PREFACE

What is a constructive approach, and why should one take it?

If you look at the table of contents for this book, you'll see mostly familiar topics, but with a slightly different emphasis. There's a long chapter on the real numbers, followed by one on "An Inverse Function Theorem." The chapter on limits, sequences and series is followed by one on *uniform* continuity—why not just *pointwise* continuity? A chapter on the Riemann integral is followed by one on differentiation—but it's actually *uniform* differentiation. All of these departures from the structure of the usual real analysis text result from a careful reassessment of the role of the course in the technical education of undergraduates.

Not every student in Real Analysis is a math major, and, in many schools, only a small percentage of math majors intend to do graduate work in mathematics. A modern course is populated by a wide range of students. Some are headed for careers in secondary education, while there is often a large contingent from the physical sciences and an even larger group from computer science. These students are in the course because they need or want more than a cookbook calculus course. Some need to know more about computability and calculability of floating-point numbers, hence more about the actual nature of the reals. They also need to know about continuity because they need to know about approximations; some need to know about convergence and improper integrals because they need to know about computing special functions and transforms.

But real analysis is not primarily focused on computing. It is, significantly, a course that shapes the way students think about mathematics. Very often it is a student's introduction to precise reasoning and writing.

So: I begin with a careful construction of the real numbers, the field on which most of analysis is played. The approach here, due to Gabriel Stolzenberg, is via intervals of rational numbers and the arithmetic of such intervals. The many elementary theorems about the properties of this arithmetic later reappear as properties of the real numbers, and verifying them provides a gentle introduction to the art and practice of devising and writing readable and correct proofs. Furthermore, there is a useful metaphor: a rational interval is exactly what is obtained when a scientist uses instruments of limited (but known) accuracy to measure something. Families of rational intervals then correspond to multiple measurements, and the condition on a family that any two of its intervals must meet establishes the consistency of its measurements. Finally, a real number is defined to be a family of rational intervals that is consistent in this sense, and that contains intervals of arbitrarily small length. Interval arithmetic, carried over to families of intervals, now becomes real arithmetic, and conditions on the lengths of intervals become the properties of approximation of reals by rationals.

At this point, the students see that the reals have a far more complex structure than the rationals. One important example is the traditional Law of Trichotomy, namely that precisely one of $x < y$, $y < x$, or $x = y$ must hold. This property holds for the rationals, since rational arithmetic is basically integer arithmetic. However, reals can, in general, only be approximated by rationals. Modern computer algebra systems allow the user to specify a tolerance, which is expressed as the number of decimal places. This number can be chosen as large as one pleases, but not infinitely large. The corresponding tolerance, say ϵ, tells us how closely we can distinguish reals using the computer's rational representation. These considerations lead to the formulation of real number comparison that we prove and use throughout the book.

ϵ-Trichotomy *Given any tolerance $\epsilon > 0$, then for any reals x and y, $x < y$, $y < x$, or x and y are within ϵ of each other.*

Thus, a construction of the reals based on rational measurement and an analysis of what we can actually calculate produces a concordance of theory and practice that students of the sciences easily relate to.

Using the notion of ϵ-trichotomy as a tool for comparing real numbers enables us to describe a bisection-like algorithm for finding the inverse of a function f, providing it satisfies upper and lower bounds on its difference quotient $\frac{f(y)-f(x)}{y-x}$. This leads directly to the construction of nth root, exponential, and logarithm functions.

Another hallmark of the constructivist program is its emphasis on uniform vs. pointwise continuity:

- *f is pointwise continuous at a if, given any $\epsilon > 0$ we can find a $\delta_a(\epsilon) > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta_a(\epsilon)$.*

- *f is uniformly continuous on $\mathcal{S}$ if, given any $\epsilon > 0$ we can find a $\delta(\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta(\epsilon)$ and $x, y \in \mathcal{S}$.*

Uniform continuity on $\mathcal{S}$ implies pointwise continuity at each point of $\mathcal{S}$, but the converse is not true: there is no general procedure for *constructing* a single δ from the infinitely many δ_a. Not only is uniform continuity a stronger notion, it is the more desirable version of continuity since it is the one most useful in studying convergence and integrability. It turns out that the usual proofs that the basic functions of analysis are pointwise continuous also prove that these functions are uniformly continuous on appropriate intervals. We exploit this fact from the very beginning and only use the stronger and more important uniform version of continuity.

We take a similar approach to differentiability. Instead of talking about the derivative of a function at a point, we talk about the derivative *function* on an interval. As with uniform continuity, this notion of uniform differentiability is the one that is of most importance in later theory and applications. In fact, it is an approach that generalizes readily to vector-valued functions of several variables.

An important consequence of using uniform notions is that they produce transparent proofs of important theorems such as the existence of the Riemann integral and the Fundamental Theorem of Calculus.

The pointwise versions of continuity and differentiability do lead to a number of classical examples of functions which are or aren't continuous or differentiable on various dense or nowhere dense subsets of intervals. Since we are emphasizing uniform notions, these examples are relegated to discussions in an appendix and a few exercises, which can be covered at the discretion of the instructor.

In summary, then, this is neither a text in numerical analysis nor one intended solely to prepare students to be professional mathematicians. It is a thoroughly rigorous modern account of the theoretical underpinnings of calculus; and, being constructive in nature, every proof of every result is direct and ultimately computationally verifiable (at least in principle). In particular, existence is never established by showing that the assumption of non-existence leads to a contradiction. By looking through the index or table of contents, you'll see that nothing of importance for undergraduates has been left off or compromised by our approach. The payoff of the constructive approach, however, is that it makes sense—not just to math majors, but to students from all branches of the sciences.