PREFACE

This book is intended for students of mathematics, physics, and engineering at the advanced undergraduate level or beyond. It is primarily a text for a course at the advanced undergraduate level, but I hope it will also be useful as a reference for people who have taken such a course and continue to use Fourier analysis in their later work. The reader is presumed to have (i) a solid background in calculus of one and several variables, (ii) knowledge of the elementary theory of linear ordinary differential equations (i.e., how to solve first-order linear equations and second-order ones with constant coefficients), and (iii) an acquaintance with the complex number system and the complex exponential function $e^{x+iy} = e^x(\cos y + i \sin y)$. In addition, the theory of analytic functions (power series, contour integrals, etc.) is used to a slight extent in Chapters 5, 6, 7, and 9 and in a serious way in Sections 8.2, 8.4, 8.6, 10.3, and 10.4. I have written the book so that lack of knowledge of complex analysis is not a serious impediment; at the same time, for the benefit of those who do know the subject, it would be a shame not to use it when it arises naturally. (In particular, the Laplace transform without analytic functions is like Popeye without his spinach.) At any rate, the facts from complex analysis that are used here are summarized in Appendix 2.

The subject of this book is the whole circle of ideas that includes Fourier series, Fourier and Laplace transforms, and eigenfunction expansions for differential operators. I have tried to steer a middle course between the mathematics-for-engineers type of book, in which Fourier methods are treated merely as a tool for solving applied problems, and the advanced theoretical treatments aimed at pure mathematicians. Since I thereby hope to please both the pure and the applied factions but run the risk of pleasing neither, I should give some explanation of what I am trying to do and why I am trying to do it.

First, this book deals almost exclusively with those aspects of Fourier analysis that are useful in physics and engineering rather than those of interest only in pure mathematics. On the other hand, it is a book on *applicable* mathematics rather than *applied* mathematics: the principal role of the physical applications herein is to illustrate and illuminate the mathematics, not the other way around. I have refrained from including many applications whose principal conceptual content comes from Subject X rather than Fourier analysis, or whose appreciation requires specialized knowledge from Subject X; such things belong more properly in a book on Subject X where the background can be more fully explained. (Many of my favorite applications come from quantum physics, but in accordance with this principle I have mentioned them only briefly.) Similarly, I have not worried too much about the physical details of the applications studied here. For example, when I think about the 1-dimensional heat equation I usually envision a long thin rod, but one who prefers to envision a 3-dimensional slab whose temperature varies only along one axis is free to do so; the mathematics is the same.

Second, there is the question of how much emphasis to lay on the theoretical aspects of the subject as opposed to problem-solving techniques. I firmly believe that theory — meaning the study of the ideas underlying the subject and the reasoning behind the techniques — is of intellectual value to everyone, applied or pure. On the other hand, I do not take "theory" to be synonymous with "logical rigor." I have presented complete proofs of the theorems when it is not too onerous to do so, but I often merely sketch the technical parts of an argument. (If the technicalities cannot easily be filled in by someone who is conversant with such things, I usually give a reference to a complete proof elsewhere.) Of course, where to draw the line is a matter of judgment, and I suppose nobody will be wholly satisfied with my choices. But those instructors who wish to include more details in their lectures are free to do so, and readers who tire of a formal argument have only to skip to the end-of-proof sign ∎. Thus, the book should be fairly flexible with regard to the level of rigor its users wish to adopt.

One feature of the theoretical aspect of this book deserves special mention. The development of Lebesgue integration and functional analysis in the period 1900–1950 has led to enormous advances in our understanding of the concepts underlying Fourier analysis. For example, the completeness of L^2 and the shift from pointwise convergence to norm convergence or weak convergence simplifies much of the discussion of orthonormal bases and the validity of series expansions. These advances have usually not found their way into application-oriented books because a rigorous development of them necessitates the building of too much machinery. However, most of this machinery can be ignored if one is willing to take a few things on faith, as one takes the intermediate value theorem on faith in freshman calculus. Accordingly, in §3.3–4 I assert the existence of an improved theory of integration, the Lebesgue integral, in the context of which one has (i) the completeness of L^2, (ii) the fact that "nice" functions are dense in L^2, and (iii) the dominated convergence theorem. I then proceed to use these facts without further ado. (The dominated convergence theorem, it should be noted, is a wonderful tool even in the context of Riemann integrable functions.) Later, in Chapter 9, I develop the theory of distributions as linear functionals on test functions, the motivation being that the value of a distribution on a test function is a smeared-out version of the value of a function at a point. Discussion of functional-analytic technicalities (which are largely irrelevant at the elementary level) is reduced to a minimum.

With the exception of the prerequisites and the facts about Lebesgue integration mentioned above, this book is more or less logically self-contained. However, certain assertions made early in the book are established only much later:

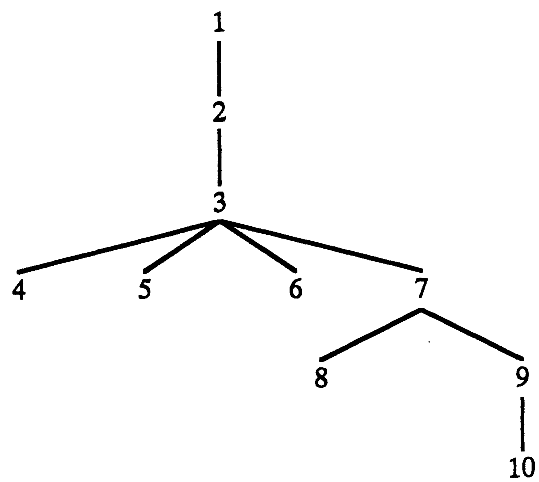
(i) The completeness of the eigenfunctions of regular Sturm-Liouville problems is stated in §3.5 and proved, in the case of separated boundary conditions, in §10.3.

(ii) The asymptotic formulas for Bessel functions given in §5.3 are proved via Watson's lemma in §8.6.

(iii) The proofs of completeness of Legendre, Hermite, and Laguerre polynomials in Chapter 6 rely on the Weierstrass approximation theorem and the Fourier inversion theorem, proved in Chapter 7.

(iv) The discussion of weak solutions of differential equations in §9.5 justifies many of the formal calculations with infinite series in the earlier chapters.

Thus, among the applications of the material in the later part of the book is the completion of the theory developed in the earlier part.



CHAPTER DEPENDENCE DIAGRAM

The main dependences among the chapters are indicated in the accompanying diagram, but a couple of additional comments are in order.

First, there are some minor dependences that are not shown in the diagram. For example, a few paragraphs of text and a few exercises in Sections 6.3, 7.5, 8.1, and 8.6 presuppose a knowledge of Bessel functions, but one can simply omit these bits if one has not covered Chapter 5. Also, the discussion of techniques in §4.1 is relevant to the applied problems in later chapters, particularly in §5.5.

Second, although Chapter 10 depends on Chapter 9, except in §10.2 the only part of distribution theory needed in Chapter 10 is an appreciation of delta functions on the real line and the way they arise in derivatives of functions with jump discontinuities. Hence, one could cover Sections 10.1 and 10.3–4 after an informal discussion of the delta function, without going through Chapter 9.

There is enough material in this book for a full-year course, but one can also select various subsets of it to make shorter courses. For a one-term course one could cover Chapters 1–3 and then select topics *ad libitum* from Chapters 4–7. (If one wishes to present some applications of Bessel functions without discussing the theory in detail, one could skip from the recurrence formulas in §5.2 to the statement of Theorem 5.3 at the end of §5.4 without much loss of continuity.) I have taught a one-quarter (ten-week) course from Chapters 1–5 and a sequel to it from Chapters 7–10, omitting a few items here and there.

One further point that instructors should keep in mind is the following. Most of the book deals with rather concrete ideas and techniques, but there are two places where concepts of a more general and abstract nature are discussed in a serious way: Chapter 3 (L^2 spaces, orthogonal bases, Sturm-Liouville problems) and Chapter 9 (functions as linear functionals, generalized functions). These parts are likely to be difficult for students who have had little experience with abstract mathematics, and instructors should plan their courses accordingly.

Fourier analysis and its allied subjects comprise an enormous amount of mathematics, about which there is much more to be said than is included in this book. I hope that my readers will find this fact exciting rather than dismaying. Accordingly, I have included a sizable although not exhaustive bibliography of books and papers to which the reader can refer for more information on things that are touched on lightly here. Most of these references should be reasonably accessible to the students for whom this book is primarily intended, but a few of them are of a considerably more advanced nature. This is inevitable; the topics in this book impinge on a lot of sophisticated material, and the full story on some of the things discussed here (singular Sturm-Liouville problems, for instance) cannot be told without going to a deeper level. But these advanced references should be of use to those who have the necessary background, and may at least serve as signposts to those who have yet to acquire it.

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