INTRODUCTION (MOSTLY FOR INSTRUCTORS)

Formally, Real Analysis—the course—is a presentation of the theoretical underpinnings of calculus. It is about the Big Three: continuity, differentiability, and convergence. Yet it is also, for many, an introduction to reading, writing, and thinking mathematics. I have tried to address all of these issues in this book.

In the first chapter we construct the real numbers, starting with the rationals. This lays the groundwork for the entire book. The basic concept here is that of a family of rational intervals. A real number is a family of rational intervals which satisfies two important conditions: *consistency* (any two intervals in the family intersect) and *fineness* (the family contains intervals of arbitrarily small length). These conditions, together with the arithmetic that families inherit from the rationals, lead to all of the familiar algebraic properties of the reals. We establish these properties via quite a few propositions and one main theorem (completeness). Proving these results requires

- knowing simple properties of the arithmetic of rational numbers,
- applying elementary algebra and simple logic, and
- learning to apply new definitions and newly proved results.

Thus, Chapter 1 is critical because it provides not only the mathematical ideas that permeate the rest of the text, but also the introduction to the reasoning and writing skills necessary for doing and communicating mathematics. Nothing is more boring than having to read a seemingly endless theorem-proof sequence, so I have tried to provide just enough sample proofs and hints so that readers can proceed on their own. Many propositions are left as exercises; indeed, the exercises provide a vital part of the whole pedagogical process. I take this chapter at a leisurely pace, allowing students to write, critique, and rewrite their work. It is an investment of time well worth making early in the course.

Chapter 1 ends with what may be the central result of any real analysis course: the completeness of the reals. This is expressed in terms of families of real intervals, but in Chapter 3 it is rephrased in the language of Cauchy sequences.

Chapter 2 uses the Completeness Theorem to prove the useful Inverse Function Theorem. This, in turn, is used to construct nth roots, general exponential functions, and logarithms. A section is devoted to the Euler number e and the natural logarithm.

Chapter 3 introduces sequences, limits, and series and derives basic formulas and inequalities for the various functions already constructed.

In Chapter 4 we encounter uniform continuity. Since this version of continuity is the one most used in more advanced courses, we relegate the idea of pointwise continuity to the exercises. Nothing is lost, however, since the usual verifications of pointwise continuity for the basic functions of calculus are used with little modification to establish uniform continuity of these functions on intervals. We also encounter many interesting and important consequences of uniform continuity, among them boundedness and the extension of uniformly continuous functions from dense subsets—for example, extending functions from a punctured interval $[a, b] - \{x_1, \ldots, x_n\}$ to the closed interval $[a, b]$.

In Chapter 5, we use the Completeness Theorem again, this time to construct the Riemann integral $\int_a^b f$ for functions uniformly continuous on an interval $[a, b]$. The results previously established for limits and extensions of uniformly continuous functions can now be applied to define and calculate improper integrals. It is here that we introduce the important idea of functions defined as integrals. This includes the definition of the arctangent as an integral, an alternate definition of the natural logarithm (previously defined as an inverse function), and the use of improper integrals to construct the Gamma function and Laplace transforms.

Chapter 6 on differentiation emphasizes the derivative as a function rather than a pointwise limit. All the usual formulas from calculus are derived. In particular, the uniform version of differentiability that we use makes for very short and illuminating proofs of two central results of calculus:

- **The Law of Bounded Change**, which says that bounds for the derivative (i.e. $A \le f'(x) \le B$) are bounds for the difference quotient (i.e. $A \le \dfrac{f(y) - f(x)}{y - x} \le B$). (This is sometimes called the "Mean Value Inequality.")
- **The Fundamental Theorem of Calculus**.

In this chapter, we also derive some rather more difficult results on differentiating under the integral sign. In the case of improper integrals, we introduce "dominated convergence" assumptions, which we will also use later in studying series of functions.

In Chapter 7, nearly all of the ideas developed in the course are applied to studying the properties of sequences and series of continuous and differentiable functions. The particular case of power series is given special attention. The chapter ends with the definition of the periodic (trigonometric) functions as power series and a derivation of their properties (including a definition of π)—*all without pictures.* My students invariably enjoy this; in fact, with just a few simplifications and detours, it has even worked well for high school students taking AP calculus.

The last chapter of the book is organized around Fourier series, but it also provides an introduction to some of the more advanced ideas in functional analysis: inner products of functions, the Bessel and Cauchy-Schwartz inequalities and their applications, kernels and convolutions, and Abel summability. The early sections also introduce the complex numbers and the properties of complex-valued functions of a real variable.

There is enough material in the eight chapters to give a full-year course, especially if a lot of the more challenging exercises are assigned and discussed in class. Some of the exercises which have several parts and require more extensive work are labeled "projects."

I have usually given Real Analysis as a one-semester course. I generally get to cover the following.

1. Chapter 1: sections 1.0 through 1.7 (omitting 1.8 and skimming some of the material on absolute value and betweenness).
2. Chapter 2: in which I skip the more technical results—especially the 1- and 2-sided versions of the Inverse Function Theorem and some of the inequalities relating to the Euler number e.
3. Chapter 3: just what I need to talk about convergence of series.
4. Chapter 4: section 4.1 and the beginning of section 4.2 (omitting extensions of continuous functions), some material on limits from Chapter 3.
5. Chapter 5: sections 5.1 and 5.2.
6. Chapter 6: sections 6.1 through 6.3.
7. Chapter 7: just the material on power series.

Having done this for one semester, if there is enough student interest in a second semester, or a student wants to do a reading course, I can cover the more technical topics such as improper integrals, general convergence of sequences of functions, complex numbers, and Fourier series.

After teaching Real Analysis for many years, I'd say that my general experience has been that there is no general experience. Student ability, background, and motivation can vary a lot from year to year, and I think it is a mistake to commit to a strict syllabus before you know your class. What is critical is that students do lots of problems and write lots of proofs. It is also very important that the central definitions and examples be memorized. I give several quizzes devoted exclusively to this. On the other hand, the more difficult material (proofs) is best tested via problem sets. Students seem to do these best—and enjoy them more—when working with one or two others. (But I do require independent write-ups!)

In terms of submitting mathematical work, most students initially write it out by hand. Since I typically require rewrites, many soon learn to use an equation editor with their word-processor. The software package *Scientific Notebook* is a good alternative, especially if you can get your school to underwrite its purchase. I have even had a few ambitious students learn to use TeX or LaTeX.

It is important to remember that this is an undergraduate course and that most students taking it are probably not intending to go to graduate school in theoretical mathematics. The goal here is to have students understand the mathematics, be able to create some on their own, and come away with happy memories of the experience. There is also plenty of challenging material here, especially in the problems, for the talented and highly motivated student. The approach I have taken in this book has worked well over the years for me and my students. I hope it does for you and yours as well.