^D + \frac{1}{12}$	z^D
u^E	$\frac{1}{4}\left(\frac{1}{4} + u^E + z^E\right)$	z^E
u^F	$z^F - \frac{1}{6}$	z^F
u^G	$u^G - \frac{1}{12}$	z^G

Table 1.11. R_3 for Store Wars II

u	v	z
u^H	v^H	$\frac{1}{4}\left(\frac{5}{12} + u^H + v^H\right)$
u^J	v^J	$v^J - \frac{1}{6}$

at $z = \max\left(0, v - \frac{1}{6}\right)$ if $\max\left(0, v - \frac{1}{6}\right) \geq \hat{z}$, but at $z = \hat{z}$ if $\max\left(0, v - \frac{1}{6}\right) < \hat{z}$. Because zero cannot exceed a positive number, the first of these two inequalities is satisfied if, and only if, $v - \frac{1}{6} \geq \hat{z}$ or $36v \geq 12u + 13$ (in addition to $|u - v| \leq \frac{1}{12}$). In other words, the first inequality is satisfied when the point (u, v) belongs to region J of Figure 1.13(c), which extends all the way to infinity in the northeasterly direction. Then f_3 is maximized by $z = v - \frac{1}{6}$ (as indicated in Figure 1.13). Correspondingly, f_3 is maximized by $z = \hat{z}$ when $36v < 12u + 13$, or when (u, v) belongs to region H. Thus, R_3 contains the strategy combinations in Table 1.11, where (u^K, v^K) is an arbitrary point in region K of Figure 1.13(c) for $K = H, J$.

Next we calculate R_1. From (1.61) in the limit as $\alpha \to \infty$, for all v, z satisfying $|v - z| \leq \frac{1}{6}$, we must maximize f_1 as a function of u subject to $-\frac{1}{12} \leq u - v \leq \frac{1}{12}$ and $u \geq 0$ or

$$\max\left(0, v - \tfrac{1}{12}\right) \leq u \leq v + \tfrac{1}{12}. \tag{1.66}$$

Again, because the maximum of f_1 on $[0, \infty)$ occurs at $u = \hat{u}$ and f_1 is increasing on $[0, \hat{u}]$ but decreasing on $[\hat{u}, \infty)$, the maximum of f_1 for (1.66) must occur where $u = \max\left(v - \frac{1}{12}, 0\right)$ if $\max\left(v - \frac{1}{12}, 0\right) \geq \hat{u}$, where $u = v + \frac{1}{12}$ if $v + \frac{1}{12} \leq \hat{u}$ and where $u = \hat{u}$ if $\max\left(v - \frac{1}{12}, 0\right) < \hat{u} < v + \frac{1}{12}$. The first of these three inequalities is satisfied where $v - \frac{1}{12} \geq \hat{u}$ or $9v \geq 3z + 2$ (in addition to $|v - z| \leq \frac{1}{6}$), i.e., where (v, z) belongs to region C of Figure 1.13(a); then f_1 is maximized by $u = v - \frac{1}{12}$. Similarly, the second inequality is satisfied when $z \geq 3v$ or (v, z) belongs to region A of Figure 1.13(a), with f_1 maximized by $u = v + \frac{1}{12}$; and the remaining pair of inequalities is satisfied when (v, z) belongs to region B, with f_1 maximized by $u = \hat{u}$. Thus, R_1 contains the strategy combinations in Table 1.9, where (v^K, z^K)

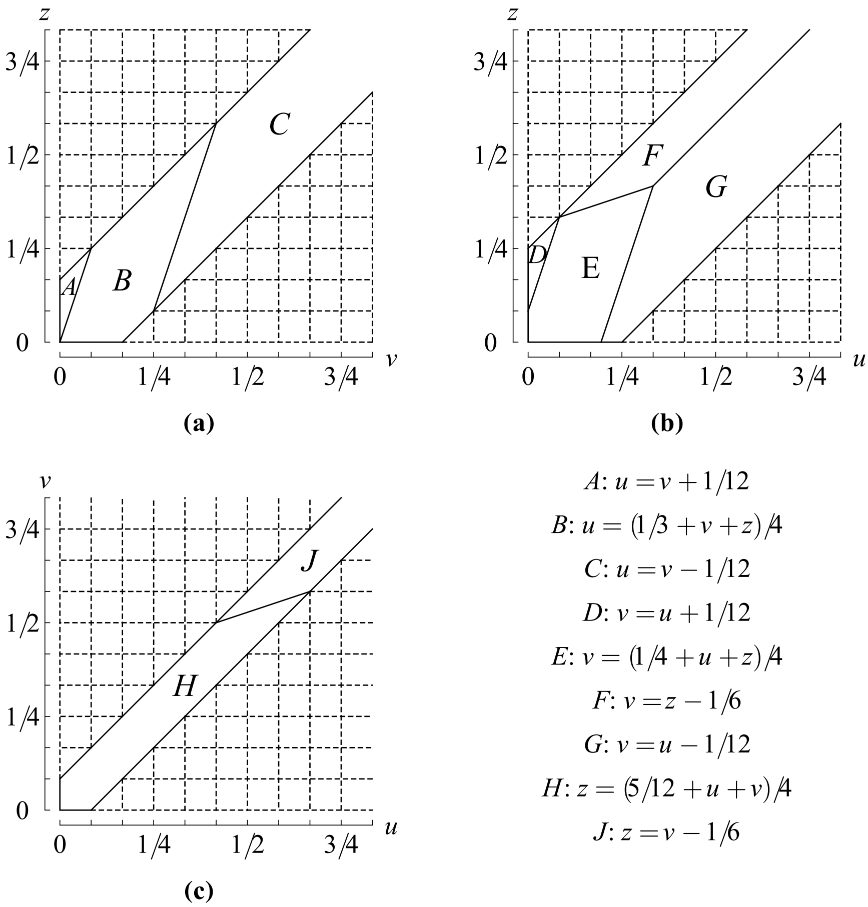

Figure 1.13. Best replies to other players' prices in Store Wars II for **(a)** Player 1, **(b)** Player 2, and **(c)** Player 3. In each case, the triangular regions criss-crossed by dashed grid lines lie outside the decision set. In **(a)**, region B is bounded by parallel line segments with equations $z = 3v$ between $(0,0)$ and $(1/12, 1/4)$ and $9v = 3z + 2$ between $(1/4, 1/12)$ and $(5/12, 7/12)$. In **(b)**, region E is bounded by parallel line segments with equations $12z = 36u + 1$ from $(0, 1/12)$ to $(1/12, 1/3)$ and $36u = 12z + 7$ from $(7/36, 0)$ to $(1/3, 5/12)$, and by a line segment with equation $36z = 12u + 11$ between $(1/12, 1/3)$ and $(1/3, 5/12)$; and region F is separated from region G by a line segment with equation $12z = 12u + 1$ extending from $(1/3, 5/12)$ to infinity. In **(c)**, regions H and J are separated by a line segment with equation $36v = 12u + 13$ from $(5/12, 1/2)$ to $(2/3, 7/12)$.

is an arbitrary point in region K of Figure 1.13(a) for $K = A, B, C$. A similar calculation shows that R_2 contains the strategy combinations in Table 1.10, where (u^K, z^K) is an arbitrary point in region K of Figure 1.13(b) for $K = D, E, F, G$; see Exercise 1.14.

To satisfy $(u^*, v^*, z^*) \in R_1 \cap R_2 \cap R_3$, we must identify all possible ways in which a row from Table 1.9 can also lie in both Table 1.10 and Table 1.11. There are three ways to choose the row from Table 1.9, four ways to choose the row from Table 1.10, and two ways to choose the row from Table 1.11, yielding 24 choices in all. It turns out, however, that 23 of these choices are impossible; see Exercise 1.15. For example, we cannot match the first rows of Tables 1.9 and 1.10 because $u = v^A + \frac{1}{12}$ implies $u \geq \frac{1}{12}$, which—by inspection of Figure 1.13—allows $u = u^D$ only if $z^D = \frac{1}{3}$, which precludes $z^D = z^A$ because $z^A = \frac{1}{3}$ is impossible. This argument rules out two of the 24 choices because it is valid for either row from Table 1.11. More obviously, we cannot match the second row of Table 1.10 (which requires $u \leq 1/3$) with the second row of Table 1.11 (which requires $u \geq 5/12$), regardless of the row we choose from Table 1.9; this observation rules out a further three possibilities. Continuing in this manner, we find that the only legitimate possibility is for the second rows of Tables 1.9 and 1.10 to match the first row of Table 1.11. Hence (see Exercises 1.15 and 1.16), the unique Nash equilibrium is $(u^*, v^*, z^*) = \left(\frac{1}{6}, \frac{3}{20}, \frac{11}{60}\right)$. Because of its uniqueness, we can safely regard it as the solution of Store Wars II. In this case, Nan should charge $\frac{4}{3}\pi ac$ dollars, Van should charge $\frac{6}{5}\pi ac$ dollars, and San should charge $\frac{22}{15}\pi ac$ dollars for the product in question.

The concepts of optimal reaction set and Nash equilibrium generalize readily to n-player continuous noncooperative games. Let the players correspond to the integers between 1 and n, and set

$$N = \{1, 2, \ldots, n\}. \tag{1.67}$$

Let Player k's strategy be denoted by w^k for all $k \in N$. Thus, for example, in Store Wars II we have $w^1 = u, w^2 = v$ and $w^3 = z$. Possibly w^k is a vector: for example, in Four Ways we have $w^1 = (u_1, u_2)$ and $w^2 = (v_1, v_2)$. Let $w = \left(w^1, w^2, \ldots, w^n\right)$ be the players' joint strategy combination. Note that w is a "vector of vectors"—if Player k controls s_k variables, i.e., w^k is an s_k-dimensional vector, then the dimension of w is $s_1 + s_1 + \cdots + s_n$. The n rewards can now be written succinctly as $f_1(w), f_2(w), \ldots, f_n(w)$.[29] Let $w||\overline{w}^k$ denote the joint strategy combination that is identical to w except for Player k's strategy, which is $\overline{w}^k$, i.e., define

$$w||\overline{w}^k = \left(w^1, \ldots, w^{k-1}, \overline{w}^k, w^{k+1}, \ldots, w^n\right). \tag{1.68}$$

Thus, in particular, $w||w^k = w$. Let the set of all feasible w—the decision set—be denoted as usual by D. Then, for $k \in N$, Player k's optimal reaction set is

$$R_k = \left\{w \in D \mid f_k(w) = \max_{\overline{w}^k} f_k(w||\overline{w}^k)\right\}. \tag{1.69}$$

If we define $w \backslash w^k$ to be that part of the joint strategy combination which is not under the control of Player k, i.e., if we define $w \backslash w^k = \left(w^1, \ldots, w^{k-1}, w^{k+1}, \ldots, w^n\right)$, then $w^* = \left((w^*)^1, (w^*)^2, \ldots, (w^*)^n\right)$ is a Nash equilibrium if, for all $k \in N$, Player k's $(w^*)^k$ is a best reply to the other players' $w^* \backslash (w^*)^k$—no player has an incentive to deviate from her Nash-equilibrium strategy if all other players adhere to theirs. In other words, w^* is a Nash equilibrium if, for any $\overline{w} \in D$,

$$f_k(w^*) \geq f_k(w^*||\overline{w}^k) \tag{1.70}$$

[29] The game is symmetric when $f_k(w)$ is independent of k (in the sense that the reward functions are identical except for cyclic permutation of their arguments).

where the vector x and the vector-valued function G are defined by

$$x(n) = (x_1(n), \dots, x_m(n)), \quad G = (G_1, \dots, G_m). \tag{2.117}$$

In the particular examples that appear in this book, the dynamics of (2.116) will turn out to be rather simple: as n increases, the vector $x(n)$ will progress towards an equilibrium vector $x(\infty) = x^*$ satisfying $x^* = G(x^*)$. Moreover, for given $x(0)$, it will be easy to generate the sequence $x(1), x(2), x(3), \dots$ by computer, recursively from (2.116). Nevertheless, one should still be aware that the dynamics of equations of type (2.116) are potentially very complicated, and a considerable variety of periodic and even chaotic behavior is possible; see, e.g., Chapter 15 of [**142**].

2.9. Continuously stable strategies

We already know from §2.8 that, whenever a discrete population game has more than one ESS, initial conditions determine which ESS the population adopts. In continuous population games, however, another issue arises. To see why, let us suppose that drivers in Crossroads II (§2.3) are not slow, and that a population of drivers is at the strong ESS defined by (2.31), i.e., the universally adopted critical lateness above which a driver initially goes is

$$v^* = 1 - \frac{\theta}{1+\lambda} = \frac{\delta + \frac{1}{2}(\eta - 1)\tau}{\delta + \frac{1}{2}\eta\tau + \epsilon} \tag{2.118}$$

with $\lambda < \theta < 1$, where δ, ϵ $(< \delta)$, η (< 1), and τ $(< 2\delta)$ denote impetuosity, ditheriness, discount factor, and junction transit time, respectively. Let us further suppose that there is a small but sudden shift in the value of any of these four parameters. For example, perhaps η rises slightly because everyone listens on their car radio to advice from a lifestyle guru. Then the population is no longer at its ESS; instead it remains at v defined by using the *old* parameter values in (2.118), whereas the ESS v^* requires the new values. What happens now? After this small perturbation, will the population of drivers converge on the new ESS? We first obtain a general answer for a two-player continuous population game, and then we apply it to Crossroads II.

Because $(v^*, v^*) \in R$, $f(v^*, v^*) = \max f(u, v^*)$ and hence

$$\left.\frac{\partial f}{\partial u}\right|_{u=v=v^*} = 0, \tag{2.119}$$

as long as f is sufficiently differentiable—which we assume, and which f defined by (2.27) clearly is.[33] We regard the current population strategy v as a small perturbation to v^*, and any fresh mutant u as an even smaller perturbation to v. We can therefore write

$$u = v + h, \qquad v = v^* + k, \tag{2.120}$$

[33] By contrast, in the cake-cutting game (Exercise 2.22), f is not differentiable, and so the concept of continuous stability could not apply. However, it would be irrelevant in any case, because the cake-cutting game has no parameters.

where h and k are both infinitesimally small, but with h much smaller than k. Let us recall Taylor's theorem, which states that

$$g(a+\xi_1, b+\xi_2) \;=\; g(a,b) + \xi_1 \frac{\partial g}{\partial u}\Big|_{\substack{u=a\\ v=b}} + \xi_2 \frac{\partial g}{\partial v}\Big|_{\substack{u=a\\ v=b}} + \mathcal{O}(\xi^2), \tag{2.121}$$

where $\xi = \max(|\xi_1|, |\xi_2|)$ and g is any sufficiently differentiable function of u and v, implying in particular that $\frac{\partial g}{\partial u}$ is another function to which Taylor's theorem can in turn be applied, and where $\mathcal{O}(\chi)$ is "order notation"—specifically, "big oh"—denoting terms so small that you can divide them by χ and the result still remains bounded as $\chi \to 0$. From (2.121) with $g = \frac{\partial f}{\partial u}$, $a = b = v^*$, and $\xi_1 = \xi_2 = k$, we obtain

$$\begin{aligned} \frac{\partial f}{\partial u}\Big|_{\substack{u=v=\\ v^*+k}} &= \frac{\partial f}{\partial u}\Big|_{\substack{u=v^*\\ v=v^*}} + k\,\frac{\partial^2 f}{\partial u^2}\Big|_{\substack{u=v^*\\ v=v^*}} + k\,\frac{\partial^2 f}{\partial u \partial v}\Big|_{\substack{u=v^*\\ v=v^*}} + \mathcal{O}(k^2) \\ &= k\Big(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial u \partial v}\Big)\Big|_{u=v=v^*} + \mathcal{O}(k^2) \end{aligned}$$

by (2.119). So, from (2.120) and (2.121) with $g = f$, $a = b = v^* + k$, $\xi_1 = h$, and $\xi_2 = 0$, we find that playing mutant strategy $u = v + h$ against the current population strategy $v = v^* + k$ yields reward

$$\begin{aligned} f(u,v) &= f(v+h, v^*+k) \\ &= f(v^*+k+h, v^*+k) \\ &= f(v^*+k, v^*+k) \;+\; h\,\frac{\partial f}{\partial u}\Big|_{u=v=v^*+k} \;+\; \mathcal{O}(h^2) \\ &= f(v,v) \;+\; hk\Big(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial u \partial v}\Big)\Big|_{u=v=v^*} \end{aligned} \tag{2.122}$$

plus terms of the form $h\mathcal{O}(k^2) + \mathcal{O}(h^2)$, which are negligible compared to those above because h is so small compared to k. For mutant strategies to move the population from its current strategy $v = v^* + k$ to the ESS v^*, those with $h > 0$ must be favored when $k < 0$ (to increase a value that is too low) and those with $h < 0$ must be favored when $k > 0$ (to decrease a value that is too high); in other words, we require $f(u,v)$ to exceed $f(v,v)$ when $hk < 0$. So a non-v^* population converges to a v^* population when

$$\Big(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial u \partial v}\Big)\Big|_{u=v=v^*} \;<\; 0. \tag{2.123}$$

If (2.123) holds, then the ESS v^* is said to be either *continuously stable* [**91**] or *convergence stable* [**65**].[34] Thus, in particular, when $\delta > \frac{1}{2}\tau$, Crossroads II has a continuously stable strong ESS defined by (2.118) because (2.27) implies that

$$\Big(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial u \partial v}\Big)\Big|_{u=v=v^*} \;=\; -(\delta+\epsilon)(1+\lambda) \;<\; 0, \tag{2.124}$$

and so (2.123) is satisfied.

The significance of continuous stability becomes more apparent in terms of the *invasion fitness*

$$i(u,v) \;=\; f(u,v) - f(v,v), \tag{2.125}$$

[34] Or even m-stable for "mutant-stable" [**333**]. But even Taylor [**333**, p. 141] was not happy with this terminology, and anyhow it didn't catch on.

that is, the relative fitness advantage of a mutant u-strategist in a population of v-strategists. If v is a strong ESS, then the invasion fitness of every mutant is negative by (2.14), so that no mutant can invade (which is why the dashed line in Figure 2.9 must lie in the shaded region). For other v, however, u can invade if, and only if, $i(u, v) > 0$. Correspondingly, the zero contour of the function i partitions the decision set into four distinct regions with different signs for $i(u, v)$. In the case of Crossroads II, it follows from (2.27) that

$$i(u, v) = \tfrac{1}{2}(\delta + \epsilon)(u - v)\{2(1 - \theta + \lambda) - \lambda u - (2 + \lambda)v\}, \tag{2.126}$$

and so the zero contour consists of the line $u = v$ from $(0, 0)$ to $(1, 1)$ and the line $\lambda u+(2+\lambda)v = 2(1-\theta+\lambda)$ from $(0, \omega_2)$ to $(1, \omega_3)$ in Figure 2.9. If $v < v^*$ (or $k < 0$), then $i(u, v) > 0$ only for $u > v$ (or $h > 0$), represented by moving from (v, v) along the dotted line in Figure 2.9(b) to (u, v); because the mutant strategy has higher fitness in the unshaded region, its frequency will keep increasing until it reaches fixation, represented by moving from (u, v) to $(u, u) = (v + h, v + h)$ along the dashed line. Because h is so small, however, the population has effectively moved upward along the line $u = v$ from (v, v) to (u, u), as indicated by the arrow: u has become the new v. Any further mutation repeats this process, until eventually (v, v) converges to (v^*, v^*), as indicated in Figure 2.9(a). If instead $v > v^*$ (or $k > 0$), then $i(u, v) > 0$ only for $u < v$ (or $h < 0$) but the process is the same, except that convergence is downward, as shown in Figures 2.9(c) and 2.9(a). The more fitness changes with u, the more rapidly v approaches the ESS. So it is reasonable to assume that $\frac{dv}{dt}$ is proportional to $\partial i/\partial u|_{u=v}$ (or $\partial f/\partial u|_{u=v}$, which is the same thing), where t denotes time and the constant of proportionality is set to 1 without loss of generality.[35] Thus, after perturbation from the old ESS, v evolves to v^* according to

$$\frac{dv}{dt} = \frac{\partial i}{\partial u}\bigg|_{u=v} = (\delta + \epsilon)(1 + \lambda)(v^* - v), \tag{2.127}$$

by (2.126) and (2.118), with $\frac{dv}{dt} > 0$ if $v < v^*$ and $\frac{dv}{dt} < 0$ if $v > v^*$. Note that R lies entirely within the unshaded region of Figure 2.9: v can be invaded by the optimal reply to v unless v is an ESS.

The derivation of (2.123) assumed that $v^* \in (0, 1)$ or that v^* is an *interior* ESS—which holds for Crossroad II if $\delta > \frac{1}{2}\tau$, as we assumed. Indeed drivers need not be quite that fast: v^* continues to be an interior ESS as long as $\delta > \frac{1}{2}(1 - \eta)\tau$. But the result fails to hold when $\delta < \frac{1}{2}(1 - \eta)\tau$, because then $\theta > 1 + \lambda$ and $\partial f/\partial u$ is invariably negative, by (2.27). So the unique best reply to any v is $u = 0$, making $v^* = 0$ a strong ESS—but a *boundary* ESS, as opposed to an interior one. What happens now to the concept of continuous stability?

For Crossroads II, the only possible boundary ESS is $v^* = 0$; however, to make our analysis apply more broadly to any population game with strategy set $[0, 1]$, we denote a boundary ESS by b (where b is either 0 or 1). Again we regard the current population strategy v as a small perturbation to the ESS b, and any fresh mutant u as an even smaller perturbation to v. We can therefore write

$$u = v + h, \qquad v = b + k, \tag{2.128}$$

[35] See Footnote 30 on p. 93.

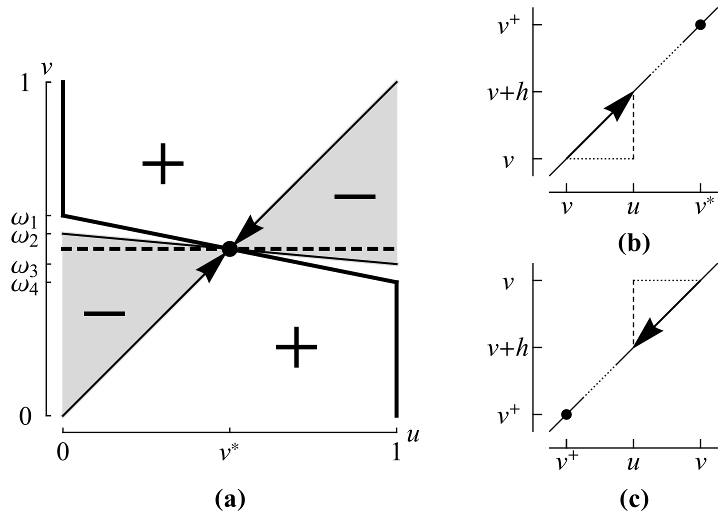

Figure 2.9. **(a)** The zero contour of invasion fitness i (thin solid lines), the optimal reaction set (thick solid) and the line $v = v^*$ (dashed) for Crossroads II when drivers are not slow, with v^* defined by (2.118); $i < 0$ in the shaded region, $i > 0$ in the unshaded region, $\omega_1 = 1-\theta+\lambda$, $\omega_2 = 1-\{2\theta-\lambda\}/\{2+\lambda\}$, $\omega_3 = 1-2\theta/\{2+\lambda\}$, and $\omega_4 = 11-\theta$. The figure is drawn for $\delta = 3, \epsilon = 2, \tau = 2$, and $\eta = 1$ (but would have the same topology for any $\delta > \tau/2$). **(b)** Convergence to v^* from $v < v^*$. **(c)** Convergence to v^* from $v > v^*$.

where h and k are both infinitesimally small, but in such a way that h is much smaller than k. From Taylor's theorem applied to f, we find that playing mutant strategy $u = v + h$ against the current population strategy $v = b + k$ yields reward

$$(2.129)\quad \begin{aligned} f(u,v) &= f(v+h, b+k) = f(b+k+h, b+k) \\ &= f(b+k, b+k) + h\,\tfrac{\partial f}{\partial u}\big|_{u=v=b+k} + \mathcal{O}(h^2). \end{aligned}$$

We assume that k and $h+k$ are positive if $b = 0$ but negative if $b = 1$, to ensure that $u \in [0,1]$. From Taylor's theorem applied to $\frac{\partial f}{\partial u}$ (as opposed to f) we obtain

$$(2.130)\quad \frac{\partial f}{\partial u}\bigg|_{\substack{u=v=\\b+k}} = \frac{\partial f}{\partial u}\bigg|_{\substack{u=b\\v=b}} + k\frac{\partial^2 f}{\partial u^2}\bigg|_{\substack{u=b\\v=b}} + k\frac{\partial^2 f}{\partial u \partial v}\bigg|_{\substack{u=b\\v=b}} + \mathcal{O}(k^2).$$

Now, if $b = 0$ is an ESS because $f(u,0)$ achieves a maximum at $u = 0$ with $\frac{\partial f}{\partial u}\big|_{u=v=0} < 0$ or if $b = 1$ is an ESS because $f(u,1)$ achieves a maximum at $u = 1$ with $\frac{\partial f}{\partial u}\big|_{u=v=1} > 0$, then (2.130) reduces to $\frac{\partial f}{\partial u}\big|_{u=v=b+k} = \frac{\partial f}{\partial u}\big|_{u=v=b} + O(k)$. Thus (2.129) reduces to

$$(2.131)\quad f(u,v) = f(b+k, b+k) + h\,\tfrac{\partial f}{\partial u}\big|_{u=v=b}$$

plus terms of the form $h\mathcal{O}(k) + \mathcal{O}(h^2)$, which are negligible compared to those above because h and k are so small. From (2.128) and (2.131) we see that $f(u,v) > f(v,v)$ for $h < 0$ if $b = 0$ but for $h > 0$ if $b = 1$. Mutant strategies that move the population from its current strategy v towards the boundary—and hence the ESS—are favored in either case; whereas mutant strategies that would move the population away from the boundary are disfavored, and hence disappear. So a non-b population always converges to a b population.[36] In sum, continuous stability is a nonissue for a boundary ESS but a very desirable property for an interior ESS. And when a continuous population game has more than one interior ESS, continuous stability yields a criterion for distinguishing among them.[37]

2.10. State-dependent dynamic games

We have allowed individuals to condition their behavior on role (e.g., owner or intruder, §2.4) or on state (e.g., lateness, §2.3), but not on time. Here we relax that restriction to touch on a discrete-time approach to dynamic games developed by Houston and McNamara [**148**], whose notation we largely adopt. This approach in turn is based on state-dependent dynamic games against nature,[38] in which a focal individual is the only decision maker, and which we consider first. We start with some general terms and notation.

Accordingly, let n-dimensional vector $x = (x_1, \ldots, x_n)$ represent an animal's state at time t; frequently $n = 1$, in which case, x and x_1 coincide. For example, an animal's state could be its energy reserves, as in the example that follows (p. 101). Correspondingly, let s-dimensional vector $u = (u_1(x,t), \ldots, u_s(x,t))$ denote a focal individual's strategy, which specifies the action to be taken when in state x at time t; again, often $s = 1$, in which case, u and u_1 coincide. Thus a strategy specifies a sequence $\{u(x,0), u(x,1), u(x,2), u(x,3), \ldots\}$ of state-dependent actions for $t = 0$ onwards, and in practice no difficulty arises from using the same symbol u for both the action $u(x,t)$ that is specified to be taken when in state x at time t and the complete tabulation of such actions, that is, the strategy itself.[39] Let $V(x,t)$ denote the fitness—expected future reproductive success—of an animal in state x at time t, predicated on the assumption that the animal behaves optimally at time t and *at all later times*. Finally, let $H(x,t,u)$ denote the (expected) reward to a u-strategist in state x at time t (not assuming future optimal behavior).

There are three ways in which the strategy u can influence this fitness. First, from the action u taken at time t, there may be a direct contribution to fitness through birth of offspring,[40] which we denote by $B(x,t,u)$. Second, the action taken can influence the probability of survival to time $t+1$, which we denote by $S(x,t,u)$. Third, the action taken can influence the state in which the animal finds itself at time $t+1$. We denote this new state—which in general is a random variable—by X', using the prime to denote a later time.[41] It now follows that

$$H(x,t,u) \;=\; B(x,t,u) + S(x,t,u)\mathrm{E}_u\big[V(X',t+1)\big], \tag{2.132}$$

where E_u denotes expected value over all possible states resulting from action u being taken at time t, conditional on survival to time $t+1$. Let $u^*(x,t)$ denote the optimal action when in state x at time t, i.e., the action specified for state x

[36] Assuming that $\frac{\partial f}{\partial u}\big|_{u=v=b} \neq 0$; if $\frac{\partial f}{\partial u}\big|_{u=v=b} = 0$, then we proceed as for an interior ESS.

[37] For an illustration of this point, see §8.3 (p. 352).

[38] See Footnote 9 on p. 5.

[39] Put differently, no difficulty arises from using the same symbol for both the image of $u : \Re^{n+1} \to \Re^s$ and the function itself, where $\Re$ denotes the real numbers.

[40] More precisely, offspring that are not still dependent on their parents at time $t+1$ [**148**, p. 13].

[41] In §2.10 only; elsewhere in the book, a prime denotes differentiation with respect to argument.

Now, the number of genes that our animal transmits to the second generation is proportional to its number of surviving grandchildren, and hence to

$$F \;=\; \sigma_f(1-u)C\cdot C + \sigma_m uC\cdot CM, \tag{6.4}$$

because $\sigma_f(1-u)C$ of its daughters and $\sigma_m uC$ of its sons survive to maturity, and because daughters all produce C offspring, whereas sons produce C offspring per mating.[3] The expected value of this payoff is $\mathrm{E}[F] = \sigma_f(1-u)C^2 + \sigma_m uC^2\mathrm{E}[M]$. Thus, on setting $\mathrm{E}[M] = d/s$ and using (6.3), our animal's reward is

$$f(u,v) \;=\; C^2\sigma_f\{v + (1-2v)u\}/v, \tag{6.5}$$

provided of course that $v \neq 0$.

We have glossed over some details in our eagerness to obtain the above reward. We have implicitly assumed that the population outbreeds—no individual mates with a brother or sister—and that all the males suffice to mate with all the females. Both assumptions are of negligible consequence if $v \neq 0$ and $N \to \infty$. If $v = 0$, however, then there are no males in the population to mate with our animal's daughters, and so it can have grandchildren only through its $\sigma_m Cu$ surviving sons when they mate with the $N\sigma_f C$ surviving females in the population, so that $d/s = N\sigma_f \div \sigma_m u$ in place of (6.3), $F = \sigma_m uC\cdot CM$ in place of (6.4), and thus $f(u,0) = NC^2\sigma_f$ when $u > 0$. But $f(0,0) = 0$: when $v = 0$, no son means no grandchildren! So the best response to $v = 0$ is any positive u—in theory. In practice, however, it is unlikely that one or two sons could mate with the entire female population, so the best response to $v = 0$ is surely $u = 1$.

Either way, it is now easy to calculate the optimal reaction set R and show that $v = \frac{1}{2}$ is a continuously stable, weak ESS (Exercise 6.1). Thus, in the highly idealized circumstances described, the population should evolve to produce equal numbers of sons and daughters—irrespective of the proportions of sons and daughters that survive to maturity.

6.2. Damselfly duels: a war of attrition

The study of the fitness consequences of behavior has come to be known as behavioral ecology.[4] Research in this field poses the basic question, What does an animal gain, in fitness terms, by doing *this* rather than *that*? [**28**, p. 9] So behavioral ecology thrives on paradoxes—baffling inconsistencies between intuition and evidence that engage our attention and stimulate further investigation. A paradox arises because evidence fails to support an intuition, which (assuming the evidence to be sound) can happen only if the intuition relies on a false assumption about behavior, albeit an implicit one. So the way to resolve the paradox is to spot the false assumption. In other words, if a paradox of animal behavior exists, then we have wrongly guessed which game best models how a real population interacts, and to resolve this paradox we must guess again—if necessary, repeatedly—until eventually we guess correctly. Assuming the validity of our solution concept for

[3] If the sons and daughters mated, then the daughter's genes would be counted by the first term in (6.4) and the son's genes by the second; see Exercise 6.28. Here, however, we simply assume that the population outbreeds.

[4] For an introduction to the subject at large, see [**42**, **81**].

population games, i.e., assuming that observed behavior corresponds to some ESS, our task is to construct a game whose ESS corresponds to the observed behavior. Then the resolution of the paradox lies in the difference between the assumptions of this new model and the assumptions we had previously been making about the observed behavior (whether we realized it or not). Of course, a model population is only a caricature of a real population. But a paradox is only a caricature of real ignorance. So, in terms of realism, a game and a paradox are a perfect match.

In this regard, our next example will illustrate how a population game can help to resolve a paradox. The game is a simple model of nonaggressive contest behavior, in which animals vie for an indivisible resource by displaying only—unlike in the Hawk-Dove game of §2.4, where the animals sometimes fight. The cost of displaying increases with time, and the winner is the individual whose opponent stops displaying first. So the game is a war of attrition.

A common expectation for such contests, confirmed by experimental studies on a variety of animals, is that each animal compares its own strength to that of its opponent and withdraws when it judges itself to be the probable loser.[5] We call this expectation the mutual-assessment hypothesis. The duration of such contests is greatest when opponents are of nearly equal fighting ability, so that it is more difficult to judge who is stronger and therefore the likely winner. But a series of contests over mating territories between male damselflies, staged by Marden and Waage [**179**], failed to follow this logic. Although the weaker animal ultimately conceded to its opponent in more than 90% of encounters, there was no significant negative correlation between contest durations and differences in strength. Why? We explore this question in terms of game theory.

A possible answer is that the damselflies were not assessing one another's strength. Animals who contest indivisible resources will vary in reserves of energy and other factors, but an animal's state need not be observable to its opponent, and so we will assume that an animal has information only about its own condition. Let us also assume that variation in reserves is continuous, and that an animal's state is represented by the maximum time it could possibly display before it would have to cede the resource for want of energy. Let this time be denoted by T_{max}, and let T be the time for which the animal has already displayed. Then the larger the value of $T_{\text{max}} - T$, the more likely it is that continuing to display will gain the animal the resource. Thus the animal should persist if its perception of $T_{\text{max}} - T$ is sufficiently large. We will refer to $T_{\text{max}} - T$ as the animal's current reserves and to T_{max} as its initial reserves.[6]

Our war-of-attrition model requires a result from psychophysics. Let π (> 0) denote the intensity of the stimulus of some physical magnitude, e.g., size or time, and let $\rho(\pi)$ be an animal's subjective perception of π. Then, in general, $\rho'(\pi) > 0$ and $\rho''(\pi) < 0$; that is, perception increases with intensity of stimulus, but the greater that intensity, the greater the increment necessary for perception. Now, for all sensory modalities in humans, the ratio between a stimulus and the increment required to make a difference just noticeable is approximately constant over the usable (middle) range of intensities. Therefore, to the extent that human

[5] See, e.g., [**225**, p. 66].

[6] An implicit assumption here is that energy is proportional to time [**225**, p. 72].

psychophysics also applies to other animals,[7] it is reasonable to assume that if a is the least observable intensity and π is increased steadily beyond a, then a first difference will be noticed when $\pi = a(1+b)$, a second when $\pi = a(1+b)^2$, a third when $\pi = a(1+b)^3$, and so on, where b is the relevant constant. If it is further assumed that these just noticeable differences are all equal—to c, say—and if zero on the subjective scale corresponds to a on the objective scale, then an animal's subjective perception of the stimulus $\pi = a(1+b)^k$ is $\rho = kc$. Thus $\pi = a(1+b)^{\rho/c}$ or

$$\rho(\pi) \;=\; \gamma \ln(\pi/a)\,, \tag{6.6}$$

where $\gamma = c/\ln(1+b)$ is a constant. In psychophysics, (6.6) is usually known as Fechner's law (but occasionally as the relativity principle), and it clearly satisfies $\rho'(\pi) > 0$, $\rho''(\pi) < 0$. We will assume throughout that it provides an adequate model of the relationship between sensory and physical magnitudes.

Now, let H and L be physical magnitudes whose difference determines the probability that a favorable outcome will be achieved if a certain action is taken. Then an animal should take the action if it perceives H to be sufficiently large compared to L. If it perceives each magnitude separately,[8] then it should take the action if $\rho(H)-\rho(L)$ is sufficiently large. But (6.6) implies that $\rho(H)-\rho(L) = \gamma\ln(H/L)$. So the action should be taken if H/L is sufficiently large. Thus, on setting $H = T_{\max}$ and $L = T$ above, an animal should persist if $T_{\max}/T$ is sufficiently large—bigger than, say, $1/w$—but otherwise give up. Accordingly, we define strategy w to mean

$$\begin{aligned} &\textit{stay} \quad \text{if} \quad T < wT_{\max}, \\ &\textit{go} \quad \text{if} \quad T \geq wT_{\max}. \end{aligned} \tag{6.7}$$

We interpret w as the proportion of an animal's initial reserves that it is prepared to expend on a contest. Thus, if the initial reserves of a u-strategist and a v-strategist are denoted by X and Y, respectively, where X and Y are both drawn randomly from the distribution of $T_{\max}$, then the u-strategist will depart after time uX, and the v-strategist will depart after time vY.

We assume that the value of the contested resource is an increasing function of current reserves, a reasonable assumption when the resource is a mating territory (although the assumption would clearly be violated—effectively reversed—in a contest for food). For simplicity, we assume that the increase is linear, i.e., the value of the resource is $\alpha(T_{\max} - T)$ where $\alpha > 0$. The parameter α has the dimensions of fitness (p. 71) per unit of time, and so we can think of α as the rate at which the victor is able to translate its remaining reserves into future offspring. Let β (> 0) denote the cost per unit time of persisting, in the same units. Then, because X and Y are (independent) random variables, the payoff to a u-strategist against a

[7] See, for example, [**296**, pp. 15–16]. Although humans are still the subject of most empirical work, human psychophysics does seem to apply to many other animal taxa—including invertebrates [**5**].

[8] As argued by [**225**, p. 75]. If the animal actually perceives the difference, of course, then it should instead take the action if $\rho(H-L)$ is sufficiently large, and hence if $H-L$ is sufficiently large, but results are qualitatively the same [**225**].

v-strategist is also a random variable, say F defined by

$$F(X,Y) \;=\; \begin{cases} \alpha(X - vY) - \beta vY & \text{if } uX > vY, \\ -\beta uX & \text{if } uX < vY, \\ 0 & \text{if } u = 0 = v, \end{cases} \tag{6.8}$$

because the u-strategist wins if $uX > vY$ but loses if $uX < vY$, and the cost of display is determined by the loser's persistence time for both contestants. We ignore the possibility that $uX = vY$ (other than where $u = 0 = v$) because reserves are continuously distributed, and so it occurs with probability zero. We also assume the resource is sufficiently valuable that $\alpha > \beta$ or $\theta < 1$, where

$$\theta \;=\; \beta/\alpha. \tag{6.9}$$

Let us now assume that initial reserves $T_{\max}$ are distributed over $(0, \infty)$ with probability density function g. Then the reward to a u-strategist is $f(u,v) = \mathrm{E}[F] = \int_0^\infty \int_0^\infty F(x,y)dA$, where we have used (2.22) and E denotes expected value. On using (6.8), we obtain

$$\begin{aligned} f(u,v) \;=\; & \int_0^\infty g(x) \left\{ \int_0^{ux/v} \{\alpha x - (\alpha+\beta)vy\} g(y)\, dy \right\} dx \\ & - \beta u \int_0^\infty g(y) \left\{ \int_0^{vy/u} x g(x)\, dx \right\} dy \end{aligned} \tag{6.10}$$

if $(u,v) \neq (0,0)$ but $f(0,0) = 0$. For example, if $T_{\max}$ is uniformly distributed with mean μ according to

$$g(t) \;=\; \begin{cases} \frac{1}{2\mu} & \text{if } 0 < t < 2\mu, \\ 0 & \text{if } 2\mu < t < \infty, \end{cases} \tag{6.11}$$

then (Exercise 6.2)

$$f(u,v) = \begin{cases} \frac{\alpha\mu u}{3v}\{2 - 3\theta v - (1-\theta)u\} & \text{if } u \le v, v \neq 0, \\ \frac{\alpha\mu\{3(1-\{1+\theta\}v)u^2 + (\{2+\theta\}u - 1)v^2\}}{3u^2} & \text{if } \quad u > v, \\ 0 & \text{if } \; u = 0 = v, \end{cases} \tag{6.12}$$

and R is defined by

$$u \;=\; \mathcal{B}(v) \;=\; \begin{cases} \frac{2-3\theta v}{2-2\theta} & \text{if } u \le v, v \neq 0, \\ u = \frac{2}{2+\theta} & \text{if } \quad u > v, \\ \text{any } u \in (0,1] & \text{if } \quad v = 0. \end{cases} \tag{6.13}$$

Note that $(0,0) \notin R$. Thus the unique ESS is $v = v^*$ where

$$v^* \;=\; \frac{2}{2+\theta}, \tag{6.14}$$

as illustrated by Figure 6.1. At this ESS, each animal is prepared to expend at least two thirds of its initial reserves, although only the loser actually does so. Moreover, because each animal is prepared to persist for time $v^* T_{\max}$ at the ESS, the victor is always the animal with the higher value of $T_{\max}$—even though an opponent's reserves are assumed to be unobservable. Thus victory by the stronger animal need not imply that an opponent's reserves are being assessed. But animals still

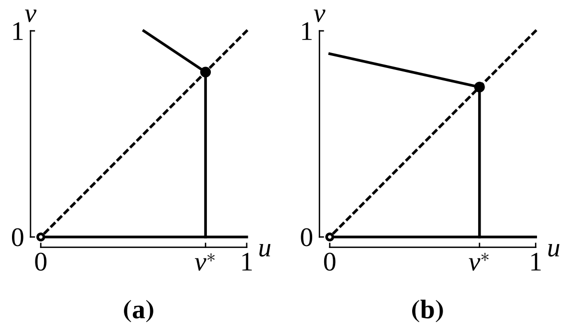

Figure 6.1. Optimal reaction set for the war of attrition with uniform distribution of initial reserves when **(a)** $0 < \theta < \frac{2}{3}$ and **(b)** $\frac{2}{3} < \theta < 1$, where θ is defined by (6.9) and v^* by (6.14). R is drawn for **(a)** $\theta = 1/2$ and **(b)** $\theta = 3/4$.

assess their own reserves (and respond to the distribution of reserves among the population). So we refer to the expectation that an opponent's reserves are not being assessed as the self-assessment hypothesis.

As noted on p. 72, an effect that is small in a real population is typically absent from a model. So we would expect a 90% win rate for stronger males in the real world to translate into a 100% win rate for stronger males in a model world. And this is precisely what we have predicted: because both contestants are prepared to deplete their initial reserves by the same proportion, the weaker one invariably gives up first. But it is also what the mutual-assessment hypothesis predicts, at least in the absence of assessment errors, because the weaker animal withdraws as soon as it has judged itself to be the probable loser. So are the two competing hypotheses indistinguishable? We return to this question below (p. 223).

Although the uniform distribution is not the only one for which an ESS always exists (Exercise 6.3), for many distributions an ESS exists only if θ is sufficiently small. For example, suppose that $T_{\max}$ is distributed parabolically with mean μ according to

(6.15)
$$g(t) = \begin{cases} \frac{3t(2\mu-t)}{4\mu^3} & \text{if } 0 < t < 2\mu, \\ 0 & \text{if } 2\mu < t < \infty. \end{cases}$$

Then, on interchanging the order of integration in the second term of (6.10) and substituting from (6.15), we have

(6.16a)
$$\begin{aligned} f(u,v) &= \int_0^{2\mu} g(x) \left\{ \int_0^{ux/v} \{\alpha x - (\alpha+\beta)vy\} g(y)\,dy \right\} dx \\ &\quad - \beta u \int_0^{2\mu} x g(x) \left\{ \int_{ux/v}^{2\mu} g(y)\,dy \right\} dx \\ &= \alpha\mu u \left\{ \frac{u(5u\{(3-\theta)u-4\}+14v\{3-(2-\theta)u\})}{35v^3} - \theta \right\} \end{aligned}$$

for $u \le v, v \ne 0$; whereas, on interchanging the order of integration in the first term of (6.10) before substituting from (6.15), we have

$$\begin{aligned} f(u,v) &= \int_0^{2\mu} g(y)\left\{\int_{vy/u}^{2\mu}\{\alpha x-(\alpha+\beta)vy\}g(x)\,dx\right\}dy \\ &\quad -\beta u\int_0^{2\mu} g(y)\left\{\int_0^{vy/u} xg(x)\,dx\right\}dy \\ &= \alpha\mu\left\{1-(1+\theta)v+\frac{(14u\{(3+\theta)uv-2v\}+5v^2\{3-(4+\theta)u\})v^2}{35u^4}\right\} \end{aligned} \tag{6.16b}$$

for $u \ge v$. Differentiating, we obtain

$$\left.\frac{\partial f}{\partial u}\right|_{u=v=v^*} = \frac{(24+13\theta)(v^*-v)}{35v}\alpha\mu, \tag{6.17}$$

where

$$v^* = \frac{24}{24+13\theta} \tag{6.18}$$

(Exercise 6.4), and it follows from (2.47)–(2.49) that v^* is the only candidate for ESS. To show that v^* is indeed an ESS, we must verify (2.17). For $u > v^*$, it follows from (6.16b) that

$$f(v^*,v^*)-f(u,v^*) = \tfrac{1}{35}\,\alpha\mu\,Q_+(u)\left\{\frac{u-v^*}{(24+13\theta)u^2}\right\}^2, \tag{6.19}$$

where $Q_+(u) = (216+47\theta)u\{48+(24+13\theta)u\}-8640$ is strictly increasing with respect to u, and so, for $u > v^*$, $Q_+(u) > Q_+(v^*) = 10368(16-3\theta)/(24+13\theta) > 0$. Hence $f(v^*,v^*) > f(u,v^*)$ for all $u > v^*$. For $u < v^*$, it follows from (6.16a) that

$$f(v^*,v^*)-f(u,v^*) = \tfrac{1}{483840}\,\alpha\mu\,(24+13\theta)Q_-(u)(v^*-u)^2, \tag{6.20}$$

where $Q_-(u) = 48(108-169\theta)+4(108+41\theta)(24+13\theta)u-5(3-\theta)(24+13\theta)^2u^2$. Because $Q_-''(u) < 0$ and $Q_-(v^*)-Q_-(0) = 96(18+71\theta) > 0$, the minimum of Q_- on $[0,v^*]$ occurs at $u=0$. Hence $f(v^*,v^*)-f(u,v^*)$ is guaranteed to be positive for all $u < v^*$ only if $Q_-(0) > 0$, or $\theta < \frac{108}{169}$. Then $f(v^*,v^*) > f(u,v^*)$ for all $u \ne v^*$, implying that v^* is a strong ESS. If $\theta = \frac{108}{169}$, then it can be shown directly that v^* is a weak ESS (Exercise 6.4). Thus v^* is an ESS if (and only if)

$$\theta \le \tfrac{108}{169}. \tag{6.21}$$

When $\theta > \frac{108}{169}$, however, a population of v^*-strategists can be invaded by mutants who play $u=0$, i.e., give up immediately. These results are illustrated by Figure 6.2, where R is sketched.

Although giving up immediately can invade strategy v^* if $\theta > \frac{108}{169}$, it is advantageous only when rare, and so it does not eliminate v^*. Suppose that giving up immediately (strategy 1) and expending proportion v^* (strategy 2) are found at frequencies x_1 and x_2, respectively. Then the reward to strategy 1, which entails neither benefits nor costs, is $W_1 = 0$, and for W_2 (the reward to strategy 2) is $f(v^*,0) = \alpha\mu$ times the probability of meeting strategy 1 plus $f(v^*,v^*)$ times the

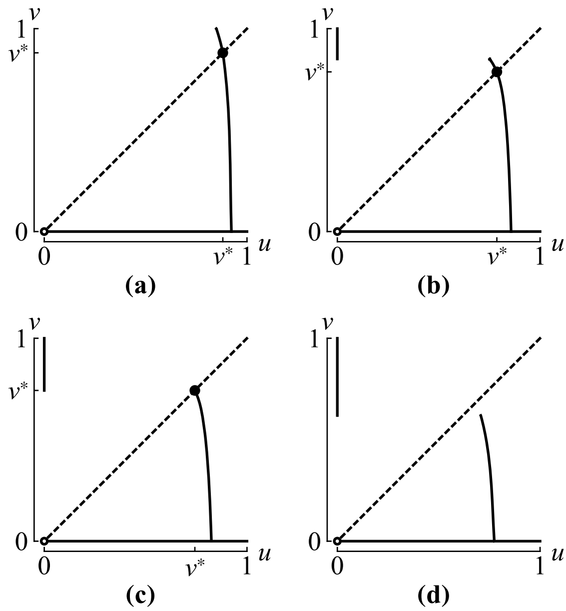

Figure 6.2. Optimal reaction set (solid curve) for the war of attrition with parabolic distribution of initial reserves when **(a)** $\theta = \frac{1}{4}$, **(b)** $\theta = \frac{1}{2}$, **(c)** $\theta = \frac{108}{169}$, and **(d)** $\theta = \frac{7}{8}$, where θ is defined by (6.9) and v^* by (6.18).

probability of meeting strategy 2. So

$$W_1 - W_2 = -\alpha\mu x_1 - f(v^*, v^*)x_2 = -f(v^*, v^*) - \{\alpha\mu - f(v^*, v^*)\}x_1, \tag{6.22}$$

where $f(v^*, v^*) = -2(169\theta - 108)\alpha\mu/(455\theta + 840) < 0$. For small x_1, $W_1 - W_2$ is positive, so that strategy 1 increases in frequency; for large x_1, $W_1 - W_2$ is negative, so that x_1 decreases. The population stabilizes where $W_1 = W_2$ or

$$x_1 = \frac{2(169\theta - 108)}{13(48 + 61\theta)} < \frac{122}{1417}. \tag{6.23}$$

In other words, although there is no monomorphic ESS when $\theta > \frac{108}{169}$, there still exists a polymorphic ESS at which the proportion of those who give up immediately is invariably less than 9%. The polymorphism persists because negative payoffs to strategy 2 on meeting itself are balanced by large positive payoffs on rarer occasions when it meets strategy 1. Thus the alternative strategies do equally well on average, and there is no incentive to switch from one to the other.

Neither a uniform nor a parabolic distribution of initial reserves is especially realistic. But the results of this section generalize to other distributions (Exercise 6.5), enabling one to find an acceptable fit to the data on damselfly energy reserves collected by Marden, Waage and later Rollins [**179**], [**178**]. Now, we discovered earlier that, with respect to victory by the stronger contestant, these data are consistent with both the mutual-assessment and the self-assessment hypotheses. But there is also a difference. In our model of pure self-assessment, which has acquired the acronym WOA-WA,[9] a contest ends when the loser gives up after using a fixed proportion of its reserves, and so we predict a positive correlation between final loser reserves and contest duration; whereas, as discussed earlier, the mutual-assessment hypothesis predicts a negative correlation between strength difference and contest duration. This difference[10] demonstrates that game-theoretic models are capable of yielding testable predictions.

Although the WOA-WA was originally developed to address a damselfly paradox, the model applies to other species. Empirical support for self-assessment has since emerged from staged contests in dragon lizards [**196**] and, to various degrees, in wasps [**339**], pigs [**55**], sheet-web spiders [**352**], and cave-dwelling wetas [**95**].

6.3. Games among kin versus games between kin

In §6.2 we found that the war of attrition need have no monomorphic ESS if the cost of display is sufficiently high.[11] We assumed, however, that contestants are unrelated. Here we study how nonzero relatedness modifies the conditions for a strategy to be an ESS with particular reference to the war of attrition. For simplicity, we assume that the strategy set is $[0, 1]$, as in §6.1 and §6.2.

According to Darwin, animals will behave so as to transmit as many as possible of their genes to posterity. By descent, any two blood relations share a nonnegligible proportion of genes for which there is variation in the population at large, and hence have a tendency to exhibit the same behavior—or to have the same strategy. But animals may also behave identically because of cultural association. Accordingly, let r be the probability that a strategy encounters itself by virtue of kinship, where kinship can be interpreted to mean either blood-relationship or similarity of character (as in common parlance); then $1 - r$ is the probability that the strategy encounters an opponent at random (still possibly itself). We call r the *relatedness*, and assume that $r < 1$. Nothing in our analysis will depend on whether animals tend to behave identically by virtue of shared descent or shared culture—except, as remarked in §2.1 (p. 62), the time scale of the dynamic by which an ESS can be reached.

Let the population contain proportion $1 - \epsilon$ of an orthodox strategy v and proportion ϵ of a mutant strategy u, and let $f(u, v)$ denote, as usual, the reward to a u-strategist against a v-strategist. Then, because u and v are encountered with probabilities ϵ and $1 - \epsilon$, respectively, the reward to strategy s against a random opponent is

$$w(s) = \epsilon f(s, u) + (1 - \epsilon) f(s, v), \tag{6.24}$$

[9] For war of attrition without assessment, i.e., without mutual assessment; see, e.g., [**9**, **335**].

[10] It has been explored elsewhere [**211**, **225**], although with inconclusive results. A difficulty is that in the absence of mutual assessment, an apparent negative relationship between strength difference and contest duration could arise as an artefact of a stronger positive relationship between loser reserves—or, more generally, loser RHP (p. 165)—and contest duration [**335**]. The matter is discussed at length by Briffa et al. [**46**, pp. 63–65].

[11] And variation in reserves is sufficiently low; see [**211**, **225**].

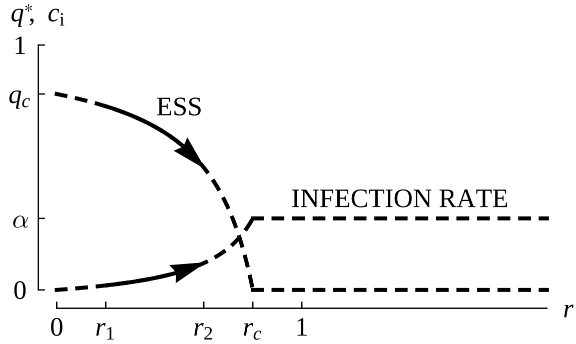

Figure 6.21. The ESS proportions of Type I parents and infected children for $\mathcal{R}_0 > 1$

vaccination—whether (6.134) or zero—will always be lower than the critical level $q_c = 1 - \mathcal{R}_0^{-1}$ needed to eradicate the disease. From (6.127) and (6.130) with $q = q^*$, the corresponding proportion of sick children is

$$c_i \;=\; \frac{\mu}{\gamma + \mu}(1 - \mathcal{R}_0^{-1} - q^*) \;=\; \frac{r\alpha}{(1-r)(\mathcal{R}_0 - 1)}, \tag{6.135}$$

which is lower than α (defined by (6.133)) because $r < r_c$.

For $\mathcal{R}_0 > 1$, Figure 6.21 illustrates the variation with respect to r of both the ESS q^* and the infection rate c_i. Because (6.120), (6.131), and (6.134) yield

$$\left(\frac{\partial^2 f}{\partial p^2} + \frac{\partial^2 f}{\partial p \partial q}\right)\bigg|_{p=q=q^*} \;=\; 0 + \phi'(q^*) = -\mathcal{R}_0(1-r)^2 < 0, \tag{6.136}$$

the ESS is continuously stable by (2.123). Thus if a vaccine scare should increase perceived relative risk, say from $r = r_1$ to $r = r_2 > r_1$, then the population can be expected to shift accordingly to the new ESS, as indicated in Figure 6.21: the level of pre-emptive vaccination will fall, and the infection rate will correspondingly rise.

6.9. Stomatopod strife: a threat game

Several decades ago, Adams and Caldwell [**1**] observed a series of contests between crustaceans called stomatopods, or mantis shrimps, who occupy cavities in coral rubble. If one stomatopod intrudes upon another, then the resident often defends its cavity by threatening with a pair of claw-like appendages. These threat displays often cause intruders to flee, so that contests are settled without any physical contact. In this regard, a surprising observation is that when stomatopods are weakened by molting, and are thus completely unable to fight, they threaten more frequently than stronger animals who are between molts. Moreover, threats by weaklings often deter much stronger intruders, who would easily win a fight if there were one; that is, weaklings often bluff. But if the very weakest residents in the population can threaten and get away with it, then why don't all residents threaten? And if the threat display can be given by animals who cannot back it up, then why do their opponents respect it?

To explore this paradox, we develop a game in which a resident possessing a resource of value V can either threaten or not threaten in defense of it, and an intruder responds by either attacking or fleeing. We assume that, if there's a fight, then the stronger animal wins. Furthermore, both contestants pay a combat cost, which is higher for the weaker animal; specifically, an animal of strength s pays $C(s)$, where $C'(s) < 0$. Threats increase the vulnerability of a resident to injury inflicted by an intruder. Thus a threat is a display of bravado that bears no cost if the resident is not attacked, but which carries a cost T in addition to the combat cost if the resident is attacked by a stronger opponent.[41] Because the molt condition of stomatopods is not externally visible, we assume (as in the war-of-attrition game) that each contestant is unaware of its opponent's strength. So its own strength must determine its behavior. In terms of §6.2 (p. 221), the animal uses self-assessment.

Let fighting strengths be continuously distributed between 0 and 1 with probability density function g, and consider a focal individual or protagonist with fighting strength X, called Player 1 for convenience. In the role of resident, Player 1 threatens if either $X < u_1$ or $X > u_2$, but does not threaten if $u_1 < X < u_2$.[42] In the role of intruder, on the other hand, Player 1 attacks when $X > u_4$ if its opponent threatens but when $X > u_3$ if its opponent does not threaten; correspondingly, Player 1 flees when $X < u_4$ or when $X < u_3$, according to whether its opponent threatens or not. Thus Player 1's strategy is a four-dimensional vector $u = (u_1, u_2, u_3, u_4)$, whose first two components govern its behavior as resident, while its last two govern its behavior as intruder. The corresponding fighting strength and strategy of Player 1's opponent, called Player 2, are denoted by Y and $v = (v_1, v_2, v_3, v_4)$, respectively; e.g., Player 2 threatens as resident if either $Y < v_1$ or $Y > v_2$. Thus threats occur only when a resident's strength is either above or below a certain threshold. Nevertheless, potential ESSs include ones such that only the strongest residents threaten ($v_2 \gg v_1 = 0$), such that the weakest residents also threaten ($v_2 \gg v_1 > 0$) or such that residents always threaten ($v_1 = v_2$). In the first case threats would carry reliable information, in the second case threats could be either honest or deceitful, and in the third case threats would carry no information at all.

Using notation that temporarily suppresses dependence on u and v, let $F(X, Y)$ denote the payoff to a u-strategist (Player 1) against a v-strategist (Player 2), let $F_k(X, Y)$ denote the payoff to u against v in role k, and let p_k be the probability of occupying role k. Then, if r stands for resident and i for intruder, we have $p_r + p_i = 1$ and

$$F(X, Y) \;=\; p_r F_r(X, Y) + p_i F_i(X, Y). \tag{6.137}$$

Note that F, F_r, and F_i are random variables, because X and Y are random variables, and it follows from (6.137) that the reward to a u-strategist in a population of v-strategists is

$$f(u, v) \;=\; \mathrm{E}[F(X, Y)] \;=\; p_r f_r(u, v) + p_i f_i(u, v), \tag{6.138}$$

[41] T is for threat cost, but it is also called a vulnerability cost [**2**], [**42**, p. 408].

[42] Because X is continuously distributed, the event that $X = u_1$ or $X = u_2$ occurs with zero probability, and so we ignore it. Similarly for $X = u_3$ or $X = u_4$.

Table 6.1. Payoff to a protagonist of strength X with strategy $u = (u_1, u_2, u_3, u_4)$ against an opponent of strength Y with strategy $v = (v_1, v_2, v_3, v_4)$. Whether its role is resident or intruder is denoted by r or i; ρ and σ are defined by (6.140).

k	Relative magnitudes of X and Y	$F_k(X,Y)$
r	$X < u_1$ or $X > u_2$ and $Y > v_4$	$\rho(X,Y)$
r	$X < u_1$ or $X > u_2$ and $Y < v_4$	V
r	$u_1 < X < u_2$ and $Y > v_3$	$\sigma(X,Y)$
r	$u_1 < X < u_2$ and $Y < v_3$	V
i	$Y < v_1$ or $Y > v_2$ and $X > u_4$	$\sigma(X,Y)$
i	$Y < v_1$ or $Y > v_2$ and $X < u_4$	0
i	$v_1 < Y < v_2$ and $X > u_3$	$\sigma(X,Y)$
i	$v_1 < Y < v_2$ and $X < u_3$	0

where E denotes expected value and, for $k = r$ or $k = i$,

$$f_k(u,v) = \int_0^1 \int_0^1 F_k(x,y)\, dA, \tag{6.139}$$

with dA denoting $g(x)\, g(y)\, dx\, dy$ as in (2.22).

It is convenient at this juncture to define ρ, σ, and τ as

$$\sigma(X,Y) = \begin{cases} V - C(X) & \text{if } X > Y, \\ -C(X) & \text{if } X < Y, \end{cases} \tag{6.140a}$$

$$\tau(X,Y) = \begin{cases} 0 & \text{if } X > Y, \\ -T & \text{if } X < Y, \end{cases} \tag{6.140b}$$

$$\rho(X,Y) = \sigma(X,Y) + \tau(X,Y). \tag{6.140c}$$

Thus ρ or σ, respectively, is the payoff to a threatening or nonthreating resident protagonist of strength X against an attacking opponent of strength Y. The protagonist's payoff from any contest is now defined by Table 6.1, in which the first, second, fifth, and sixth rows correspond to threatening behavior by the resident, and the remaining four rows correspond to nonthreatening behavior.

For $u_1 \le u_2$, substitution from (6.140) into (6.139) now yields

$$\begin{aligned} f_r(u,v) = {} & \int_0^{u_1}\int_{v_4}^{1} \rho(x,y)\, dA + \int_{u_2}^{1}\int_{v_4}^{1} \rho(x,y)\, dA + \int_0^{u_1}\int_0^{v_4} V\, dA \\ & + \int_{u_2}^{1}\int_0^{v_4} V\, dA + \int_{u_1}^{u_2}\int_{v_3}^{1} \sigma(x,y)\, dA + \int_{u_1}^{u_2}\int_0^{v_3} V\, dA \end{aligned} \tag{6.141}$$

and

$$f_i(u,v) = \int_{u_4}^1 \int_0^{v_1} \sigma(x,y)\, dA + \int_{u_4}^1 \int_{v_2}^1 \sigma(x,y)\, dA + \int_{u_3}^1 \int_{v_1}^{v_2} \sigma(x,y)\, dA. \tag{6.142}$$

In each case, the first integral sign corresponds to integration variable x and the second to variable y. If $u_1 > u_2$, however, then the u-strategist always threatens, and in place of (6.141) we have

$$(6.143) \qquad f_r(u,v) \;=\; \int_0^1\int_{v_4}^1 \rho(x,y)\,dA \;+\int_0^1\int_0^{v_4} V\,dA \;=\; Q,$$

say, where Q is independent of u. In other words, any strategy satisfying $u_1 > u_2$ is equivalent mathematically to any strategy satisfying $u_1 = u_2$. Therefore, from now on we constrain u to satisfy

$$(6.144) \qquad 0 \le u_1 \le u_2 \le 1, \quad 0 \le u_3 \le 1, \quad 0 \le u_4 \le 1.$$

It is a moot point whether the strategies thus excluded are all equivalent biologically: an animal who always threatens because its "high" threshold (for reliable communication) is normal but its "low" threshold (for deceitful communication) is abnormally high may be said to behave very differently from an animal who always threatens because its low threshold is normal but its high threshold is abnormally low, whereas our game does not distinguish between them. Nevertheless, the question becomes irrelevant, because $u_1 < u_2$ at the only ESS.

To calculate the optimal reaction set R, we must maximize f defined by (6.138) with respect to u. We first observe that the game is separable, because (6.138) may be written as (6.1) with $p = 4$ and

$$(6.145a) \qquad f_1(u_1,v) \;=\; p_r\int_0^{u_1}\int_{v_3}^{v_4}\{V-\sigma(x,y)\}\,dA \;+\; p_r\int_0^{u_1}\int_{v_4}^1 \tau(x,y)\,dA,$$

$$(6.145b) \qquad f_2(u_2,v) \;=\; p_rQ - f_1(u_2,v),$$

$$(6.145c) \qquad f_3(u_3,v) \;=\; p_i\int_{u_3}^1\int_{v_1}^{v_2}\sigma(x,y)\,dA,$$

$$(6.145d) \qquad f_4(u_4,v) \;=\; p_i\int_{u_4}^1\int_0^{v_1}\sigma(x,y)\,dA \;+\; p_i\int_{u_4}^1\int_{v_2}^1\sigma(x,y)\,dA,$$

where Q is defined by (6.143). Thus maximization with respect to u_3 may be performed separately from that with respect to u_4, and both independently of that with respect to u_1 or u_2. This separability of the reward function makes the game analytically tractable.

Some general features of R require only that V, C, T, and g are all positive, which we assume. From (6.140a) and (6.145c),

$$\frac{\partial f_3}{\partial u_3} = -p_i\int_{v_1}^{v_2}\sigma(u_3,y)g(u_3)g(y)\,dy = p_iC(u_3)g(u_3)\int_{v_1}^{v_2} g(y)\,dy$$

is positive for $u_3 < v_1 < v_2$, so the maximum of f_3 for $0 \le u_3 \le 1$ must occur where $v_1 \le u_3 \le 1$. Again, (6.140a)–(6.140b) and (6.145a) imply that if $v_4 \le v_3$, then $\frac{\partial f_1}{\partial u_1} < 0$ for all $0 < u_1 < 1$ unless $v_4 = v_3 = 1$. Thus if $v_4 \le v_3$, then the maximum of f_1 must occur at $u_1 = 0$, unless $v_4 = v_3 = 1$, in which case, f_1 is independent of u_1. Correspondingly, (6.145b) yields $\frac{\partial f_2}{\partial u_2} > 0$ for all $0 < u_2 < 1$ unless $v_4 = v_3 = 1$, and so the maximum of f_2 must occur at $u_2 = 1$, unless $v_4 = v_3 = 1$, in which case, f_2 is independent of u_2.

Nevertheless, we cannot fully calculate R until we specify both C and g in (6.140)–(6.143). In this regard, we make two assumptions. First, combat cost

Table 6.2. Quantities that appear in Tables 6.3–6.4. Note that all but the last two depend on $v = (v_1, v_2, v_3, v_4)$.

$\delta = \frac{(a+b)(v_1-v_2+1)-v_1}{v_1-v_2+1}$	$\omega_1 = \frac{(1+a+b)v_4-(a+b)v_3-t(1-v_4)}{1+b(v_4-v_3)}$	
$\omega_4 = \frac{(a+b)(v_1-v_2+1)}{1+b(v_1-v_2+1)}$	$\gamma_4 = \frac{(a+b)(v_1-v_2+1)-v_1+v_2}{1+b(v_1-v_2+1)}$	
$\theta_1 = \frac{(1+a+b)(v_4-v_3)-t(1-v_4)}{b(v_4-v_3)}$	$\theta_2 = 1 - \frac{a(v_4-v_3)}{t-b(v_4-v_3)}$	
$\theta_3 = \frac{v_1+(a+b)(v_2-v_1)}{1+b(v_2-v_1)}$	$\theta_4 = \frac{(a+b)(v_1-v_2+1)-v_1}{b(v_1-v_2+1)} = \frac{\delta}{b}$	
$\Delta = \frac{t(1-v_4)}{v_4-v_3}$	$\omega_3 = 1 - \frac{1-a}{b}$	$\gamma_3 = \frac{a+b}{1+b}$

decreases linearly with fighting strength according to

$$C(s) \;=\; V\{a + b(1-s)\}, \tag{6.146}$$

with $0 < a < 1$ and $b > 0$. Thus $V > C$ for the strongest animal and $V > C$ for every animal in the limit as $b \to 0$, but in general there may be (weaker) animals for which $C > V$. Second, fighting strength is uniformly distributed between 0 and 1, i.e., $g(x) = g(y) = 1$ or $dA = dx\,dy$ in (6.145). Furthermore, it is convenient to introduce a dimensionless threat-cost parameter

$$t \;=\; T/V. \tag{6.147}$$

We can now proceed to calculate R.[43]

We obtain $f_3(u_3, v) = \frac{1}{2}V(v_2-v_1)(1-u_3)\{2(1-a)-b(1-u_3)\} - \frac{1}{2}V(v_2-u_3)^2$ if $v_1 \le u_3 \le v_2$, whereas the last (squared) term must be omitted to obtain the correct expression for f_3 if $v_2 \le u_3 \le 1$. It follows from Exercise 6.24 that the maximum of f_3 for $v_1 \le u_3 \le 1$ (and hence also for $0 \le u_3 \le 1$) occurs at $u_3 = \theta_3$ if $b(1-v_2) \le 1-a$ but at $u_3 = \omega_3$ if $b(1-v_2) > 1-a$, where θ_3 and ω_3 are defined in Table 6.2. Also, $f_4(u_4, v) = \frac{1}{2}V(1-u_4)\{2v_1-(2a+b\{1-u_4\})(v_1-v_2+1)\} + \frac{1}{2}V(1-v_2)^2 - \frac{1}{2}V(v_1-u_4)^2$ when $0 \le u_4 \le v_1$, but the last (negative squared) term must be omitted to obtain the correct expression for f_4 when $v_1 \le u_4 \le v_2$, and for $v_2 \le u_4 \le 1$ we obtain $f_4(u_4, v) = \frac{1}{2}V(1-u_4)\{2v_1 - 2v_2 + u_4 + 1 - (2a + b\{1-u_4\})(v_1 - v_2 + 1)\}$. Provided $v_1 - v_2 + 1 \ne 0$, the maximum of f_4 for $0 \le u_4 \le 1$ can now be shown to occur at $u_4 = \omega_4$ if $\delta < bv_1$, at $u_4 = \theta_4$ if $bv_1 \le \delta \le bv_2$ and at $u_4 = \gamma_4$ if $\delta > bv_2$, where δ, γ_4, θ_4, and ω_4 are defined in Table 6.2. If $v_1 - v_2 + 1 = 0$, which can happen only if $v_1 = 0$ and $v_2 = 1$, then f_3 is maximized at $u_3 = \gamma_3$ (defined in Table 6.2) and any u_4 maximizes f_4. These results imply that the maximum of f_i—defined by (6.142)—is given by Table 6.3.

We have already seen that when $v_3 \ge v_4$, f_r is maximized for $0 \le u_1 \le u_2 \le 1$ where $u_1 = 0$, $u_2 = 1$ (unless $v_3 = v_4 = 1$, in which case both u_1 and u_2 are arbitrary). Moreover, it is clear from (6.140a) and (6.145a)–(6.145b) that when $v_3 < v_4 = 1$, f_r is maximized where $u_1 = u_2$. Let us therefore assume that $v_3 < v_4 < 1$, and hence that

$$b(1-v_4) \;<\; a + b(1-v_4) \;<\; a + b(1-v_3). \tag{6.148}$$

[43]This calculation is rather complicated; readers who would prefer to take its outcome on trust are advised to skip ahead to p. 272.

Table 6.3. Maximizers u_3 and u_4 of f_i for $0 \le u_3, u_4 \le 1$

Constraints on v_1, v_2 $(\ge v_1)$	u_3	u_4	Constraints on u_3, u_4
$\delta < bv_1, b(1-v_2) \le 1-a$	θ_3	ω_4	$v_1 \le u_3 \le v_2, u_4 < v_1$
$bv_1 \le \delta \le bv_2, b(1-v_2) \le 1-a$	θ_3	θ_4	$v_1 \le u_3, u_4 \le v_2$
$\delta > bv_2, b(1-v_2) \le 1-a$	θ_3	γ_4	$v_1 \le u_3 \le v_2, u_4 > v_2$
$\delta < bv_1, b(1-v_2) > 1-a$	ω_3	ω_4	$u_3 > v_2, u_4 < v_1$
$bv_1 \le \delta \le bv_2, b(1-v_2) > 1-a$	ω_3	θ_4	$u_3 > v_2, v_1 \le u_4 \le v_2$
$\delta > bv_2, b(1-v_2) > 1-a$	ω_3	γ_4	$u_3 > v_2, u_4 > v_2$
$v_1 = 0, v_2 = 1$	γ_3	u_4	u_4 arbitrary

Then $f_1(u_1, v) = \frac{1}{2}Vu_1\{2(1+a+b)(v_4-v_3) - 2t(1-v_4) - b(v_4-v_3)u_1\}$ for $0 \le u_1 \le v_3$, $\frac{1}{2}V(u_1-v_3)^2$ must be subtracted to obtain the correct expression for $v_3 \le u_1 \le v_4$, and, for $v_4 \le u_1 \le 1$,

$$f_1(u_1, v) = \frac{1}{2}V\{(v_4-v_3)(v_4+v_3+u_1\{2(a+b)-bu_1\}) \\ - 2tu_1(1-v_4) + t(u_1-v_4)^2\}.$$

From Exercise 6.24, f_1 varies between $u_1 = 0$ and $u_1 = 1$ as follows. If $\Delta \ge 1+a+b$, then f_1 decreases between $u_1 = 0$ and $u_1 = \theta_2$ $(> v_4)$ but increases again between $u_1 = \theta_2$ and $u_1 = 1$. If $1+a+b > \Delta \ge 1+a+b(1-v_3)$, then f_1 increases between $u_1 = 0$ and $u_1 = \theta_1$ $(\le v_3)$, decreases between $u_1 = \theta_1$ and $u_1 = \theta_2$, and increases again between $u_1 = \theta_2$ and $u_1 = 1$. If $1+a+b(1-v_3) > \Delta > a+b(1-v_4)$, then f_1 increases between $u_1 = 0$ and $u_1 = \omega_1$ (which satisfies $v_3 < \omega_1 < v_4$), decreases between $u_1 = \omega_1$ and $u_1 = \theta_2$, and increases again between $u_1 = \theta_2$ and $u_1 = 1$.[44] Finally, if $\Delta \le a+b(1-v_4)$, then f_1 increases monotonically between $u_1 = 0$ and $u_1 = 1$; its concavity is always downward for $0 \le u_1 \le v_4$, but it is upward or downward for $v_4 \le u_1 \le 1$ according to whether $\Delta > b(1-v_4)$ or $\Delta < b(1-v_4)$.

Correspondingly, from (6.145b), f_2 varies between $u_2 = 0$ and $u_2 = 1$ as follows. If $\Delta \ge 1+a+b$, then f_2 increases between $u_2 = 0$ and $u_2 = \theta_2$ and decreases again between $u_2 = \theta_2$ and $u_2 = 1$. If $1+a+b > \Delta \ge 1+a+b(1-v_3)$, then f_2 decreases between $u_2 = 0$ and $u_2 = \theta_1$, increases between $u_2 = \theta_1$ and $u_2 = \theta_2$, and decreases again between $u_2 = \theta_2$ and $u_2 = 1$. If $1+a+b(1-v_3) > \Delta > a+b(1-v_4)$, then f_2 decreases between $u_2 = 0$ and $u_2 = \omega_1$, increases between $u_2 = \omega_1$ and $u_2 = \theta_2$, and decreases again between $u_2 = \theta_2$ and $u_2 = 1$. Finally, if $\Delta \le a+b(1-v_4)$, then f_2 decreases monotonically between $u_2 = 0$ and $u_2 = 1$. Thus the maximum of f_r—defined by (6.141)—is given by Table 6.4. Note that the maximum corresponds to unconditional threatening if $\Delta \le a+b(1-v_4)$.

Now, if v is an ESS, then the maximum in Table 6.3 must occur where $u_3 = v_3$ and $u_4 = v_4$, the maximum in Table 6.4 must occur where $u_1 = v_1$ and $u_2 = v_2$, and all conditions on u must be satisfied. Let us first of all look for a strong ESS. Then v must be the only best reply to itself. This immediately rules out the fourth and sixth rows of Table 6.4, where u_1 and u_2 do not yield a unique best reply to v_3 and v_4, and although the fifth row of Table 6.4 does yield a unique best reply, it corresponds to the bottom row of Table 6.3, where u_4 is not unique.

[44] Note that $\Delta > a+b(1-v_4)$ and (6.148) imply $t > b(v_4-v_3)$ in θ_2 (and hence, eventually, that $\eta'(s) > 0$ for $L < s < 1$ in (6.152)).

Table 6.4. Maximizers u_1 and u_2 of f_r for $0 \le u_1 \le u_2 \le 1$

Constraints on v_3, v_4 ($> v_3$)	u_1	u_2	Constraints on u_1, u_2
$\Delta \ge 1 + a + b$	0	θ_2	$u_1 \le v_3, u_2 > v_4$
$1 + a + b > \Delta \ge 1 + a + b(1 - v_3)$	θ_1	θ_2	$u_1 \le v_3, u_2 > v_4$
$1 + a + b(1 - v_3) > \Delta > a + b(1 - v_4)$	ω_1	θ_2	$v_3 < u_1 \le v_4, u_2 > v_4$
$\Delta \le a + b(1 - v_4), v_4 > v_3$	u_1	u_2	$u_1 = u_2$, u_2 arbitrary
$v_3 \ge v_4, v_4 \ne 1$	0	0	
$v_3 = v_4 = 1$	u_1	u_2	u_1, u_2 both arbitrary

Accordingly, we restrict our attention to the first three rows of Table 6.4. Then, for the maximum to occur at $u_2 = v_2$, each possibility requires $v_2 > v_4$. Thus the maximum at $u_4 = v_4$ in Table 6.3 must satisfy $v_4 < v_2$, excluding the third and sixth row of that table. Again, the relative magnitudes of v_3 and v_4 in the first three rows of Table 6.4 all imply $v_3 < v_4 < 1$, so that the maximum at $u_3 = v_3$ in Table 6.3 cannot satisfy $v_3 \ge v_4$, and hence (because $v_4 < v_2$) cannot satisfy $v_3 \ge v_2$; thus the fourth and fifth rows of Table 6.3 are excluded. The first row of the table is likewise excluded, because the maximum where $u_3 = v_3$ and $u_4 = v_4$ would have to satisfy $v_4 < v_1 \le v_3$, which is impossible because $v_4 > v_3$. Only the second row of Table 6.3 now remains. Because the maximum at $u_3 = v_3$ must therefore satisfy $v_1 \le v_3$, we have to exclude the third row of Table 6.4. But the maximum where $u_1 = v_1$ and $u_2 = v_2$ in Table 6.4 cannot now occur where $u_1 = 0$ and $u_2 = \theta_2$ because the second row of Table 6.3 would then imply $0 \le a + b \le b\theta_2$, which is impossible for $a > 0$. We have thus excluded the top row of Table 6.4, and only the second remains. We conclude that a strong ESS must correspond to the second row in each table.

Let us now set $v = (I, J, K, L)$ in Table 6.2, so that θ_3, θ_4, δ, ω_4, and γ_4 depend on I and J, whereas θ_1, θ_2, Δ, and ω_1 depend on K and L. Then what we have shown is that (I, J, K, L) is a strong ESS if it satisfies the equations $I = \theta_1$, $J = \theta_2$, $K = \theta_3$, and $L = \theta_4$. The last two equations yield

$$K = \tfrac{I+(a+b)(J-I)}{1+b(J-I)}, \quad L = \tfrac{(a+b)(I-J+1)-I}{b(I-J+1)}. \tag{6.149}$$

Substituting into $I = \theta_1$ and $J = \theta_2$, we obtain a pair of equations for I and J. The first has the form

$$tab(1 - J)^2 + d_1(1 - J) + d_0 = 0, \tag{6.150}$$

where

$$d_0 = (1 - a)\{(1 + t)(1 - bI + b) + a\}I$$

is quadratic in I and

$$d_1 = -(a + b + at)(1 - bI + b) - bt(1 - a)I - a(a + b)$$

is linear in I. The second equation is cubic in J and can be used in conjunction with the first to express J as a quotient of cubic and quadratic polynomials in I. Substitution back into the first equation yields a sextic equation for I, of which three solutions—namely, $I = 0$, $I = 1 + a/b$, and $I = 1 + (1 + a)/b$—can be found

Table 6.5. Coefficients for cubic equation (6.151)

c_0	$=$	$-a\{(a+b)(1+2a+b)+at(1+a+b)\}$
c_1	$=$	$(1+a+b)\{(1+t)\{a+(1+b)(1+t)+b^2\}+2ab\}$ $+\ a(1+t)\{1+b+b(3a+2b)\}+a^2$
c_2	$=$	$-b\{(1+t)\{(2+b)t+2b^2+(3b+2)(1+a)\}+a(1+b)\}$
c_3	$=$	$b^2(1+b)(1+t)$

by inspection. None of these solutions satisfies $0 < I < 1$. Thus, removing the appropriate linear factors, we find that I must satisfy the cubic equation

$$c_3I^3 + c_2I^2 + c_1I + c_0 \;=\; 0, \tag{6.151}$$

whose coefficients are defined by Table 6.5. Because $c_0 < 0$ and $c_0+c_1+c_2+c_3 > 0$, it is clear at once that there is always a real solution satisfying $0 < I < 1$. It is not difficult (but a bit tedious) to show that this solution is the only solution satisfying $0 < I < 1$; the other two solutions are either complex conjugates or, if they are real, satisfy $I > 1$. Moreover, only one solution of quadratic equation (6.150) for J satisfies $J > I$. Thus the strategy $v = (I, J, K, L)$ defined by (6.149)–(6.151) is the only strong ESS.

Nevertheless, there are several candidates for a weak ESS. First, from the last row of Table 6.3 and the fifth row of Table 6.4, we find that $v^* = (0, 1, \gamma_3, \lambda)$ satisfies (2.17a) for any $\lambda \le \gamma_3$ (where γ_3 is defined in Table 6.2); however, v^* fails to satisfy (2.17b), because $f(u, v^*) = f(v^*, v^*)$ for $u = (0, 1, \gamma_3, u_4)$, $0 \le u_4 \le 1$. From (6.141)–(6.142), we then find that $f(v^*, u) = f(u, u) = 0$, so that (2.17c) fails to hold. Thus v^* is not a weak ESS. Intuitively, never threatening cannot be an evolutionarily stable behavior because in equilibrium the threshold λ is irrelevant; even if the population strategy v satisfies $v_4 \le \gamma_3$ to begin with, there is nothing to prevent v_4 from drifting to $v_4 > \lambda$, in which case, never threatening is no longer a best reply. (In particular, there is nothing to prevent v_4 from drifting to 1, and never threatening cannot be a best reply to an opponent who never attacks when threatened.)

Second, from the last row of Table 6.4, we must investigate the possibility that there is a weak ESS of the form $v = (v_1, v_2, 1, 1)$. Because $a < 1$, however, we see from Table 6.3 that this possibility requires $\theta_3 = 1$ or $(1-a)v_1 + av_2 = 1$, which implies $v_1 = v_2 = 1$. But then, from the first three rows of Table 6.3, either $v_4 = \gamma_3$ or $v_4 = \omega_3$, contradicting $v_4 = 1$. Hence there is no such ESS.

The remaining possibility for a weak ESS is an always-threatening equilibrium with $v_1 = v_2 = \lambda$, say, which corresponds to the fourth row of Table 6.4, and therefore satisfies $v_4 > v_3$. This equilibrium cannot correspond to the first row of Table 6.3, because $v_1 \le v_3 \le v_2$ and $v_4 < v_1$ then imply $v_4 < \lambda \le v_3$, contradicting $v_4 > v_3$. For similar reasons, the equilibrium cannot correspond to either the second row of Table 6.3 (which would require $v_4 = \lambda = v_3$) or the fourth or fifth row (each of which would require $v_3 > v_4$). Thus the equilibrium must correspond to either the third or sixth row of Table 6.3, and hence have the form $v^* = (\lambda, \lambda, \zeta, \gamma_3)$, where $\zeta = \max(\lambda, \omega_3)$ satisfies $\zeta < \gamma_3$. Then $f(u, v^*) = f(v^*, v^*)$ and $f(v^*, u) = f(u, u)$ for any u such that $u_1 = u_2$ and $u_4 = \gamma_3$: although v^* satisfies (2.17a), it fails to

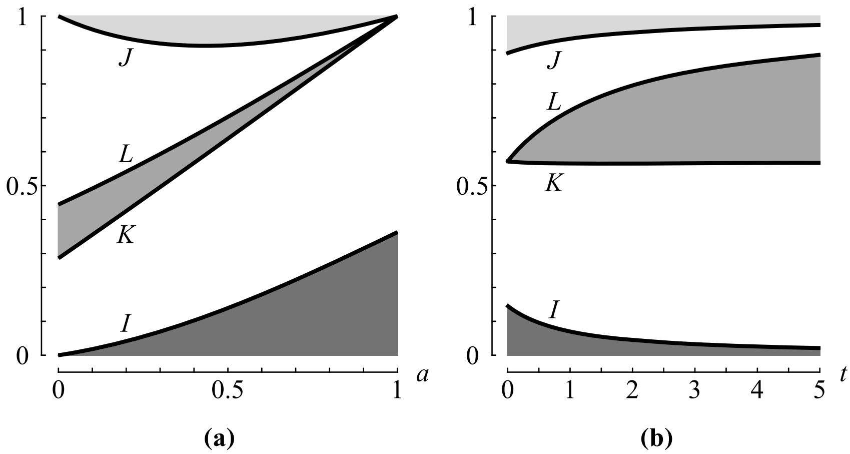

Figure 6.22. The effect of varying **(a)** the fixed cost of fighting or **(b)** the threat cost on the ESS thresholds. In these examples $b = 0.4$ and **(a)** $t = 0.4$ or **(b)** $a = 0.4$. Light shading corresponds to the proportion of residents who threaten honestly, dark shading corresponds to the proportion of residents who threaten deceptively, and intermediate shading corresponds to the proportion of intruders deterred by threats.

satisfy (2.17b)–(2.17c), and so is not a weak ESS. Intuitively, always threatening cannot be an evolutionarily stable behavior because in equilibrium the threshold ζ is irrelevant; even if the population strategy v satisfies $v_3 < \gamma_3$ to begin with, there is nothing to prevent v_3 from drifting to $v_3 \geq \gamma_3$, in which case, always threatening is no longer a best reply. (In particular, there is nothing to prevent v_3 from drifting to 1, and always threatening cannot be a best reply to an opponent who never attacks when not threatened.)

The upshot is that the sole ESS is the strong ESS, for which $J > L > K > I$ (Figure 6.22). At this ESS, the weakest and strongest animals both threaten when resident, whereas those of intermediate strength do not. The proportion of intruders deterred by threats is $L - K$ (because an intruder attacks if its strength exceeds K when not threatened, but only if its strength exceeds L when threatened). All such intruders would lose against a resident whose strength exceeds J (because $J > L$); the threats of the strongest residents are therefore honest. On the other hand, all deterred intruders would win against a resident whose strength does not exceed I (because $I < K$); the threats of the weakest residents are therefore deceptive—but they cannot be distinguished from honest threats without escalation. The proportions of residents who threaten deceptively, who do not threaten, and who threaten honestly are I, $J - I$, and $1 - J$, respectively, and it is readily shown that $J - I > \frac{1}{2}$, so that fewer than half of the residents threaten (Exercise 6.25). Nevertheless, the proportion of threats that are deceptive can be considerable (Exercise 6.25). In other words, not only is bluffing a part of the ESS, but also it can persist at high frequency.

To see why the weakest and strongest animals both threaten when resident while those of intermediate strength do not, it is instructive to compute the expected difference in net gain between threatening and not threatening to a resident of known strength s against an intruder whose unknown strength S is drawn at random from the uniform distribution (so that $\text{Prob}(S \le z) = z$). We compute this quantity by subtracting the expected difference in total cost (i.e., combat cost plus threat cost) from the expected difference in benefit.

On the one hand, the resident's combat cost—if paid (i.e., if $S > L$ or if $S > K$, according to whether the resident threatens or not)—is $C(s)$. Thus the expected difference in combat cost between threatening and not threatening is $C(s)\{\text{Prob}(S > L) - \text{Prob}(S > K)\} = \{K - L\}C(s)$. Also, the resident's threat cost is T if $S > \max(s, L)$ but zero otherwise, with expected value $T\text{Prob}(S > \max(s, L)) = T\{1 - \max(s, L)\}$—which is also the expected *difference* in threat cost, because the cost is avoided by not threatening. Adding, we find that the expected difference in total cost is $\{K - L\}C(s) + T\{1 - \max(s, L)\}$.

On the other hand, the benefit to a threatening resident, who wins if the intruder is weaker or does not attack, is V if $S < \max(s, L)$ but 0 if $S > \max(s, L)$, with expected value $V\text{Prob}(S < \max(s, L))$. For a nonthreatening resident, the corresponding expected value is $V\text{Prob}(S < \max(s, K))$, and their difference is the expected difference in benefit. So the expected difference in net gain is $V\{\text{Prob}(S < \max(s, L)) - \text{Prob}(S < \max(s, K))\} - \{K - L\}C(s) - T\{1 - \max(s, L)\}$. Let us denote this quantity by $V\eta(s)$, so that η is dimensionless. Then, from (6.146)–(6.147) and Exercise 6.26, we obtain

$$\eta(s) = \begin{cases} \{1 + a + b(1-s)\}(L-K) - t(1-L) & \text{if } 0 \le s \le K, \\ L - s + \{a + b(1-s)\}(L-K) - t(1-L) & \text{if } K \le s \le L, \\ \{a + b(1-s)\}(L-K) - t(1-s) & \text{if } L \le s \le 1. \end{cases} \tag{6.152}$$

Furthermore, $\eta(I) = 0 = \eta(J)$, $\eta(s) < 0$ if $I < s < J$, and $\eta(s) > 0$ if $s < I$ or $s > J$. So the strongest and the weakest residents both threaten because the expected net gain from doing so exceeds that from not threatening; however, their threats are profitable for different reasons. The strongest residents threaten because, although their expected benefit (of avoiding combat costs) is low, their expected cost is even lower—they are very unlikely to meet an opponent strong enough to inflict the threat cost. At the other extreme, the weakest residents threaten because, although their expected cost is high—an intruder who attacks invariably inflicts the threat cost—their expected benefit from threatening is even higher. They are able thereby to deter some considerably stronger intruders (who would win a fight if there were one), and to do so without the cost of combat (which is highest for the weakest animals).

Note, finally, that our partial-bluffing ESS arises only in special circumstances. The ESS does not persist in the limit as $b \to 0$, or if we change the reward structure so that a threatening resident pays the threat cost either regardless of whether it is attacked, or only if it is attacked, but regardless of whether it wins or loses.[45] From (6.146), we have $Vb = -C'(s)$ where s is strength and C is cost of combat. Thus our model predicts a partial-bluffing ESS only if the combat cost is higher for weaker animals *and* a threatening resident pays an additional cost only when it is attacked and loses. And this is a strength of the model: it helps to identify the

[45] For details, see Exercise 6.27.

particular circumstances in which we might expect to observe a high frequency of bluffing in nature.

6.10. Commentary

In this chapter we used continuous population games to study a variety of topics in behavioral ecology and epidemiology, namely, sex allocation (§6.1), kinship (§6.3), sperm competition (§6.4), resource partitioning (§6.5, §6.6), landmarks as conventions (§6.6), vaccination behavior (§6.8), and aspects of animal contest behavior (§6.2, §§6.5–6.7, §6.9). Within this, §6.2 is based on [**225**], §6.5 on [**233**], §6.6 on [**212**], §6.7 on [**221**], and §6.9 on [**2**]. This list of topics is by no means exhaustive, but it exemplifies the scope and variety of applications. Other topics for continuous population games include anisogamy (i.e., sexual reproduction by fusion of unlike gametes [**53**]), parent-offspring conflict [**122**], timing of arrival to breeding grounds in migratory birds [**154**], and seed dispersal [**100**] or other aspects of plant ecology [**151**, **199**], although in this book we consider only games among animals.

Evolutionary game theory did not fully emerge as a field of study in its own right until 1982 when Maynard Smith's definitive monograph [**188**] consolidated the advances that he and others had made during the 1970s. But the application of game-theoretic reasoning to the study of sex ratios (§6.1) is significantly older, and it can be traced through Hamilton [**125**] all the way back to Fisher [**98**] in 1930. The study of sex allocation has since advanced considerably; for a masterly synthesis, see West [**359**].

The application of game theory to animal contest behavior (§6.2, §§6.5–6.7, §6.9) began with two basic models introduced by Maynard Smith [**185**, **192**], the Hawk-Dove game (§2.4), and a war-of-attrition model. Both models have since been developed in various ways by numerous authors; see [**77**, **126**, **128**, **160**, **166**] and references therein. Other models of contest behavior include Enquist and Leimar's sequential assessment model of mutual assessment or SAM [**90**] and Payne's cumulative assessment model or CAM [**266**]. In Payne's model, an opponent's strength—despite not being directly assessed—affects the decision to withdraw because it determines the opponent's ability to inflict costs (e.g., injuries) on its rival. These and other models are discussed in the context of current research by Kokko [**160**], as well as in most later chapters of Hardy and Briffa [**130**], which comprehensively covers both theoretical and empirical work on animal contest behavior.

In particular, the CAM, SAM, and WOA-WA (§6.2), together with a more recent model [**222**] allowing for both self- and mutual assessment within the same game, have been central to an ongoing debate over the question of whether animals' decisions to withdraw from contests rely on self- or mutual assessment. Even if assessment of opponents—for which evidence is sparse [**88**]—is cognitively possible, it need not be cost-effective [**222**]. Studies have shown that pure self-assessment (as in the WOA-WA) is far more common than once thought; yet different species conform to different models, and some conform to no existing model [**95**, p. 86]. Indeed assessment mechanisms can vary not only across species but even within contests [**9**]. So the question has no simple answer, and existing models seem unlikely to settle the debate; rather, newer models are needed. Here lies a golden opportunity for game theorists.