Preface

Graph theory is a fascinating and inviting branch of mathematics. It is full of results, both simple and profound. Natural visual representations of many problems invite exploration by new students and professional mathematicians. The goal of this textbook is to present the fundamentals of graph theory to a wide range of readers.

Unlike many other subjects in mathematics, graph theory is quite new. While results on graphs date back to Euler's solution of the Konigsberg Bridge Problem in 1736, graph theory was first studied as a coherent theory by Julius Petersen in 1891. It remained very small until an explosion of research in the 1960s, and it has grown dramatically since then.

One consequence of this history is that teaching of this subject is much less standard than in older subjects like calculus and linear algebra. Any graph theory text must begin with many definitions in the first chapter. After that, there are a number of common topics that can be covered in almost any order. My goal in ordering these topics was to present the Four Color Theorem, the most significant theorem in the history of graph theory, early enough in the book that any class should have no difficulty in reaching it before the end. This theorem ties together vertex coloring and planarity, so I used it as a bridge between the two topics. It comes after a thorough discussion of coloring, and is used to motivate the idea of planarity.

Connectivity and degeneracy are both important in understanding vertex coloring, so they are covered before it, along with basic topics, such as trees and Eulerian graphs. The topics of isomorphisms and degree sequence characterizations are often covered near the beginning in other textbooks, but they tend to cause students problems. They have been moved back, allowing students to learn more graph theory before tackling them. This also facilitates a somewhat deeper examination of these topics. Other common topics, such as Hamiltonian graphs, matchings and domination, are covered later in the book.

Amongst many possible optional topics, I choose topics for which I have a new perspective, topics I find interesting, and topics that have not appeared in textbooks before. These topics include cores, rigid graphs, generalized vertex colorings, and Nordhaus-Gaddum Theorems.

Distinctive Features. There are many other graph theory textbooks, some very good in parts, yet I find all of them unsatisfactory in some ways. This text has a number of distinct features that make it different, and hopefully better than other texts.

(1) **Current Notation.** The fact that graph theory is relatively new means that there have been many changes in notation and terminology. Every effort has been made to use the most current notation, and when there are multiple notations in use, to use the most natural and instructive notation.

(2) **New Results.** New results are continually proved in graph theory, some settling old conjectures or open problems. This text includes a number of significant recent results, including the proof of Alspach's Conjecture, the proof (for large orders) of the 1-factorization and Hamiltonian decomposition conjectures, progress on the chromatic number of unit distance graphs, and the discovery of a quasipolynomial algorithm for the graph isomorpism problem.

(3) **New proofs.** Theorems including Menger's Theorem, Brooks' Theorem, Turan's Theorem, the Perfect Graph Theorem, Tutte's 1-Factor Theorem, Vizing's Theorem, Nash-Williams' Theorem, and the Nordhaus-Gaddum Theorem form the backbone of a first course in graph theory. However, the original proofs of theorems are often not the best possible. Over time, new proofs of classic results are discovered. Unfortunately, many textbooks recycle the same older proofs of these theorems. I have endeavored to include the shortest, most elegant, most intuitive proofs of these classic theorems, including some that have never before appeared in textbooks. Some alternative proofs of theorems are explored in the Exercises, illustrating different proof techniques.

(4) **Motivation by Applications.** The text begins with a section full of practical problems that can be solved using graph theory. There are also separate sections on applications of trees, vertex coloring, and Hamiltonian graphs. Other major topics are introduced with practical applications that motivate their development. This should encourage students who are not sold on mathematics for its own sake, while also interesting pure mathematicians.

(5) **New Approaches.** Many classic topics are presented in new ways. Some of my approaches are (I believe) better, while other are simply different, and offer new ways to think about old topics. Vertex coloring is approached using degeneracy, which simplifies the proofs of many theorems and provides bounds that are superior to most of the better known theorems presented in other textbooks. Chordal graphs and k-trees are presented as generalizations of trees, which facilitates comparisons between these classes and basic results about them. List coloring, vertex arboricity, and other generalizations of vertex coloring are presented in a chapter that illustrates how thinking about a familiar concept in different ways leads to generalizations and new research questions. Block designs, Ramsey numbers, and Nordhaus-Gaddum Theorems

are presented as decompositions, which helps to illustrate connections between these topics. There are many other new approaches scattered throughout this text.

(6) **Applying Theorems.** Many textbooks present theorems, but do not clearly explain how to use them in practice. I include many examples illustrating how to use basic theorems to solve standard problems. For example, several examples show how to apply the numerous bounds on chromatic number to verify the chromatic number of moderate-sized graphs. There is also a detailed discussion of how to find Hamiltonian cycles (or prove none exist) in sparce graphs.

(7) **Brute Force Solutions.** Whether you like it or hate it, it is a fact that many graph theory problems are solved by brute force (case-checking), that is, generating or checking a large number of graphs to verify some fact. Even when there is an elegant solution to a problem, in practice it is often found by checking many or all relevant graphs, observing a pattern, and then conjecturing and proving a theorem. This is a skill that all graph theorists need, but it is entirely ignored by other textbooks. I include several long examples illustrating this process, and some exercises require brute force solutions.

(8) **Internet Resources.** This textbook does not ignore the existence of the internet. It references appropriate internet resources to help students expand their learning. Such resources include the Online Encyclopedia of Integer Sequences (OEIS), which enumerates many graph classes and other graph theory problems.

(9) **Related Terms.** The internet has made it much easier to look up research papers. But one of the biggest problems in mathematical research is not knowing whether something has already been done and given a different name. Without knowing the name of an existing concept, it is very difficult to search for it. Many sections in this text have a list of related terms immediately before the Exercises. A reader looking to extend or modify concepts in this book should first search for these related terms to see what has already been done.

(10) **Homework.** The text has more than 1200 homework exercises, far more than most books. Most of the Exercises are new. There is a wide range of difficultly in the Exercises, allowing the text to be used at many different levels of difficulty. Especially difficult exercises are marked (+). These are generally not good homework problems for undergraduates, but may make good projects or presentation problems. There are also many types of questions, including simple conceptual questions, applications, evaluating parameters, proofs, case-checking problems, and exploration of related concepts. My goal in writing the Exercises was to ask the sort of questions that one would naturally ask when learning a topic.

(11) **Appendices.** There are some "meta-topics" that recur throughout the book. Rather than awkwardly halt a section to discuss them, they are contained in appendices at the end of the book, so they can be covered at any point in a course, or left for background reading. One appendix discusses proofs,

including common proof techniques and how they are applied in graph theory. Another discusses counting techniques and identities, which have many applications in graph theory. Graph theory contains many problems and algorithms, so there is an appendix on the increasingly important topic of computational complexity. Other appendices cover bounds, extremal graphs, and graph characterizations, which allows a deeper look at these common topics in graph theory.

Using This Book. This book is designed to be used at different levels of difficulty. There are almost no formal mathematical prerequisites, only *mathematical maturity*, that is, exposure to enough mathematics to be able to grasp the concepts presented. A previous course in discrete math would be beneficial. However, the essentials on proofs, counting techniques, and computational complexity are presented in the appendices.

A basic knowledge of matrices, including matrix multiplication, is essential. Somewhat more advanced concepts from linear algebra (rank and determinants) are used in the proof of the Perfect Graph Theorem. The concept of groups is alluded to when discussing automorphisms, Tait coloring, and Steiner triple systems. Other areas of math are occasionally mentioned as asides, but these can easily be avoided when readers are unfamiliar with them.

The following are four distinct levels of difficulty at which this book can be used.

(1) **No Proofs.** This would be a class for undergraduates, perhaps in computer science or math education rather than pure math. The students would not be expected to do proofs. The instructor could present proofs, but often an informal illustration of a theorem would be more useful than a formal proof. Such a class would focus on applications, algorithms, and more practical problems, such as evaluating parameters and finding structures like Eulerian trails and Hamiltonian cycles.

(2) **Introduction to Proofs.** This would be a class for undergraduates who have little or no experience writing proofs. It should begin by covering the appendix on proofs. Proofs should be presented formally, though informal illustrations are often good motivations for formal proofs. Since careful proofs take time, more difficult material may need to be omitted. Many easier proofs should be given as exercises.

(3) **Experienced Undergraduates.** This would be a class for math majors who have already had at least one proof-based class, such as modern algebra or number theory. Some proofs may be presented informally, assuming that students are capable of filling in the details themselves. Some more challenging exercises should be assigned.

(4) **Beginning Graduate Students.** This would be an introductory class for graduate students in math. Applications and the first chapter would be omitted as background or summarized quickly. Most of Chapters 2–9 would be presented, including the most difficult proofs. Difficult exercises would be assigned.

Each of these levels requires a careful selection of which topics to cover, and which to omit. Some professional mathematicians never take a course in graph theory and may be unsure which topics to cover or how different topics are related. The following table (Guide for Using this Book, pp. xiv–xv) describes importance (Imp.) of each topic on a scale of 1 to 5, with 5 being the most important. Less important topics are still good, interesting mathematics, but are less well known and are less likely to come up in other books and papers.

The guide also describes how sections depend on each other. Essential dependence (ED) means that an earlier section or chapter must be covered or well understood for a later section to make sense. Inessential dependence (ID) means that some concepts or theorems from the earlier section are used, but they can be quickly explained or omitted when covering the later section.

The guide also includes a recommendation of how many lectures to devote to each section for each of the four levels of difficultly (L1–L4) described above. Recommendations for a 3-credit (36 lecture) course exclude the parenthesized numbers. Recommendations for a 4-credit or two quarter (48 lecture) course include the parenthesized numbers.

Acknowledgments. I dedicate this book to Allen Schwenk, my doctoral advisor, mentor, and friend. I am grateful to him for patiently answering my many questions and providing support for my career. Many of his insights on graph theory are scattered throughout this book.

This book would not be possible without the mathematicians who developed graph theory and the earlier textbooks on this subject. I have been particularly influenced by the texts by West [**2001**] and Chartand and Lesniak [**2005**]. I am also grateful to the reviewers and editors who provided constructive suggestions for improving this text.

Feedback. Comments, corrections, and constructive criticism are welcome. Please inform me of any errors. You can contact me through my website: `allanbickle.wordpress.com`.

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Guide for Using this Book

§§	Title	Imp.	ED	ID	L1	L2	L3	L4
1.1	Graphs as Models	3			1	.5	.5	0
1.2	Representations of Graphs	5		1.1	1	1	1	.5
1.3	Graph Parameters	5	1.2		.5	.5	.5	0
1.4	Common Graph Classes	5	1.2	1.3	1	.5	.5	0
1.5	Graph Operations	5	1.4	1.3	1	1	1	0
1.6	Distance	4	1.4	1.5	1	1	1	.5
1.7	Bipartite Graphs	5	1.6	1.5	.5	1	.5	(.5)
1.8	Generalizations of Graphs	3	1.2	1.6, 1.7	1	.5	.5	0
2.1	Trees	5	1.6	1.5	1	1.5	1.5	1
2.2	Tree Algorithms	3	2.1		1	(1)	1	(.5)
2.3	Connectivity	5	1.6	1.3, 1.5, 2.1	1.5	2	1.5	1
2.4	Menger's Theorem	5	1.8, 2.3		2	2	2	2
3.1	Eulerian Graphs	5	C1	2.2	1.5	1.5	1	1
3.2	Graph Isomorphism	4	C1		2	2	2	2
3.3	Degree Sequences	3	1.3, 1.4	1.5	(1)	(1)	(2)	(2)
3.4	Degeneracy	4	C1	2.1, 2.4, 3.3	1	1	1.5	2
4.1	Applications of Coloring	2	C1		1	(.5)	1	0
4.2	Coloring Bounds	5	2.3, 3.4		3	3	2	2
4.3	Coloring and Operations	4	4.2	2.3	1	(1)	1	1
4.4	Extremal k-Chromatic Graphs	5	4.2	4.3	1	1	1	1
4.5	Perfect Graphs	4	4.2	3.4	1	(2)	(2)	2
5.1	Four Color Theorem	5	4.2	5.2	1	1	1	1
5.2	Planar Graphs	5	C1	2.1	2	2	2	2

Guide for Using this Book (continued)

§§	Title	Imp.	ED	ID	L1	L2	L3	L4
5.3	Kuratowski's Theorem	4	5.2	2.3, 5.1	1	1	2	2
5.4	Dual Graphs and Geometry	3	5.2	4.2, 5.3	2	2	2	2
5.5	Genus of Graphs	3	4.2, 5.2	5.3	(2)	0	(2)	1
6.1	Finding Hamiltonian Cycles	5	2.3, 2.4	1.3, 3.1	2	2	1.5	1.5
6.2	Hamiltonian Applications	3	6.1		1	.5	0	0
6.3	Hamiltonian Planar Graphs	3	5.2, 6.1		(.5)	.5	.5	.5
6.4	Tournaments	3	1.8, 6.1		0	(.5)	(.5)	0
7.1	Bipartite Matchings	5	C1	2.4	2	2	2	2
7.2	Tutte's 1-Factor Theorem	3	7.1	3.1	(1)	(2)	(1.5)	2
7.3	Edge Coloring	4	7.1	4.2	1	(1.5)	1	2
7.4	Tait Coloring	3	7.3	5.1, 5.2, 6.3	(1)	(1)	1	1
7.5	Domination	3	2.1	3.4, 7.1	(2)	(1.5)	(2)	(1)
8.1	List Coloring	3	4.2, 5.2		0	0	0	1
8.2	Vertex Arboricity	2	2.1, 4.2	6.3	0	0	0	(.5)
8.3	Grundy Numbers	1	4.2		0	0	0	(.5)
8.4	Distance and Sets	1	4.2		0	0	0	(1)
9.1	Decomposing Complete Graphs	3	C1	5.2, 5.5, 6.1	(2)	0	(2)	(3)
9.2	General Decompositions	2	2.1	3.4	0	0	0	(1)
9.3	Ramsey Numbers	3	C1	2.1, 3.4, 4.2	0	0	0	2
9.4	Nordhaus-Gaddum Theorems	2	3.4, 4.2		0	0	0	(2)
10.1	Proofs	5			0	3	0	0
10.2	Counting Techniques	5		10.1	(2.5)	2	2	0