Chapter 1

Basics of Graphs

1.1. Graphs as Models

Social networks have become increasingly prominent in modern life. On the internet, a social network has many people as members. These people are friends with some people and not with others. This is a virtual version of real-life friendships. Instead of friendship, the relationship between people could be acquaintance or it could be biological relation.

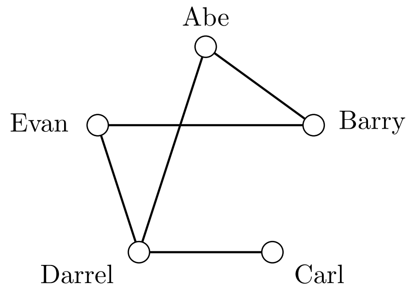
In other industries, more distinct relationships are possible. In the entertainment industry, we can ask whether two people have ever collaborated on some project. In academia, a natural relationship to consider is whether two people have ever coauthored a paper.

There are many questions we can ask about social networks. Who has the most friends? What is the largest number of people who are all friends? Given two people, what is the smallest number of people required to "connect" them?

Rather than try to solve equivalent problems separately in different contexts, we should use a mathematical model that can describe all of these situations. Then we can solve problems once in this abstract setting and apply the results to many different real-world problems.

We can model a social network by drawing a dot to represent each person and a line between two dots when two people are friends.

Example. Abe is friends with Barry and Darrel. Barry is friends with Abe and Evan. Carl is friends with Darrel. Darrel is friends with Abe, Carl, and Evan. Evan is friends with Barry and Darrel. These friendships are illustrated in the figure below.



The same model works just as well for other relationships between people, and many other real-world problems.

Definition 1.1. A **graph** G is a mathematical object consisting of a finite non-empty set of objects called **vertices** $V(G)$ (the **vertex set**), and a set of **edges** $E(G)$ (the **edge set**). An **edge** is two-element subset of the vertex set.

We commonly use G and H for graphs; u, v, $w, \ldots$, for vertices; and e and f for edges. An edge $e = \{u, v\}$ will typically be written uv or vu, dropping the inconvenient braces.

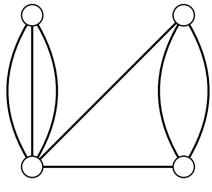
The name "graph" should not be confused with the graph of a function, an unrelated mathematical concept. The term "network", which is common in computer science and technology, would probably be a more intuitive name. However, "graph" is now standard in mathematics. The terms "vertex" and "edge" come from geometry, as they can be used to represent the geometric objects with the same names.

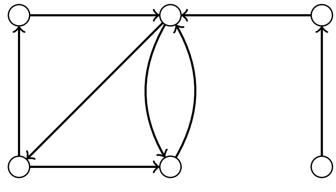
Several variations on the concept of a graph are possible, allowing multiple edges or directed edges. If we allow multiple edges between vertices, the edges must be in a **multiset**, which allows multiple copies of the same object. For directed edges, we replace unordered pairs with ordered pairs of vertices.

Definition 1.2. A **multigraph** G is a mathematical object consisting of a finite nonempty set of objects called vertices $V(G)$ and a multiset $E(G)$ of pairs of vertices.

A **directed graph** (digraph) D is a mathematical object consisting of a finite nonempty set of objects called vertices $V(D)$ and a set $E(D)$ of ordered pairs of distinct vertices called **directed edges**.

A **directed multigraph** replaces the set $E(D)$ with a multiset.





Examples of a multigraph and a digraph are shown above. Every graph is also a multigraph. A multigraph is allowed, but not required, to have multiple edges between pairs of vertices.

Each of these mathematical objects can be used to model many real-world situations. Which one is chosen depends on whether multiple or directed edges make sense in the context of the problem.

Graphs can model transportation networks.

Example. A network of roads can be modeled using graph theory. Vertices represent intersections, and edges represent road segments. Often, a graph will be sufficient to model this situation. However, some areas have one-way streets, which should be represented by directed edges. Thus a digraph is the appropriate model in this situation. Sometimes there may be more than one road between the same pair of intersections. In this case, we would use a multigraph with multiple edges between some vertices.

Example. Airplane flights can be modeled with vertices representing airports. If edges represent flights, they are directed and likely multiple. Edges could alternatively represent the existence of a regular flight between two airports. In this case, the edges are not multiple.

Graphs also model communication and information networks.

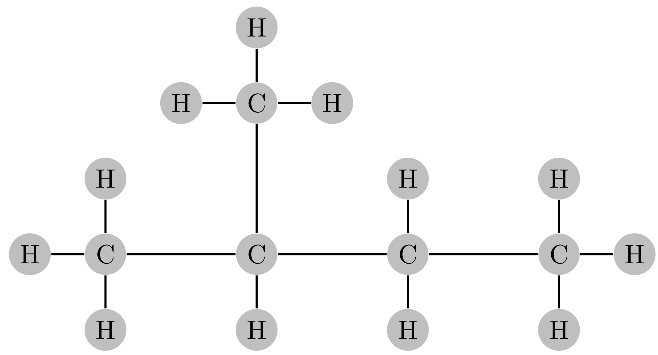
Example. A computer network has several computers connected by cables. A graph modeling this situation has vertices represent computers and edges represent cables.

Example. The **web graph** is a digraph that models the internet. Vertices represent webpages. A directed edge represents when one website links to another. The web graph is very large and growing. It changes as sites are added or deleted, and as links are added or deleted. Many of the other graphs modeling real-world situations also change over time.

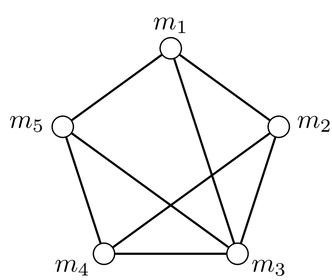
Example. A **citation graph** has vertices representing documents such as academic papers, patents, or legal opinions. A directed edge goes to the cited document from the document that cites it. Citations point toward documents further back in time. However, it is possible for two academic papers to both cite each other if they were written by the same author or if their authors corresponded while they were being written.

Graphs can model topics as diverse as chemical molecules and scheduling meetings.

Example. A chemical molecule has several atoms that are held together by chemical bonds. A graph models this situation with vertices for atoms and edges for chemical bonds. Vertex and edge labels may be necessary to identify different atoms and different types of bonds. A graph representing isopentane, C_5H_{12}, is shown below.



Example. Eight businessmen must have five meetings. The sets of people in meetings are $m_1 = \{1, 2, 3, 4\}$, $m_2 = \{1, 5, 6\}$, $m_3 = \{2, 5, 7\}$, $m_4 = \{6, 7, 8\}$, and $m_5 = \{3, 4, 7, 8\}$. We draw a graph where vertices represent meetings and there is an edge between vertices when the sets have a nonempty intersection. We would like to schedule the meetings in as few time slots as possible. Observation shows that meetings 1 and 4 can be scheduled in one slot, 2 and 5 in another, and 3 in a third. Meetings 1, 2, and 3 all need separate slots, so three slots are required. The graph in this example is called an **intersection graph**, which has vertices represent sets and edges between vertices when the sets have a nonempty intersection.



The situations that can be modeled with graph theory are seemingly endless. You can even construct a graph whose vertices represent other graphs! Many more situations that can be modeled with graph theory are explored in the Exercises.

1.2. Representations of Graphs

There are several terms associated with a graph containing an edge $e = uv$.

Definition 1.3. If $e = uv$ and $f = uw$ are edges of a graph, then we write $u \leftrightarrow v$, and we say u and v are **adjacent**, u and v are **neighbors**, u is **adjacent to** v, e **joins** u and v, u is **joined to** v, e and v are **incident**, and e and f are **adjacent**. If u and v are not adjacent, we write $u \not\leftrightarrow v$, and we say u and v are **nonadjacent.**

We can describe a graph by simply listing the vertex and edge sets. While this description is accurate, it tends not to be particularly helpful for solving problems or making new discoveries. There are alternative ways of describing graphs.

Definition 1.4. An **adjacency list** of a graph is a list of each vertex in a separate row followed by a list of the vertices that it is adjacent to. An **adjacency matrix**

of a graph G with vertex set $\{v_1, \ldots, v_n\}$ is an $n \times n$ matrix with i, j entry 1 when $v_i v_j \in E(G)$, and 0 otherwise.

The adjacency matrix of a graph is symmetric and has 0's on the diagonal.

Example. Let G be the graph with vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and edge set $E(G) = \{12, 13, 23, 24, 34, 45\}$. The adjacency list and adjacency matrix are given below.

Vertex	Adjacent to
1	2, 3
2	1, 3, 4
3	1, 2, 4
4	2, 3, 5
5	4

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For sparse graphs (those with relatively few edges), an adjacency list uses less space than an adjacency matrix. For dense graphs (those with many edges), the space usage is comparable. Each has advantages and disadvantages for computational efficiency. However, the adjacency matrix has more theoretical uses, since the theory of linear algebra can be applied. For example, in Section 1.6, it is used to count walks in graphs.

It is often beneficial to represent a graph using a drawing. Indeed, much of the appeal of graph theory comes from fact that graphs can be drawn and analyzed visually.

Definition 1.5. A **drawing of a graph** is a diagram with a small circle (either open or solid) in the plane representing each vertex and a curve (often a straight line) joining two circles representing each edge.

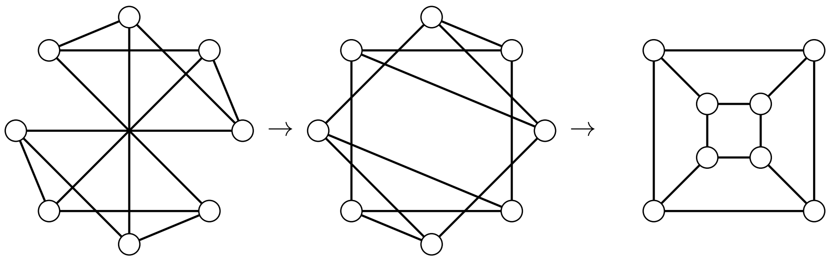
The names of the vertices can be written near them in the drawing. Less commonly, the names of edges can also.

But how should a graph be drawn? At a minimum, vertices must have distinct locations, and edges must not intersect vertices to which they are not incident. Edges may cross, but usually they should not cross more than once. But there are still (infinitely) many ways to draw a graph. We may prefer a drawing that has other desirable properties, such as no edge crossings (if possible), reflective or rotational symmetry, and no long edges. There is no one best drawing; different drawings may emphasize different properties of a graph. The field of **graph drawing** studies how best to visually represent graphs.

One way to satisfy the essential conditions above is to place all the vertices on a circle and make all the edges straight lines that are chords of the circle. Some computer programs use this method when not given any information about where to place the vertices. This method tends to lead to many edge crossings and long edges, however.

Another possibility is to start with one vertex, draw its neighbors close to it, try to draw their neighbors close to them, and so on. No matter what method is used, it usually takes several attempts to find a desirable drawing. An undesirable drawing can be improved by moving one or more vertices to another location and repeating this step until a better drawing is found.

Example. Suppose we draw the graph below left, with its vertices initially around a circle. We notice that the graph contains several 4-cycles. We swap two pairs of vertices to unravel two 4-cycles, obtaining the middle drawing. Finally, we move one 4-cycle inside the other so that no edges cross, obtaining the drawing at right. We obtain a symmetric, visually pleasing drawing.



The fact that a graph can be drawn in many different ways raises the issue that two graphs that appear to be different may be essentially the same. One way this may be is that the vertex sets are different, but the vertices can be renamed so that the graphs are the same. Equivalently, the graphs can be drawn identically except for the vertex names. Another possibility is that the vertices may have the same names, but the graphs may be drawn differently.

We say that two such graphs are **isomorphic**. Loosely speaking, this means that the vertices can be renamed or moved so that the graphs are the same. It is usually not too hard to tell whether or not two graphs are the same, but finding a method that always works is a difficult problem. We consider this problem in depth in Section 3.2, where we also state a precise definition of **isomorphism**.

It is possible to draw the diagram of a graph without naming its vertices. We call this an **unlabeled graph**. An unlabeled graph can be viewed as representing all possible labeled graphs with the same structure. Two unlabeled graphs are isomorphic if it is possible to label their vertices so that they are the same. If two unlabeled graphs G and H are isomorphic, we write $G = H$.

We will refer to a graph with named vertices as a **labeled graph**. Without additional context, "graph" will mean an unlabeled graph. Practical applications usually involve labeled graphs, while theoretical problems more commonly involve unlabeled graphs. When we work with a labeled graph that does not derive from a specific application, we simply need n distinct names for the vertices, where n is the number of vertices in the graph. The natural choice is to use the set $[n] = \{1, \ldots, n\}$ as the vertex set.

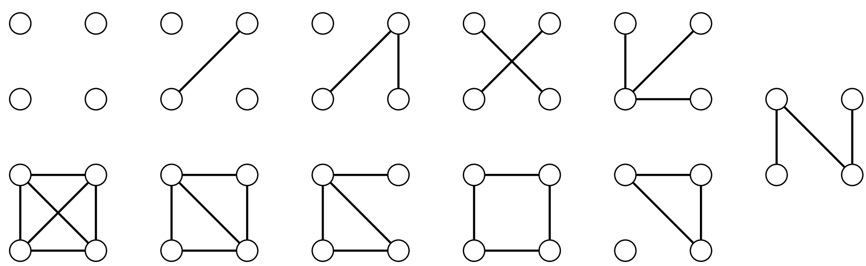
Example. Suppose we want to count the number of labeled graphs with vertex set $[n]$. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of vertices, each of which may be joined by an edge or not. There are two possibilities for each pair, and the choices are independent. Thus there are $2^{\binom{n}{2}}$ labeled graphs with vertex set $[n]$.

Basic counting techniques, including those just used, are discussed in Section 2 of the Appendix.

There are $n!$ ways to permute the vertices of a labeled graph. Thus there are at least $\frac{2^{\binom{n}{2}}}{n!}$ unlabeled graphs. This is not exact since the number of labeled graphs corresponding to each unlabeled graph varies. Using generating functions, it is possible to generate the sequence of the number of unlabeled graphs with n vertices: 1, 2, 4, 11, 34, 156, 1044, 12346, ... (sequence A000088 in the Online Encyclopedia of Integer Sequences (OEIS)). This sequence is asymptotic to $\frac{2^{\binom{n}{2}}}{n!}$.

Example. To find all unlabeled graphs with four vertices, we could just start drawing graphs. But then we could not be certain that we had found all of them. This is one of many problems in graph theory that require a careful "case-checking" argument to be certain that all solutions have been found. We need to break down the possibilities using the properties of the graphs.

With four vertices, the number of edges must be between 0 and 6. There is only one graph when the number of edges is 0, 1, 5, or 6. When there are two edges, they can either be adjacent or not. When there are four edges, the two missing edges can either be adjacent or not. When there are three edges, they could all be incident with the same vertex. When two are adjacent, the third can either be adjacent to both of their other ends, or only one. Thus there are 11 graphs with four vertices (graphed below), and we can be confident we have them all.



1.3. Graph Parameters

We need simple notation for the number of vertices and edges of a graph.

Definition 1.6. The **order** $n(G) = |V(G)|$ of a graph G is the number of vertices of G. The **size** $m(G) = |E(G)|$ of a graph G is the number of edges of G.

When the context is clear (typically when there is a single graph under discussion), we will simplify the notation for order and size to n and m, respectively. We will only use n and m to represent order and size (thus the complete graph K_n has order n), but caution is still warranted. For example, the statement "$n(K_n - v) = n - 1$" should be revised to "$n(K_r - v) = r - 1$" or "the order of $K_n - v$ is $n - 1$."

We note in passing that graph theory is a relatively young subject, and its notation is still evolving. Most terminology and notation have changed from past texts and papers. Some notation is still not standardized (for example, the notations p, $|G|$, n, $n(G)$, and $|V(G)|$ are all common for order). Other terms are now standard but have varied in the past. To avoid unnecessary confusion, this text will usually not mention alternate terminology or notation, but readers are urged to use caution when reading other sources.

Order and size are examples of functions that are defined on all graphs.

Definition 1.7. A **graph parameter** $f(G)$ is a function that is from some or all graphs to the real numbers.

Most parameters of interest have integer values. Thinking of parameters as functions is not especially beneficial in graph theory. For one thing, parameters of interest are almost never one-to-one, so inverse functions usually do not exist.

Some parameters are defined on the vertices of a particular graph.

Definition 1.8. The **degree** of a vertex v, written $d_G(v)$ or $d(v)$ when the graph in question is clear, is the number of edges incident with v. A vertex with degree 0 is an **isolated vertex**. A vertex of degree 1 is a **leaf**. An **even vertex** has even degree; an **odd vertex** has odd degree. The **neighborhood** of a vertex v, $N(v)$, is the set of neighbors of v.

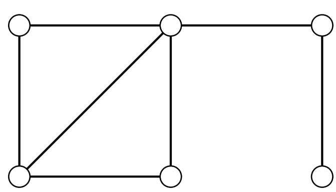
Note that the degree of a vertex is the number of vertices in its row of the adjacency list, and the number of 1's in its row (or column) of the adjacency matrix.

Example. Airlines usually do not fly directly between two smaller cities. Instead, they fly from a smaller city airport to a hub airport, which has flights to many other smaller airports and other hub airports. A hub airport can be identified by having a relatively large degree in a graph representing flights.

We can also consider the degrees of an entire graph.

Definition 1.9. The **degree sequence** of a graph G is the list of its degrees, usually written in nonincreasing order. Its **minimum degree** is $\delta(G)$. Its **maximum degree** is $\Delta(G)$. It is **regular** if every vertex has the same degree (k**-regular** if the common degree is k). A 3-regular graph is a **cubic graph**.

Example. The following graph G has degree sequence 4, 3, 2, 2, 2, 1, $\Delta(G) = 4$, and $\delta(G) = 1$.



There are many relationships among graph parameters. The following theorem is basic enough that it is called the First Theorem of Graph Theory.

Theorem 1.10 (First Theorem of Graph Theory). *If G is a graph, then* $\sum d(v_i) = 2m$.

Proof. Consider the set of all vertex-edge incidences (the "ends" of edges) in a graph. Partitioning the set by vertices shows its cardinality is the sum of the degrees. Partitioning the set by edges shows each edge appears twice, so its cardinality is $2m$. Thus $\sum d(v_i) = 2m$. □

This theorem is also known as the **degree sum formula**. The proof uses the technique known as *counting two ways*, which partitions a set two different ways and counts both ways to obtain an identity. This technique is explored in detail in Section 2 of the Appendix on counting techniques.

Corollary 1.11. *Every graph has an even number of vertices of odd degree.*

Proof. Assume to the contrary that a graph has an odd number of odd vertices. Then its degree sum must be odd, which contradicts the First Theorem. □

Both this corollary and the First Theorem are sometimes called the **Handshaking Lemma**. The results are interpreted involving people at a party shaking hands. With vertices representing people and edges representing handshakes, it follows that an even number of people shake an odd number of hands.

Several other observations follow immediately from the First Theorem. The average degree of a graph is $\frac{2m}{n}$. Hence $\delta(G) \leq \frac{2m}{n} \leq \Delta(G)$. Also, the size of a k-regular graph with order n is $m = \frac{nk}{2}$.

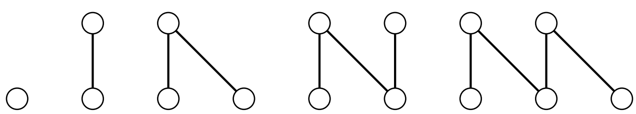
1.4. Common Graph Classes

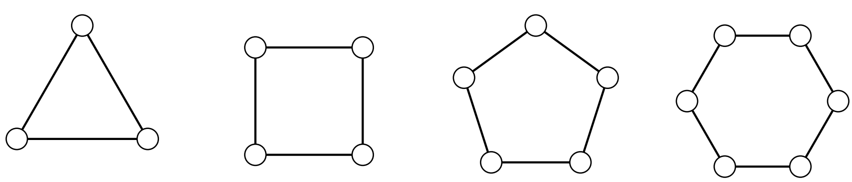
Definition 1.12. A **graph class** is a set of graphs. We denote a graph class using blackboard bold ($\mathbb{G}$ or $\mathbb{H}$) if it does not have its own notation.

We now introduce several important graph classes.

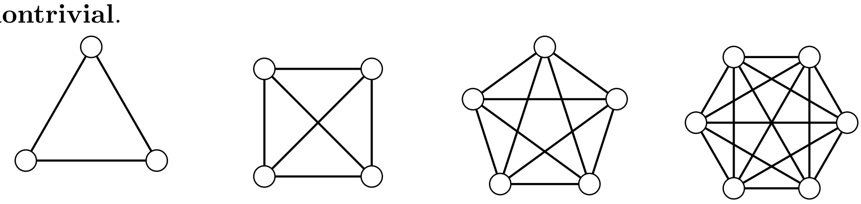
Definition 1.13. A **path** P_n is a graph whose vertices can be numbered $v_1, v_2, \ldots,$ v_n so that its edges are $v_1v_2, \ldots, v_{n-1}v_n$. A **cycle** C_n (or n-cycle) is a graph whose vertices can be numbered $v_1, v_2, \ldots, v_n$ so that its edges are $v_1v_2, \ldots, v_{n-1}v_n$, and v_nv_1. An **even cycle** has n even, and an **odd cycle** has n odd.

Small paths and cycles are illustrated in the following figures.





Definition 1.14. A **complete graph** K_n has order n and every pair of vertices is adjacent. An **empty graph** $\overline{K}_n$ has order n and no edges. The (unique) graph with one vertex K_1 is called the **trivial graph**. A graph with more than one vertex is **nontrivial**.



The complete graph K_n has size $\binom{n}{2} = \frac{n(n-1)}{2}$ since every pair of vertices produces an edge. Complete and empty graphs are opposite extremes for the size of a graph with n vertices.

These graph classes are interesting in their own right. When we study an unfamiliar parameter or property of graphs, we usually apply it to these classes first, since they are familiar and easy to define. However, these classes are also important because they are contained in other graphs.

Many mathematical objects contain subobjects. Sets contain subsets. Groups contain subgroups. Vector spaces contain subspaces. The same is true for graphs.

Definition 1.15. A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ so that the edges in $E(H)$ use only vertices in $V(H)$. We write $H \subseteq G$ and say G **contains** H. An **induced subgraph** is a subgraph $G[S]$ with vertex $S \subseteq V(G)$ and all edges with both ends in S.

Equivalently, an induced subgraph can be obtained by deleting a set of vertices and all edges incident with them.

Definition 1.16. A **clique** is a complete subgraph, or the set of vertices inducing a complete subgraph. An **independent set** of vertices is a set that induces an empty graph.

Both of these concepts are used heavily in graph coloring and elsewhere in graph theory.

Graphs that do not contain certain subgraphs are also of interest.

Definition 1.17. A graph G is H**-free** if it does not contain any induced subgraph isomorphic to H.

The graph K_3 is called a **triangle**. A K_3-free graph is called **triangle-free**.

Paths are relevant as subgraphs when discussing connectivity and distance (Section 1.6). Cycles are relevant as subgraphs in Eulerian and Hamiltonian graphs, planar graphs, and more.

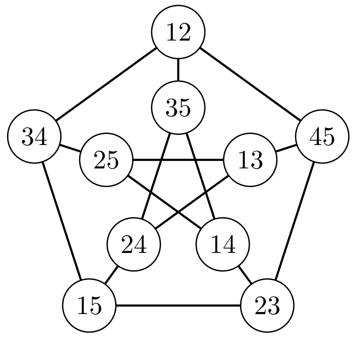
Definition 1.18. A graph, subgraph, or structure of a graph is **maximal** if no larger one contains it. It is **maximum** if it is as large as possible. The definitions of **minimal** and **minimum** are similar.

Lemma 1.19. *If a graph G has $\delta(G) \geq 2$, then it contains a cycle.*

Proof. Let P be a maximal path in G with vertices $v_1, \ldots, v_n$. Now v_n has degree at least 2, so it is adjacent to some vertex $u \neq v_{n-1}$. If u is not on the path, it could be extended, contradicting its maximality. Thus u must be on the path, forming a cycle. □

The distinction between *maximal* and *maximum* is important. Any maximum structure is maximal, but the converse may not be true. A maximal path may not be maximum. The same is true for cliques and independent sets.

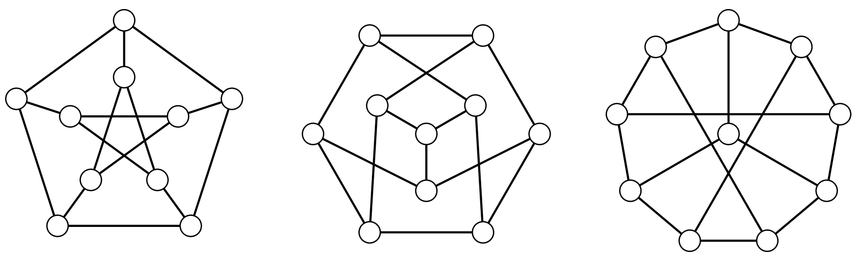
Definition 1.20. The **Petersen graph** has vertices that represent the 2-element subsets of $[5] = \{1, 2, 3, 4, 5\}$. Edges join vertices whose subsets are disjoint.



The Petersen graph is a particularly important example in graph theory. It has several surprising properties that will be revealed throughout this text. In fact, there is an entire book about it (Holton/Sheehan [**1993**])!

There are $\binom{5}{2} = 10$ 2-element subsets of [5], so the Petersen graph has order 10. There are three elements of [5] not contained in a given 2-element subset, so each vertex is adjacent to three others. Thus the Petersen graph is cubic, so it has size $\frac{10 \cdot 3}{2} = 15$. There cannot be three disjoint 2-element subsets of [5], so the Petersen graph is triangle-free.

It can be drawn to emphasize two 5-cycles with vertices 12, 34, 51, 23, 45 and 13, 52, 41, 35, 24, respectively, with edges matching vertices on each cycle. Other drawings of the Petersen graph emphasize other interesting properties.



Proposition 1.21. *Two nonadjacent vertices of the Petersen graph have exactly one common neighbor. Thus the Petersen graph has no* 4*-cycle.*

Proof. Nonadjacent vertices have sets with one element in common. Thus there are two elements of [5] in neither of them, which produces one vertex adjacent to both. A 4-cycle would require nonadjacent vertices with two common neighbors. $\square$

Definition 1.22. The **girth** of a graph is the length of its shortest cycle.

Thus the Petersen graph has girth 5.

Counting different types of graphs or subgraphs is a common problem in graph theory. The following result uses the counting technique known as counting by bijection (Appendix, Section 2).

Proposition 1.23. *The Petersen graph contains* 15 8*-cycles.*

Proof. Any 8-cycle omits two vertices. They must be adjacent, or else they would have a common neighbor that could not be on the cycle. Deleting two adjacent vertices from the Petersen graph results in a graph that clearly contains a single 8-cycle. The Petersen graph has 15 edges and, hence, 15 pairs of adjacent vertices, so it has 15 8-cycles. □

There are several classes of graphs that generalize the Petersen graph, which are explored in the Exercises.

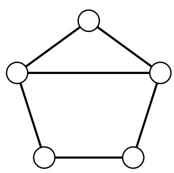
1.5. Graph Operations

Almost any type of mathematical object has corresponding mathematical operations to manipulate the objects and produce new objects. Numbers can be negated, added, or divided. Functions can be multiplied or composed. Sets can be complemented or intersected. Thus it is natural that analogous operations exist for graphs.

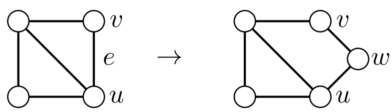
A **unary operation** produces a new graph when given a single graph.

Definition 1.24. The graph $G-e$ or $G-X$ is obtained by deleting edge e or edge set X from $E(G)$. The graph $G-v$ or $G-S$ is obtained by deleting vertex v or vertex set S and all incident edges from G. The graph $G+e$ is obtained by adding e to the edges of G.

When G is an unlabeled graph, $G-e$ will only be defined when a particular edge is specified or when deleting any edge produces the same graph. For example, $C_n - e = P_n$ no matter which edge is deleted. The same will hold for vertices. When e is unspecified, $G+e$ will only be defined when any choice yields the same graph. The graph C_5 satisfies this condition, so C_5+e is defined.



Definition 1.25. A **subdivision of an edge** $e = uv$, deletes e and adds vertex w and edges uw and wv. A graph H is a **subdivision of a graph** G if it can be obtained by some number (perhaps zero) of subdivisions of edges of G.



Loosely speaking, a subdivision can be thought of as inserting a vertex on an edge. Subdivisions are important when we are mainly interested in whether there is a path between vertices of a graph, regardless of length. This includes connectivity (Section 2.3) and planarity (Section 5.2). Note that a subdivision of an edge increases the order and size by one.

The preceding operations can be considered “local” operations that leave most of the graph unchanged. Unary operations can also change the entire graph.

Definition 1.26. The **complement** $\overline{G}$ of a graph G has the same vertex set and edge $uv \in V\left(\overline{G}\right)$ if and only if $uv \notin V(G)$. A **decomposition** of G is a set of nonempty subgraphs whose edge sets partition $E(G)$. The subgraphs are said to **decompose** G.

Note that an empty graph is the complement of a complete graph of the same order (and vice versa), explaining the notation $\overline{K}_n$ for empty graphs. A graph G of order n and its complement decompose K_n.

Example. The complement of C_4 is $2K_2$, so $\{C_4, 2K_2\}$ is a decomposition of K_4.

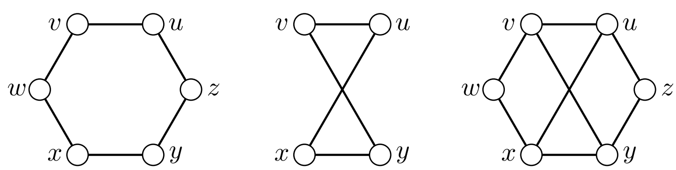
A unary operation may produce a graph with a completely different vertex set. An example of this is the line graph (Section 2.4).

A **binary graph operation** uses two graphs to produce a new graph. We consider several binary operations: the union, join, and Cartesian product of graphs.

Definition 1.27. The **union of graphs** G and H, $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The union is a **disjoint union** if the vertex sets of the graphs are disjoint. A disjoint union of k copies of G is denoted kG.

When we work with unlabeled graphs, we consider a union disjoint.

Example. The union of the two graphs at left is the graph at right.



Empty graphs, complete graphs, and cycles are all regular. In fact, 0-regular graphs are exactly the empty graphs, which can also be denoted nK_1. Any 1-regular graph must be composed of disjoint copies of K_2, that is kK_2. Next we characterize the structure of 2-regular graphs.

Proposition 1.28. *Any 2-regular graph is a disjoint union of cycles.*

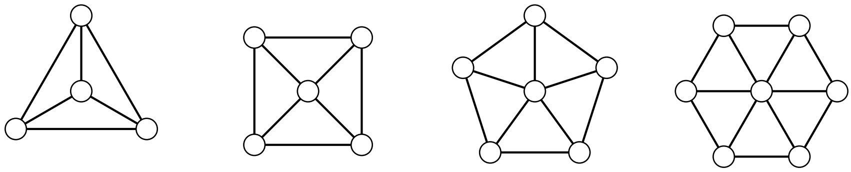
Proof. We use strong induction on the number of cycles of a 2-regular graph G. By Lemma 1.19, G must contain a cycle. If G is a single cycle, we are done. Assume the result holds for graphs with fewer than k cycles. Let G contain k cycles, one of which is C. All of the vertices of C have degree 2 in G. Deleting these vertices produces a 2-regular graph H with fewer than k cycles. Thus H must be a disjoint union of cycles, and hence so is G. $\square$

Strong induction is a common proof technique in graph theory. Section 1 of the Appendix reviews common proof techniques.

The structure of k-regular graphs is much more complicated when $k \geq 3$. This problem is considered further in later sections.

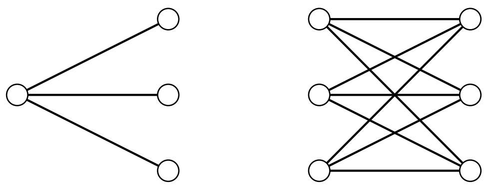
Definition 1.29. The **join of graphs** G and H, $G + H$, is obtained from the disjoint union $G \cup H$ by adding the edges $\{uv : u \in V(G), v \in V(H)\}$.

The **wheel** W_n, $n \geq 4$, is the join of a cycle and a single vertex, $W_n = C_{n-1} + K_1$.



Definition 1.30. The **complete bipartite graph** $K_{r,s}$ is the join of two empty graphs, $K_{r,s} = \overline{K}_r + \overline{K}_s$. A **star** is the complete bipartite graph $K_{1,s}$. The star $K_{1,3}$ is known a **claw** when it is an induced subgraph of another graph.

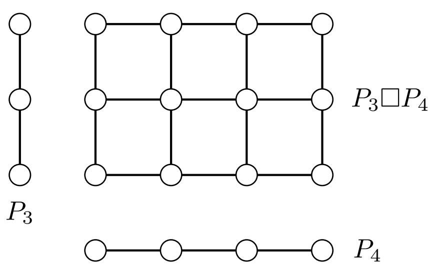
General (noncomplete) bipartite graphs will be defined in Section 1.7. The graphs $K_{1,3}$ and $K_{3,3}$ are shown below.



Definition 1.31. The **Cartesian product** of G and H, $G\Box H$, has vertex set $V(G) \times V(H)$ and (u,v) adjacent to (u',v') if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

The notation $G\Box H$ symbolizes the fact that $K_2\Box K_2 = C_4$. Note that $G\Box H$ has a copy of G for each vertex of H, and vice versa. Thus it decomposes into copies of G and H.

Example. The Cartesian product of P_3 and P_4 is shown below.



Definition 1.32. The **grid** $G_{r,s}$ is the Cartesian product of two paths, $G_{r,s} = P_r\Box P_s$.

The definitions of the union, join, and Cartesian product all extend in the natural way to more than two graphs. (Note that they are all associative and commutative operations.) This can be used to define another important class of graphs.

Definition 1.33. The **hypercube** Q_k is defined recursively by $Q_1 = K_2$ and $Q_k = Q_{k-1}\Box K_2$.

Note that $Q_2 = C_4$ and Q_3 represents the usual three-dimensional cube. The hypercube Q_k can also be defined as a graph with vertices representing k-digit

3.2. Graph Isomorphism

3.2.1. The Isomorphism Problem. Suppose a chemist notices that two molecules have similar properties. He suspects that they may actually be the same molecule. How can he tell?

We briefly introduced the idea of isomorphism between graphs in Section 1.2. Informally, two graphs are isomorphic if they have the same structure, despite having different vertex names or drawings. We now make this idea precise.

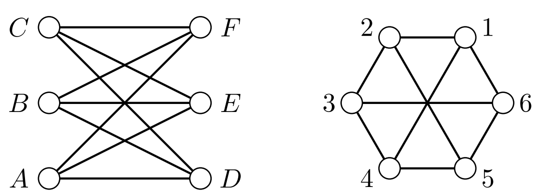
Definition 3.6. An **isomorphism** from graph G to graph H is a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. We say G is **isomorphic to** H and write $G \cong H$ for labeled graphs or $G = H$ for unlabeled graphs. Two graphs are **nonisomorphic** if they are not isomorphic.

It is a straightforward exercise to show that isomorphism of graphs is an equivalence relation. This implies that the relation partitions the collection of all graphs into equivalence classes. An unlabeled graph can be thought of as representing the isomorphism class of all graphs with the same structure.

How can we determine whether two graphs are isomorphic? The most naive answer is to check all bijections between the vertex sets of G and H. Either one is an isomorphism or none are. This is a theoretical answer but not a practical answer. For a graph of order n, there are $n!$ permutations of its vertices. Since the factorial function grows very quickly, the brute force method is not practical for even fairly small graphs.

A somewhat better approach is to pair up one vertex, then try to match up their neighborhoods, and so on until an isomorphism is found or this attempt fails.

Example. Consider the following graphs.

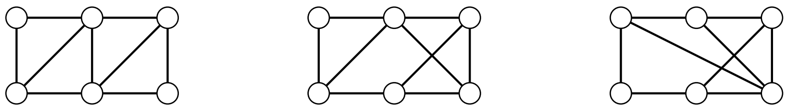


We try to construct an isomorphism by mapping vertex A to vertex 1. Then vertices D, E, and F must map onto $\{2, 4, 6\}$, so map D to 2, E to 4, and F to 6. Then an isomorphism can be completed by mapping B to 3 and C to 5. The graphs are isomorphic; both are $K_{3,3}$.

Definition 3.7. A **graph invariant** is a property of a graph that is preserved by isomorphism.

To show that two graphs are not isomorphic, a much better approach than checking all possible isomorphisms is to find a graph invariant that varies on them. What properties are invariants? It is immediate that the order and size are invariant since a bijection matches up vertices and edges. Similarly, the degree sequence is preserved since every vertex is mapped to a vertex with the same number of neighbors.

Example. Consider the following graphs. All have order 6 and size 9. However, the graph on the left has degree sequence 443322, while the others have degree sequence 433332. The graph in the middle has adjacent vertices with degrees 2 and 4, while the graph on the right does not. Thus all three are nonisomorphic.

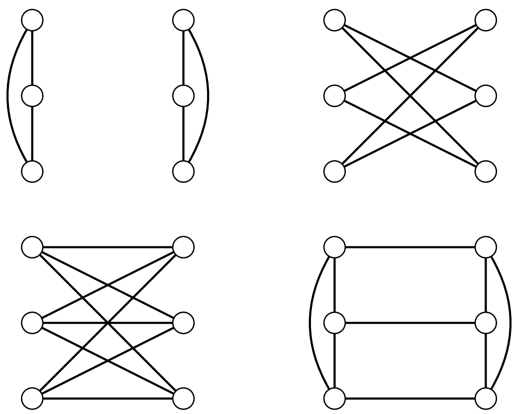


An isomorphism matches both edges and nonedges. This proves the following observation.

Proposition 3.8. *Let G and H be graphs. Then $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.*

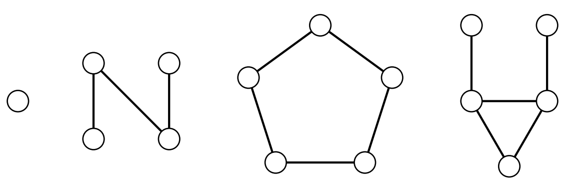
When a graph is dense, it can be easier to analyze its complement.

Example. Consider cubic graphs of order 6. The complement of a 3-regular graph of order 6 is a 2-regular graph of order 6. There are two such graphs, $2C_3$ and C_6. Thus there are two cubic graphs of order 6, $K_{3,3}$ and $K_3 \square K_2$.



Definition 3.9. A graph G is **self-complementary** if $G = \overline{G}$.

Example. The only self-complementary graphs with order at most 5 are K_1, P_4, and C_5, and a graph with order 5 is called the **bull graph**. This can be shown by examining all small graphs.



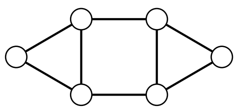
A self-complementary graph of order n has size $m = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$, which must be an integer. Then $4|n$ or $4|n-1$, so $n \equiv 0$ or $n \equiv 1 \bmod 4$. The first few values of the sequence of the number of self-complementary graphs of order n is 1, 0, 0, 1, 2, 0, 0, 10, 36, 0, 0, 720, 5600, ... (OEIS A000171). The structure of self-complementary graphs is explored in the Exercises.

Proposition 3.10. *The existence of a particular subgraph, and the number of occurrences of that subgraph, is a graph invariant.*

Proof. A subgraph is determined by its vertices and edges. Since corresponding vertices and edges are matched by an isomorphism, the existence of a subgraph is invariant. Similarly, each copy of a subgraph must match with a distinct copy in the other graph, so the number of occurrences of a subgraph is invariant. □

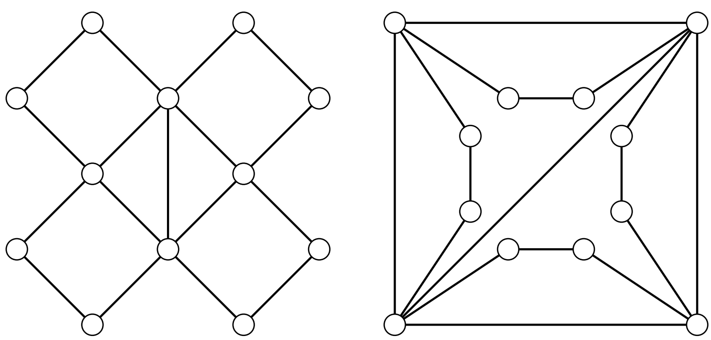
A little thought shows that graph properties that are invariant include maximum and minimum degree, number of components, diameter and radius, bipartiteness, girth, connectivity, and edge connectivity. What is not preserved? A property that depends on the particular drawing of a graph may not be preserved. For example, whether there is an edge crossing in the drawing is not preserved. The length of the regions in a plane drawing (see Section 5.2) is also not preserved.

Example. Two drawings of the same graph are shown below. The drawing at left has regions with lengths 3, 3, 4, and 6, while the drawing at right has region lengths 3, 3, 5, and 5.



When two graphs are isomorphic, isomorphic subgraphs can be used to help find the isomorphism.

Example. Consider the two graphs below. Both contain a single copy of $K_4 - e$. Mapping one of them onto the other quickly produces an isomorphism.



A good strategy to test a pair of graphs for isomorphism starts with their degree sequences. If they are the same, and the graphs are not regular, examine the subgraphs induced by edges between vertices of particular degrees. If these subgraphs are nonisomorphic, neither are the larger graphs. If they are are isomorphic, this provides information on how to find an isomorphism for the larger graphs.

Example. In the graphs above, the subgraphs induced by the vertices of degree 5 are both K_2. The subgraphs induced by the vertices of degrees 4 and 5 are $K_4 - e$. The subgraphs induced by the vertices of degrees 4 and 2 are $2P_5$.

However, this is not helpful when the graphs are regular. Since vertex degrees tell us how many vertices are distance 1 away from a vertex, it may be helpful to determine how many vertices are distance 2 from each vertex. More generally, we could compute the number of walks of distance 2 between each pair of vertices. This can be done simply by squaring the adjacency matrix (Theorem 1.35). Then we can consider the vectors counting the number of distance 2 walks starting at each vertex. Since we don't know the isomorphism (if it exists), sort the numbers in each vector in nonincreasing order.

Example. Consider the graphs with matrices shown below. Inspection quickly reveals that both are cubic graphs of order 6. However, it is not immediately clear whether they are isomorphic.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Now consider squaring the matrices. In the graph at left, each vertex has distance 2 walk vector $(3,3,3,0,0,0)$. In the graph at right, each vertex has distance 2 walk vector $(3,2,2,1,1,0)$. No matter how the vertices are permuted, the rows of the two matrices are different, so the corresponding graphs are nonisomorphic. In fact, they are $K_{3,3}$ and $K_3 \square K_2$ again.

$$\begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 & 1 & 0 & 2 & 2 \\ 1 & 3 & 1 & 2 & 0 & 2 \\ 1 & 1 & 3 & 2 & 2 & 0 \\ 0 & 2 & 2 & 3 & 1 & 1 \\ 2 & 0 & 2 & 1 & 3 & 1 \\ 2 & 2 & 0 & 1 & 1 & 3 \end{bmatrix}$$

If the squares of the adjacency matrix cannot be distinguished, higher powers of it could be analyzed. This will distinguish many, but not all regular graphs.

Definition 3.11. A graph is **strongly regular** if there are integers λ and μ such that every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. A strongly regular graph with order n and degree k is denoted $\operatorname{srg}(n,k,\lambda,\mu)$.

Example. Complete graphs are $\operatorname{srg}(n,n-1,n-2,0)$. The Petersen graph is an $\operatorname{srg}(10,3,0,1)$.

There are nonisomorphic strongly regular graphs with the same parameters. They must have the same distance 2 walk vectors, so this technique will not distinguish them.

Thus the graph isomorphism problem is both easy and hard in different senses. Two graphs are usually easy to tell apart. Two randomly chosen graphs will almost certainly have different orders; two randomly chosen graphs with the same order will almost certainly have different sizes, etc. But some pairs of strongly regular graphs exist which have many properties in common. Yet even these can be distinguished with some work. In knot theory, there are pairs of knots that are not known to be the same or different for some time; this is not the case in graph theory.

The difficulty in the graph isomorphism problem is that there is no algorithm that is known to work efficiently for any pair of graphs. For several decades, the best known algorithms had complexity essentially $\mathcal{O}\left(2^{\sqrt{n}}\right)$. In 2017, Laszlo Babai [**2017**] announced a quasipolynomial time algorithm (that has complexity $2^{\mathcal{O}((\log n)^c)}$ for some $c > 0$). The algorithm uses group theory.

The graph isomorphism problem is in class NP, but is not known to belong to either class P or NP-complete. It has been used to define its own complexity class, GI. The graph isomorphism problem can be solved in polynomial time for some special classes of graphs, including trees.

3.2.2. Applications of Isomorphisms. Isomorphism and graph invariants can be used to classify all graphs of a certain type. We characterized 2-regular graphs in Proposition 1.28. Cubic graphs are much more difficult to classify.

Example. Suppose we want to find all cubic graphs of order 8. Note that if we just start drawing graphs, we cannot be sure we have found them all, and we may draw the same graph more than once without realizing it. Instead, we use a case-checking argument based on graph invariants.

Start by asking whether a cubic graph of order 8 can be disconnected. If so, each component must have order 4, and so must be K_4. Thus we find $G_1 = 2K_4$.

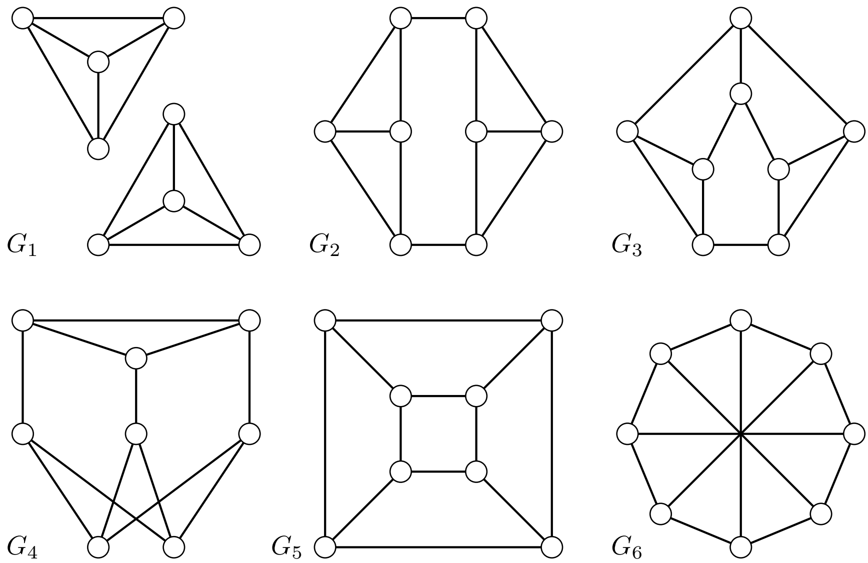
It is easily checked that a cubic graph of order 8 cannot have a cut-vertex or bridge. If it has a 2–edge cut, the components when it is deleted both have degree sequence 3, 3, 2, 2, so they must be $K_4 - e$. Thus we have a unique graph G_2.

The remaining graphs must be 3-connected. If there is an edge cut of three nonadjacent edges, then the orders of the components when they are deleted must be odd since else they would have an odd number of odd vertices. One must be 2-regular with order 3, so K_3. The other must have degree sequence 3, 3, 2, 2, 2. The vertices of degree 3 are either adjacent or not, leading to $C_5 + e$ and $K_{2,3}$. Thus we have found G_3 and G_4.

Any triangle leads to one of the edge cuts in the previous two cases, so we consider triangle-free graphs. If a graph is bipartite, each partite set has four vertices, so each vertex is not adjacent to one in the other set. There are four mutually nonadjacent nonedges, so there is one possible graph G_5. Since the 3-cube is cubic and bipartite with order 8, $G_5 = Q_3$.

If there is a triangle-free nonbipartite graph, suppose it has a 5-cycle. The other three vertices have degree sum 9, and five edges join to the 5-cycle. There must be two edges between the three vertices, so they induce P_3. The two edges from an end of the path join to nonadjacent vertices in the 5-cycle (else there would be a triangle). This leaves only one option. The final graph G_6 can be shown to be a Mobius ladder.

Note that if a graph contains a 7-cycle, the final vertex is adjacent to three vertices on it, creating a smaller odd cycle. The six cubic graphs of order 8 are shown below.



Many of the techniques used in the previous example can be applied to cubic graphs in general. These ideas are explored further in the Exercises.

Isomorphisms can be used to describe the symmetries of a graph.

Definition 3.12. An **automorphism** of G is an isomorphism from G to G. A graph is **vertex-transitive** if for every pair of vertices u and v, there is an automorphism that maps u to v. A graph is **edge-transitive** if for every pair of edges e and f, there is an automorphism that maps e to f.

In a vertex-transitive graph, every vertex resembles every other, so a statement for all vertices can be proved by just looking at one.

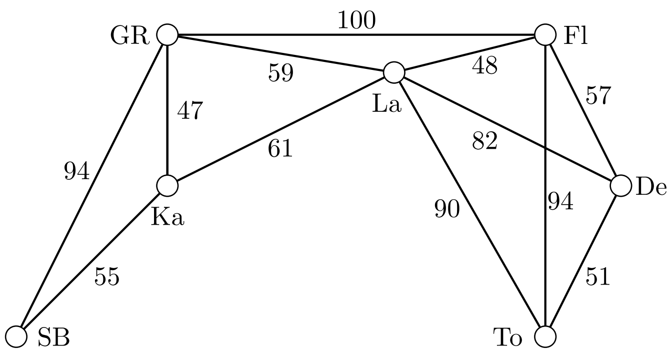
Example. A nontrivial path has two automorphisms. The identity leaves it in place. It can also be reflected so that the ends are interchanged. Thus a nontrivial path has two automorphisms, and only P_2 is vertex-transitive. Both P_2 and P_3 are edge-transitive.

Example. The Petersen graph is defined using 2-element subsets of [5]. Permuting these elements produces $5! = 120$ automorphisms. It follows that the Petersen graph is both vertex-transitive and edge-transitive. The 5-cycle 12-34-51-23-45 must be mapped to a 5-cycle with each two numbers on two nonadjacent vertices. This implies that any automorphism of the Petersen graph comes from permuting elements of [5].

The graph above models the situation. It can be colored using three colors. There are three species that all conflict (e.g., A, B, and E), so three tanks is the minimum acceptable number.

Example. Television stations may not broadcast on the same channel if they are within some distance D of each other. We can model this situation with a graph, where the vertices represent television stations and the edges join stations within D miles. To minimize the number of channels used, we want to minimize the number of colors used in a coloring of this graph.

	De	Fl	GR	La	Ka	To	SB
Detroit		57	141	82	130	51	171
Flint			100	48	109	94	160
Grand Rapids				59	47	141	94
Lansing					61	90	114
Kalamazoo						113	55
Toledo							139
South Bend							



The table above lists distances between several cities. Suppose $D = 100$. The graph above models this situation. It can be colored using four colors.

These examples illustrate the fact that graph coloring has many applications. The rest of this chapter studies how to color graphs efficiently.

4.2. Coloring Bounds

Definition 4.1. A **vertex coloring** of a graph assigns one color to each vertex. A **proper vertex coloring** requires that adjacent vertices are colored differently.

While the colors could be actual colors (red, green, blue, ...), we will typically use natural numbers $1, 2, \ldots, k$.

Definition 4.2. A k**-coloring** of a graph is a proper vertex coloring using **colors** $1, \ldots, k$ (not necessarily all of them). A graph is k**-colorable** if it has a k-coloring. The **chromatic number** $\chi(G)$ is the minimum number of colors used in any k-coloring of a graph G. A graph with $\chi(G) = k$ is said to be k**-chromatic**. A **minimum coloring** of a graph is one using $\chi(G)$ colors. A **color class** is all vertices with the same color in some coloring of the graph.

A k-coloring G can be thought of as a function $f : V(G) \to [k]$. However, using function notation is only occasionally beneficial when discussing graph coloring.

To determine the chromatic number of a graph, it is useful to have bounds that are easier to calculate. It is immediate that

$$1 \leq \chi(G) \leq n.$$

The extremal graph for the lower bound is the empty graph $\overline{K}_n$, since only a graph with no edges can be colored with one color. The extremal graph for the upper bound is K_n, since only a graph with all possible edges requires a different color on each vertex.

To determine the chromatic number exactly, we will need better bounds. The following observation is useful.

Proposition 4.3. *If $H \subseteq G$, then $\chi(H) \leq \chi(G)$.*

Proof. A coloring of G with $\chi(G)$ colors can be restricted to H. $\square$

Definition 4.4. The **clique number** $\omega(G)$ of a graph G is the size of the largest clique of G.

Corollary 4.5. *For any graph G, $\chi(G) \geq \omega(G)$.*

Definition 4.6. The **independence number** $\alpha(G)$ of a graph G is the size of the largest independent set of G.

The independence number and clique number are complementary parameters, since $\omega\left(\overline{G}\right) = \alpha(G)$ and vice versa. The notation suggests that the beginning (α) of a graph is empty and the end (ω) is complete. For small graphs, independence number and clique number can be determined by inspection. For larger graphs, a systematic argument or exhaustive search is required.

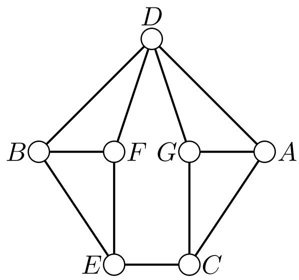
Any color class in a proper vertex coloring is an independent set. A k-coloring partitions the vertex set into k color classes. The chromatic number is the smallest number of independent sets into which $V(G)$ can be partitioned.

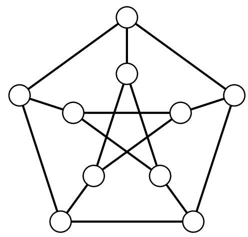
Proposition 4.7. *For any graph G, $\chi(G) \geq \frac{n}{\alpha(G)}$.*

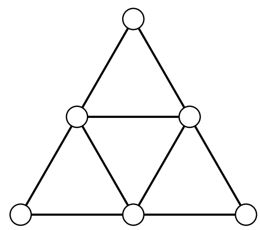
Proof. Let $k = \chi(G)$, so G has color classes $V_1, \ldots, V_k$ for some k-coloring. Then $n = \sum_{i=1}^{k} |V_i| \leq k \cdot \alpha(G)$. Thus $\chi(G) \geq \frac{n}{\alpha(G)}$. $\square$

The two basic lower bounds on the chromatic number are $\omega(G)$ and $\frac{n}{\alpha(G)}$. Which one is better depends on the graph.

Example. Find ω, α, and χ for the following graphs.



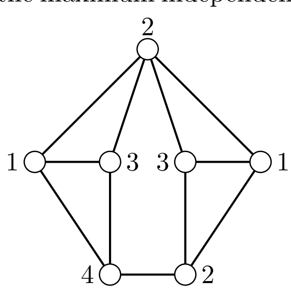


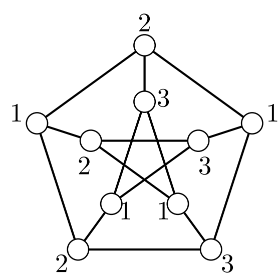


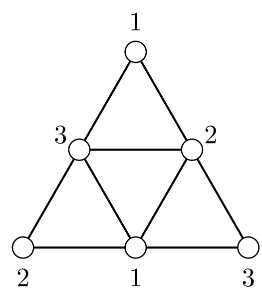
The Moser spindle has $\omega(G) = 3$, with $\{A, D, G\}$ being one of four triangles. It has $\alpha(G) = 2$, with $\{C, D\}$ one of many independent sets of size 2. Corollary 4.5 implies $\chi(G) \geq 3$, and Proposition 4.7 implies $\chi(G) \geq \frac{7}{2} = 3.5$. Since the chromatic number must be an integer, $\chi(G) \geq 4$. A 4-coloring is shown below, so $\chi(G) = 4$.

The Petersen graph has $\omega(G) = 2$ and $\alpha(G) = 4$, with the four vertices colored 1 below being one maximum independent set. The latter implies that $\chi(G) \geq \frac{10}{4} = 2.5$. A 3-coloring is shown below, so $\chi(G) = 3$.

The graph at right has $\omega(G) = 3$ and $\alpha(G) = 3$, with the three vertices of degree 2 being the unique maximum independent set. These imply lower bounds of 3 and $\frac{6}{3} = 2$ for the chromatic number. A 3-coloring is shown below. Note that the maximum independent set cannot be a color class in any 3-coloring of this graph.

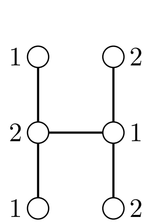


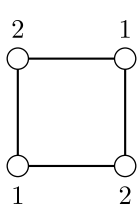


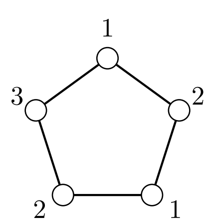


Any nonempty bipartite graph requires exactly two colors, and the color classes are the partite sets. In fact, a graph is 2-colorable if and only if it is bipartite. Thus there is a good characterization of 2-colorable graphs (Theorem 1.44); a graph is 2-colorable if and only if it contains no odd cycle.

Example. Trees are bipartite, so they are 2-colorable. Even cycles have $\chi(C_{2k}) = 2$. Odd cycles have $\chi(C_{2k+1}) = 3$.







We have good characterizations of graphs with chromatic number 1 or 2. Unfortunately, there is no good characterization of graphs with $\chi(G) = k$ when $k \geq 3$. In fact, determining $\chi(G)$ when $k \geq 3$ is an NP-complete problem.

The class of NP-complete problems could all be solved in polynomial time if any can be solved in polynomial time. The fact that these problems have been studied extensively without anyone finding a polynomial time solution for any of them suggests (but does not prove) that no such algorithm exists. See Section 3 of the Appendix for more on computational complexity and NP-complete problems.

Determining α and ω are also NP-complete problems. They are essentially equivalent due to complementation. A naive algorithm would check all 2^n vertex subsets of a graph. A better algorithm (Robson [**1986**]) runs in $\mathcal{O}(1.2108^n)$ time, but no polynomial algorithm is known.

To show that $\chi(G) = k$, we must show

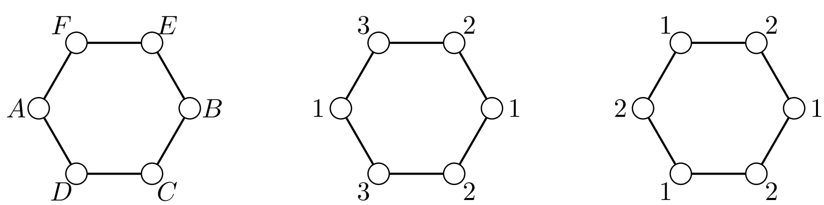
(1) $\chi(G) \geq k$. Use a lower bound, or find a contradiction to show that $\chi(G) < k$ is impossible.

(2) $\chi(G) \leq k$. Find a k-coloring, or use an upper bound (discussed below).

How can we find a k-coloring? Trial and error may work for small graphs, but larger graphs may require a more systematic approach.

Algorithm 4.8 (**Greedy Coloring**). *Given some vertex order, color each vertex with the smallest color that has not already been used on an adjacent vertex.*

Example. Color the vertices of the graph below left in order A–F. The 3-coloring produced is in the center. However, the coloring at right uses only two colors.

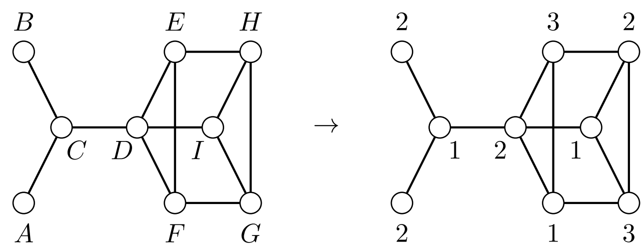


Greedy coloring must produce a proper coloring, but as with many greedy algorithms, it is not guaranteed to produce an optimal solution. How good the coloring is depends on the vertex order used. Some vertex order must produce a minimum coloring, but checking all $n!$ vertex orders is not practical. We need a vertex order that is likely to produce a good coloring, namely a construction sequence (Section 3.4).

Theorem 4.9 (**The Degeneracy Bound**). *For any graph G, $\chi(G) \leq 1 + D(G)$.*

Proof. Greedily color a construction sequence of G. Each vertex has at most $D(G)$ neighbors when colored, so at most $1 + D(G)$ colors are needed. □

Example. A deletion sequence of the graph below left is the vertices A through I in alphabetical order. Greedy coloring using the corresponding construction sequence produces the 3-coloring below right. This is a minimum coloring, one better than the 4-coloring guaranteed by the Degeneracy Bound. Note that beginning the construction sequence with HGI requires four colors.



Since $D(G) \leq \Delta(G)$, an immediate corollary to the Degeneracy Bound is $\chi(G) \leq 1 + \Delta(G)$. The maximum degree bound is more famous, but is often much worse. For nontrivial trees, the Degeneracy Bound gives two—the correct value, while the maximum degree bound may be arbitrarily large. A single vertex of large degree will determine the maximum degree bound, while only a core of many large degree vertices determines the Degeneracy Bound. Several other common upper bounds for $\chi(G)$ are worse than the Degeneracy Bound, and can be proved as corollaries of it (see the Exercises). Nonetheless, the Degeneracy Bound still fails to give good results for some graphs, such as $K_{r,r}$.

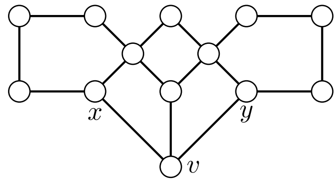
The Degeneracy Bound has been observed many times in various forms. The quantity $1 + D(G)$ has been called the **coloring number** (Erdos/Hajnal [**1966**]) and the **Szekeres-Wilf number** (Szekeres and Wilf [**1968**]). Unfortunately, the Degeneracy Bound is often presented in the confusing form $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$, which seems to imply that all 2^n induced subgraphs of G must be checked. In fact, only one subgraph (the maximum core) must be checked, which can be done in $O(m)$ time.

Among connected graphs, the Degeneracy Bound equals the maximum degree bound only for regular graphs (Proposition 3.31). The next theorem shows which regular graphs equal the maximum degree bound. We start with a lemma.

Lemma 4.10 (Lovasz [**1975**]). *Given $r \geq 3$, if G is an r-regular 2-connected noncomplete graph, then G has a vertex v with two nonadjacent neighbors x and y such that $G - x - y$ is connected.*

Proof. If G is 3-connected, let v be any vertex, and let x and y be two nonadjacent neighbors of v, which must exist since G is noncomplete.

If $\kappa(G) = 2$, let $\{u, v\}$ be any 2-vertex-cut of G. Then $\kappa(G - v) = 1$, so $G - v$ has at least two end-blocks and v has neighbors in all of them. Let x, y be two such neighbors. They must be nonadjacent, and $G - x - y$ is connected since blocks have no cut-vertices and $r \geq 3$. $\square$



Theorem 4.11 (**Brooks' Theorem**—Brooks [**1941**]). *If G is connected, then $\chi(G) = 1 + \Delta(G)$ if and only if G is complete or an odd cycle.*

Proof. ($\Leftarrow$) Equality certainly holds for cliques and odd cycles.

($\Rightarrow$) Let G satisfy the hypotheses. Then by Proposition 3.31, G is r-regular. The result certainly holds for $r \leq 2$, so we may assume $r \geq 3$. If G had a cut-vertex, each block could be colored with fewer than $r+1$ colors to agree on that vertex, so we may assume G is 2-connected and, to the contrary, not a clique.

By the lemma, we can establish a deletion sequence for G starting with some vertex v and ending with its nonadjacent neighbors x and y so that all vertices but v have at most $r-1$ neighbors when deleted. Reversing this yields a construction sequence, and coloring greedily gives x and y the same color, so G needs at most r colors. □

Thus the extremal graphs for $\chi(G) \leq 1 + \Delta(G)$ are complete graphs and odd cycles. For the Degeneracy Bound, the extremal graphs include these, and also trees, fans, maximal k-degenerate graphs, irregular graphs, and many more. No complete characterization of the extremal graphs for this bound is known.

There is another approach to graph coloring that is sometimes useful.

Algorithm 4.12. *Find a maximum independent set S of a graph, and color it with a single color. For $G - S$, repeat this step until all vertices are colored.*

This approach may not be optimal, as some graphs have no minimum coloring with any color class that is a maximum independent set. Finding a maximum independent set is not easy in general, so replacing "maximum" with "maximal" yields a faster algorithm. This algorithm does not translate directly into a bound, since most graphs have several maximum independent sets, and which one is chosen may change the number of colors used. Nonetheless, this algorithm may still yield decent results. One bound based on this approach follows.

Theorem 4.13 (Brigham/Dutton [**1985**]). *For any graph G,*

$$\chi(G) \leq \frac{\omega(G) + n + 1 - \alpha(G)}{2}.$$

Proof. We use induction on n. Note the result is true for empty graphs, which have $\chi = n$, $\omega = 1$, and $\alpha = n$. Thus it holds for $n = 1$. Assume the result holds for graphs with fewer than n vertices, and let G be a nonempty graph with $n > 1$. Let S be a maximum independent set of G and $H = G - S$.

4.5. Perfect Graphs

4.5.1. Introduction.

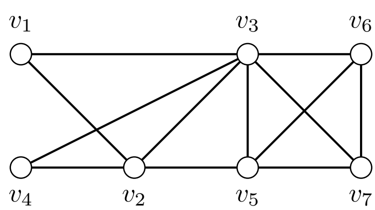
Example. A university student center has a number of identical rooms that are available for group meetings and events. They cannot be reserved ahead of time but are available when not already in use. Two natural questions are what is the minimum number of rooms needed and how should they be assigned?

Note that this problem differs from the scheduling problem in Section 4.1 because we have no control over when the meetings take place, only to which rooms they are assigned. There is a natural solution to this problem. Simply assign each meeting to the smallest-numbered room that is available when it starts. We will need k rooms if and only if k rooms are in use at once at some time.

Thus in this special situation, the chromatic number is equal to the clique number, and both can be easily determined.

Definition 4.28. An **intersection graph** has vertices representing sets and edges between sets that have nonempty intersections. An **interval graph** in an intersection graph with sets that are intervals.

Example. Suppose that meetings occur during the intervals $I_1 = [0,3]$, $I_2 = [2,6]$, $I_3 = [3,8]$, $I_4 = [4,5]$, $I_5 = [6,9]$, $I_6 = [7,10]$, $I_7 = [8,11]$. The interval graph is shown below. Its chromatic number is 4, since it contains K_4.



Any graph can be an intersection graph, but interval graphs are more restrictive. In the Exercises, you are asked to show that C_4 is not an interval graph. Note also that any induced subgraph of an interval graph can be formed by deleting some of the intervals, so it is also an interval graph.

Proposition 4.29. *If G is an interval graph, then $\chi(G) = \omega(G)$.*

Proof. Use a greedy coloring, and order the vertices by the left endpoint of the interval. For each vertex, a new color is only needed when all the existing colors are in use on intervals containing the left end of the next interval. Thus G can be colored with $\omega(G)$ colors. □

The properties shown above for interval graphs hold for a larger class of graphs.

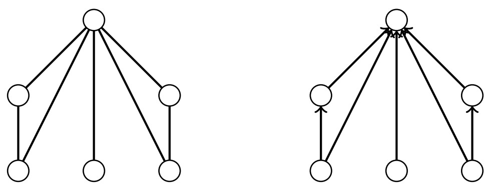
Definition 4.30. A graph G is **perfect** if for every induced subgraph H, $\chi(H) = \omega(H)$. A graph class $\mathbb{G}$ is **hereditary** if every induced subgraph of a graph in $\mathbb{G}$ is also in $\mathbb{G}$.

Perfect graphs can be viewed as the extremal graphs for the bound $\chi(G) \geq \omega(G)$. The condition on induced subgraphs is needed to avoid trivial examples.

Nonempty bipartite graphs have $\chi(G) = \omega(G) = 2$, and bipartite graphs are hereditary. Thus bipartite graphs are perfect. Bipartite graphs can have all their edges oriented from one set to the other, so they are contained in the following graph class.

Definition 4.31. A **transitive orientation** of a graph G is an orientation D such that when uv and vw are edges in D, G contains edge uw, oriented that way in D. A **comparability graph** is one with a transitive orientation.

Example. The comparability graph below left has the transitive orientation below right.



Proposition 4.32 (Berge [**1960**])**.** *Comparability graphs are perfect.*

Proof. Every induced subdigraph of a transitive digraph is transitive. Let D be a transitive orientation of a comparability graph G. Certainly D has no cycle. Color G by assigning each vertex v the number of vertices in the longest path of D ending at v in a proper coloring. If $uv \in E(D)$, then any path ending at u could be extended to v, so they must have different colors. By transitivity, the vertices of a path in D form a clique in G, so G can be colored with $\omega(G)$ colors. □

4.5.2. Chordal Graphs. What graphs are not perfect? Any odd cycle (except K_3) has $\omega(C_{2k+1}) = 2$ and $\chi(C_{2k+1}) = 3$, and so is not perfect. It is easily checked that any complement of an odd cycle (except K_3) is also not perfect. Thus any graph containing an odd cycle or its complement (except K_3) as an induced subgraph is not perfect. Thus when a perfect graph contains an odd cycle (except K_3), it must also have an edge not on the cycle joining two vertices of the cycle.

Definition 4.33. A **chord** of a cycle is an edge not on the cycle joining two vertices of the cycle. A graph is **chordal** if any cycle (except K_3) has a chord. A **simplicial vertex** is a vertex whose neighbors induce a clique.

Equivalently, a chordal graph has no induced cycle (except K_3). Any induced subgraph of a chordal graph is chordal.

Trees are (vacuously) chordal. They have the property that any minimal cutset is a single vertex. This observation can be generalized to characterize chordal graphs.

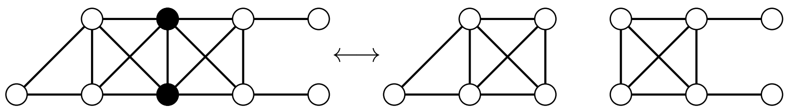
Lemma 4.34. *Any minimal cutset of a chordal graph induces a clique.*

Proof. Let S be a minimal cutset of a chordal graph G, and let X and Y be components of $G - S$. If $|S| = 1$, the result is obvious. Assume $|S| \geq 2$, and let $u, v \in S$. Then there are $u - v$ paths of minimum length through X and Y. Combining them forms a cycle whose only chord can be uv. Thus every possible edge in $G[S]$ is present, so it is a clique. □

Theorem 4.35 (Hajnal/Suranyi [**1958**], Dirac [**1961**]). *A graph is chordal if and only if it is complete or can be formed by identifying cliques of two smaller chordal graphs.*

Proof. ($\Leftarrow$) Complete graphs are chordal, and the operation cannot create a chordless cycle, so any graph formed this way is chordal.

($\Rightarrow$) Let G be chordal and not complete, and let S be a minimal cutset of G, which induces a clique. Let G_1 be an S-lobe of G, and let G_2 be the union of all other S-lobes. Both graphs are smaller than G and are chordal since they are induced subgraphs. Thus G can be formed by identifying the vertices of S in G_1 and G_2. □



Lemma 4.36. *Every chordal graph is either complete or has at least two nonadjacent simplicial vertices.*

Proof. We use induction on order n. The result is obvious for $n = 1$. Assume the result is true for graphs with fewer than n vertices. Let G be chordal with order n, let S be a minimal cutset, and let G_1 and G_2 be smaller chordal graphs that overlap on S. Then they are either complete or have at least two nonadjacent simplicial vertices. In a complete graph, every vertex is simplicial. Now $G[S]$ cannot contain all of G_1 (if it is complete) and cannot contain nonadjacent vertices, so at least one simplicial vertex of G_1 is also simplicial in G. Thus G has at least two nonadjacent simplicial vertices. □

Simplicial vertices with degree 1 are leaves. The previous lemma is analogous to the fact that nontrivial trees have at least two leaves. Along with maximal k-degenerate graphs and k-trees, connected chordal graphs are another generalization of trees. All k-trees are chordal. In the Exercises, you are asked to show that a graph is a k-tree if and only if it is maximal k-degenerate and chordal with $n \geq k+1$.

The simplicial vertices of a chordal graph can be successively deleted until only a single vertex remains. Conversely, a chordal graph can be constructed starting with K_1 and then adding simplicial vertices. This makes it easy to find minimum colorings of chordal graphs and show that they are perfect.

Theorem 4.37 (Voloshin [**1982**]). *A graph G is chordal if and only if $\omega(H) = 1 + D(H)$ for all induced subgraphs H in G.*

Proof. ($\Rightarrow$) We use induction on order n. The result holds when $n = 1$. Assume it holds for graphs with order less than n, and let G be chordal with order n. Let v be a simplicial vertex, and let $H = G - v$. Then H has $\omega(H) = 1 + D(H)$. Now v's neighborhood is a clique, so $d(v) \leq \omega(H)$. Thus, adding v to H can leave both ω and D unchanged or increase them both by one, so $\omega(G) = 1 + D(G)$.

($\Leftarrow$) (contrapositive) Let G be not chordal. Then G contains a cycle C_n, $n \geq 4$, as an induced subgraph. Then $\omega(C_n) = 2 < 3 = 1 + D(C_n)$. □

Corollary 4.38. *Chordal graphs are perfect, and any construction sequence produces a minimum coloring.*

Proof. For any graph G, $\omega(G) \le \chi(G) \le 1 + D(G)$, and coloring G with a construction sequence uses at most $1 + D(G)$ colors. By the previous theorem, these are all equalities. $\square$

Other vertex orders for chordal graphs also produce minimum colorings. Many classes of perfect graphs are related. In the exercises, you are asked to show that interval graphs are chordal, and their complements are comparability graphs.

4.5.3. The Perfect Graph Theorem. Early in the study of perfect graphs, Claude Berge noticed that many classes of perfect graphs also had perfect complements. In 1961, he conjectured that a graph is perfect if and only if its complement is perfect. This conjecture was proved in 1972 by Lazlo Lovasz. He actually proved a somewhat stronger result, which follows. The proof uses some ideas from linear algebra.

Theorem 4.39 (Lovasz [**1972B**]). *A graph G is perfect if and only if for every induced subgraph H, $\alpha(H)\,\omega(H) \ge n$.*

Proof. ($\Rightarrow$) Assume that G is perfect. Then, for every induced subgraph H, $\chi(H) = \omega(H)$. Then $\alpha(H)\,\omega(H) = \alpha(H)\,\chi(H) \ge \alpha(H)\,\frac{n}{\alpha(H)} = n$.

($\Leftarrow$) (Gasparian [**1996**]) (contrapositive) Assume that G is not perfect. Let H be a minimally imperfect subgraph of G, and let $n = n(H)$, $\alpha = \alpha(H)$ and $\omega = \omega(H)$. Then H satisfies $\omega = \chi(H - v)$ for every vertex $v \in V(H)$ and $\omega = \omega(H - S)$ for every independent set $S \subseteq V(H)$.

Let A_0 be a maximum independent set of H. Fix an ω-coloring of each of the α graphs $H - s$ for $s \in A_0$, let $A_1, \ldots, A_{\alpha\omega}$ be the independent sets occurring as a color class in one of these colorings and let $\mathbb{A} = \{A_0, A_1, \ldots, A_{\alpha\omega}\}$. Let A be the corresponding independent set versus a vertex incidence matrix. Define $\mathbb{B} = \{B_0, B_1, \ldots, B_{\alpha\omega}\}$ where B_i is an ω-clique of $H - A_i$. Let B be the corresponding clique versus a vertex incidence matrix.

Let $S_1, \ldots, S_\omega$ be any ω-coloring of $H - v$. Since any ω-clique C of H has at most one vertex in each S_i, C intersects all S_i's if $v \notin C$ and all but one if $v \in C$. Since C has at most one vertex in A_0, every ω-clique of H intersects all but one of the independent sets in $\mathbb{A}$.

In particular, it follows that $AB^T = J - I$. (J is a matrix of all 1's, and I is an identity matrix.) Now $\operatorname{rank}(J - I) \le \min\left\{\operatorname{rank}(A), \operatorname{rank}\left(B^T\right)\right\}$. Since $J - I$ is nonsingular, A and B have at least as many columns as rows, so $n \ge \alpha\omega + 1$. $\square$

Corollary 4.40 (**Perfect Graph Theorem**—Lovasz [**1972A, 1972B**]). *A graph G is perfect if and only if $\overline{G}$ is perfect.*

Proof. Let G be perfect, and let H be any induced subgraph. Since $\alpha(H) = \omega\left(\overline{H}\right)$ and $\omega(H) = \alpha\left(\overline{H}\right)$, $\alpha\left(\overline{H}\right)\omega\left(\overline{H}\right) = \alpha(H)\,\omega(H) \ge n$. Thus $\overline{G}$ is perfect. $\square$

Planar graphs are minor-closed, since contracting an edge in a plane drawing cannot produce a crossing. Wagner's Theorem says that planar graphs have only two forbidden minors.

Theorem 5.30 (**Graph Minor Theorem**—Robertson/Seymour [**2004**])**.** *Any minor-closed class of graphs has a finite set of forbidden minors.*

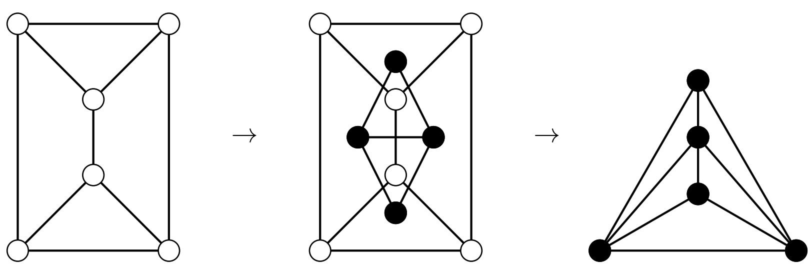
This is perhaps the deepest theorem in graph theory. It was proved in the series of papers totaling about 500 pages between 1983 and 2004 by Neal Robertson and Paul Seymour. It is an existence proof; it does not produce the set of forbidden minors for a given class. Some classes known to be minor-closed do not have complete characterizations of their forbidden minors. The Graph Minor Theorem implies that the class of all graphs having the property that all minors of graphs in the class can be $(k-1)$-colored has a finite set of forbidden minors; Hadwiger's Conjecture asserts that K_k is the only one.

5.4. Dual Graphs and Geometry

5.4.1. Dual Graphs. In the Four Color Problem, a map is modeled with a graph called a dual map. Something similar can be done for any planar graph.

Definition 5.31. The **dual of a plane drawing of a graph** G is a multigraph G^* with each vertex representing a region of the drawing, and each edge joining vertices representing regions that share an edge in the drawing. When all drawings of G have isomorphic duals, we call G^* the **dual graph** of G.

Example. To construct the dual of $G = K_3 \square K_2$, put vertices inside its interior regions and draw the corresponding edges, so that each edge of G^* crossed exactly one edge of G. Drawing the edges incident with the vertex corresponding to the exterior region will require several curved edges. We draw G^* in a more aesthetically pleasing form, finding $G^* = P_3 + K_2$.



A dual of a drawing may not be a graph. It has a loop whenever G has a bridge. It has multiple edges whenever G has a minimal 2–edge cut. A graph and its dual necessarily have the same size. For a connected graph, $(G^*)^* = G$, so the dual graph is a dual operation (it is its own inverse). The sum of the region lengths of G, $\sum r_i = 2m$, is equivalent to the First Theorem of Graph Theory for the dual graph G^*.

In the definition, we were careful to specify the dual of a drawing, rather than the dual of a graph. We saw earlier that different drawings of a graph can have different region lengths, which must produce different duals. Some restrictions on the graph will guarantee a unique dual.

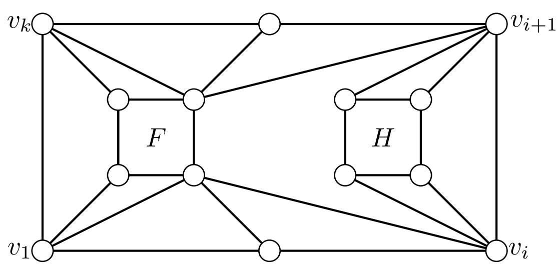
Lemma 5.32. *A cycle C of a planar graph G must bound a region in any drawing of G if and only if $G - C$ is connected.*

Proof. ($\Leftarrow$) Suppose $G - C$ is connected. Then in any plane drawing of G, all of $G - C$ is inside or outside (not both) of C. Thus C is the boundary of a region of G.

($\Rightarrow$) (contrapositive) Suppose $G - C$ is disconnected. Let F and H be subgraphs of G distinct from C with $F \cup H = G$ and $F \cap H = C$. Then G can be drawn with F inside C and H outside C, producing a plane drawing of G where C is not a region boundary. $\square$

Theorem 5.33 (Whitney [**1933**])**.** *If G is a 3-connected planar graph, then G has a unique plane drawing, up to rotation on a sphere (any plane drawing has the same regions).*

Proof. (contrapositive) Assume G has more than one distinct drawing, and let C be a cycle that is a region boundary in one drawing and not the other. By the lemma, $G - C$ is disconnected. Say F and H are components of $G - C$. Draw G with C on the outside, and let $v_1, \ldots, v_k$ be vertices of C adjacent to vertices of F. Then all vertices of C adjacent to vertices of H must be between some v_i and v_{i+1}. Then $\{v_i, v_{i+1}\}$ is a cutset of G. $\square$

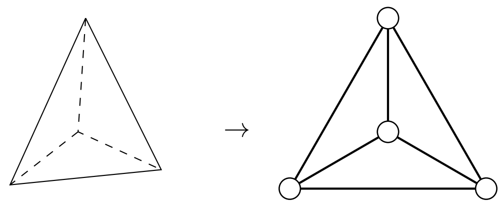


This implies that any 3-connected planar graph has a unique dual. The dual of any maximal planar graph with $n \geq 4$ is a 3-connected cubic planar graph. This partly explains our interest in cubic graphs elsewhere in this text.

5.4.2. Polyhedra. The main focus of this section is geometric applications of graph theory. In geometry, we consider figures in two or three-dimensional space. Polygons (two dimensions) have vertices at distinct points and straight line edges between some of them.

Definition 5.34. A **polyhedron** is a figure in three-dimensional space with points called vertices and straight-line edges connecting some vertices bounded by flat faces whose boundaries are polygons. A **regular polyhedron** has the same number of edges incident with each vertex and the same number of edges bounding each face.

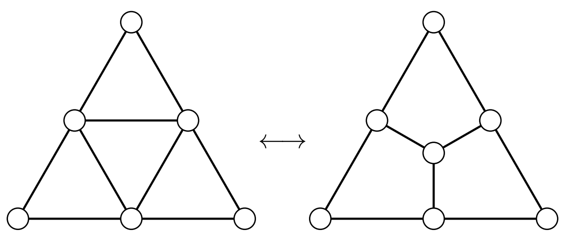
Any polygon or polyhedron can be modeled by a graph representing its vertices and edges. In fact, the names "vertex" and "edge" in graph theory come from this geometric terminology.



What can be said about a graph that represents a polyhedron? Certainly it must be planar. A vertex cut of size 2 would require the polyhedron to be "pinched", so it must be 3-connected. In fact, the converse is also true.

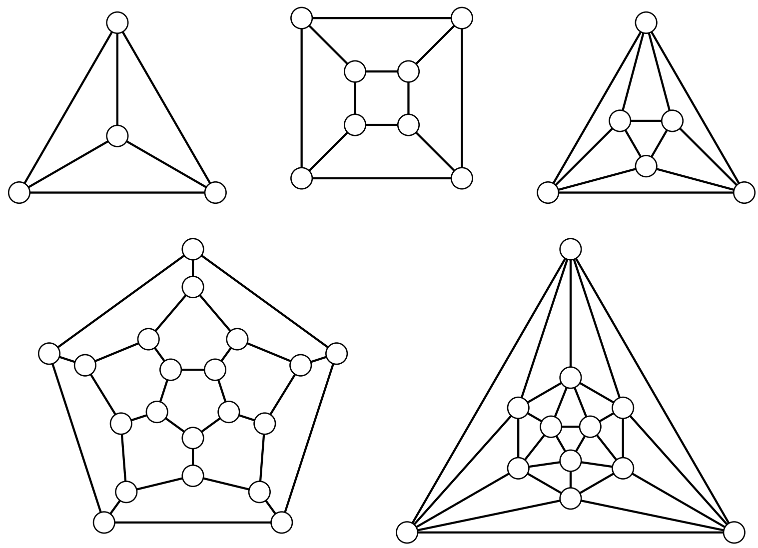
Theorem 5.35 (**Steinitz's Theorem**—Steinitz [**1922**]). *A graph represents a convex polyhedron if and only if it is planar and* 3*-connected.*

To prove the hard direction of this theorem, Steinitz observed that Euler's Polyhedron Formula implies that a planar 3-connected graph must contain either a degree 3 vertex or a triangular region. He showed that any planar 3-connected graph can be constructed from K_4 using an operation called a $\Delta - Y$ **transform** (see below). This corresponds to slicing off or adding a piece to a polyhedron. However, finding the appropriate sequence of $\Delta - Y$ transforms is not especially straightforward.



We can characterize regular polyhedra, which are also known as **Platonic solids**. In graph theory terms, the graph representing a polyhedron and its dual graph are both regular. Let G be a k-regular graph with order n, size m, and r regions, and let G^* be l-regular. Then $kn = 2m = lr$. Substituting into Euler's Formula, we see $\frac{2m}{k} - m + \frac{2m}{l} = 2$, so $m\left(\frac{2}{k} - 1 + \frac{2}{l}\right) = 2$. Since $m > 0$, $\frac{2}{k} - 1 + \frac{2}{l} > 0$. By Corollary 5.14, $3 \leq k, l \leq 5$. The five pairs of values of k and l that satisfy these inequalities are given in the table below. The corresponding values of n, m, and r are readily calculated. Each graph is 3-connected and has a unique realization. Each graph corresponds to a unique polyhedron.

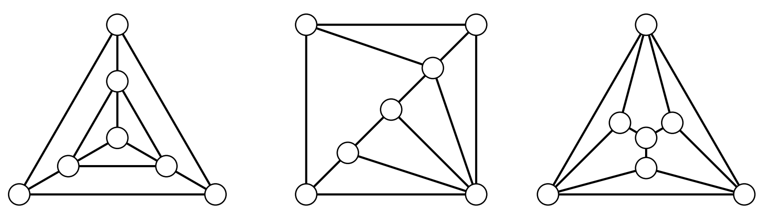
k	l	n	m	r	name	notation
3	3	4	6	4	tetrahedron	K_4
3	4	8	12	6	cube	Q_3
4	3	6	12	8	octahedron	$K_{2,2,2}$
3	5	20	30	12	dodecahedron	
5	3	12	30	20	icosahedron	



The cube and octahedron are dual graphs, as are the dodecahedron and icosahedron. The tetrahedron is its own dual.

Definition 5.36. A graph is **self-dual** if $G = G^*$.

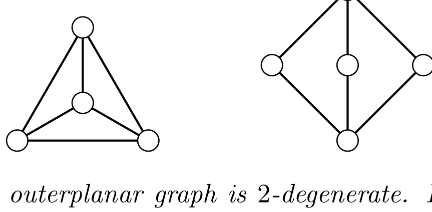
The trivial graph is self-dual, as are all wheels. Three self-dual graphs of order 7 are shown below.



5.4.3. Outerplanar Graphs. Some planar graphs can be drawn with all vertices on the outside.

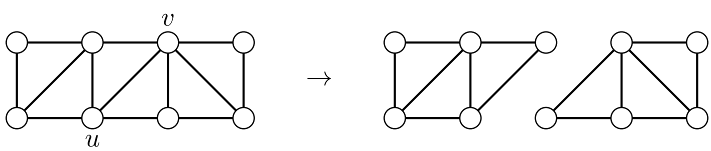
Definition 5.37. A graph is **outerplanar** if it has a plane drawing with all vertices on the exterior region.

The graphs K_4 and $K_{2,3}$ are not outerplanar, since any plane drawing of them must have one vertex in the interior.



Theorem 5.38. *Any outerplanar graph is* 2*-degenerate. In particular, any nontrivial outerplanar graph has at least two vertices of degree at most* 2*.*

Proof. The result is obvious when $2 \leq n \leq 4$, where K_4 is the only nonouterplanar graph. Assume the result holds for all maximal outerplanar graphs with order less than n, and let G be a maximal outerplanar graph of order n. Then all the vertices are on a spanning cycle C. Every other edge uv is a chord of C. The vertices of the two $u-v$ paths on C induce two maximal outerplanar graphs that only overlap on $\{u, v\}$. By induction, each of them has at least two vertices of degree at most 2, at least one of which is not u or v. Thus the result holds for G. Thus it holds for any nontrivial outerplanar graph. Since any subgraph of an outerplanar graph is outerplanar, any outerplanar graph is 2-degenerate. □



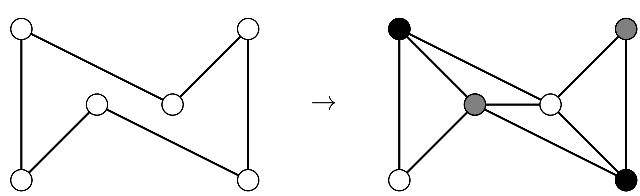
Corollary 5.39. *The size of a nontrivial outerplanar graph satisfies* $m \leq 2n - 3$.

This follows immediately from the bound on the size of k-degenerate graphs. Note that expressions of the form $k \cdot n - \binom{k+1}{2}$ occur repeatedly in graph theory. The size of a tree is $n-1$. A maximal outerplanar graph, which is a 2-tree, has size $2n-3$. A maximal planar graph has size $3n-6$. Some 3-trees are maximal planar, and Wagner [**1936**] showed that any maximal planar graph can be converted into a 3-tree via edge flips (Exercise 29).

Theorem 5.38 and the Degeneracy Bound immediately imply that any outerplanar graph is 3-colorable. This has a surprising geometric application. Imagine that an art gallery has a shape of an n-sided polygon. The gallery needs guards to prevent the art from being stolen. Guards must be positioned at vertices of the polygon so that every point of the interior is seen by some guard. How many guards are needed?

Theorem 5.40 (**Art Gallery Theorem**—Chvatal [**1975**]). *Any art gallery* (*an* n*-sided polygon*) *requires at most* $\left\lfloor \frac{n}{3} \right\rfloor$ *guards at its vertices to see all of its interior.*

Proof (Fisk [**1978**]). Add chords to the interior of the polygon so that it is split into triangles. Every vertex sees the interior of any triangle that contains it. The triangulated polygon can be represented by a maximal outerplanar graph, which is 3-colorable. In any 3-coloring, each triangle has each color on one of its three vertices. Thus each color class sees the interior of the polygon, and one of the classes has at most $\left\lfloor \frac{n}{3} \right\rfloor$ vertices. □

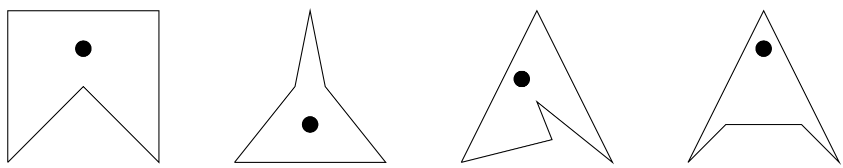


In the exercises, you are asked to show that this bound is sharp.

5.4.4. Straight Line Drawings. Kuratowski's Theorem guarantees that any graph not containing a subdivision of K_5 or $K_{3,3}$ has a plane drawing. In fact, it can be drawn in the plane using only straight lines for edges.

Lemma 5.41. *Any polygon with length at most* 5 *has a point in its interior that sees* (*has a straight line of sight to*) *all its vertices.*

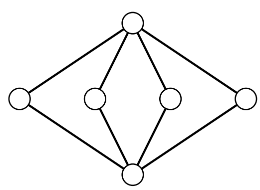
Proof. When the polygon is convex, any interior point works. Otherwise, there are several cases (see below) depending on the shape of the polygon. □



Theorem 5.42 (**Fary's Theorem**—Fary [**1948**], Wagner [**1936**]). *If a graph G is planar, then it has a drawing in the plane with straight lines for edges.*

Proof. Let G be planar, and let H be a maximal planar graph containing G. We use induction on order. The result is obvious when $1 \leq n(H) \leq 4$. Assume any maximal planar graph with order less than n has a straight line drawing. Let H have order $n \geq 4$. By Corollary 5.14, H has at least four vertices with degree at most 5, so at least one vertex v with $3 \leq d(v) \leq 5$ is not on the exterior region. If $H - v$ is not maximal planar, add one or two edges so that it is. Then this graph has order less than n, so it has a straight line drawing. Now deleting the added edges results in a region with length at most 5. By Lemma 5.41, there is a point in the interior of this region that sees all the vertices on its boundary. Add a vertex at this point, and add straight line edges to the vertices. Then H has a straight line drawing, so G does also. □

When a graph is maximal planar, every region is a triangle and, hence, convex. Otherwise, the last step in the proof of Fary's Theorem is to delete the edges added to make it maximal planar. This may result in regions that are not convex. For example, $K_{2,4}$ has no plane drawing where all interior regions are convex. Nonetheless, a stronger hypothesis can guarantee that all interior regions are convex.

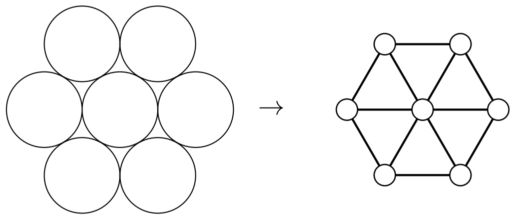


Theorem 5.43 (**Tutte's Spring Theorem**—Tutte [**1963**]). *Any* 3*-connected planar graph has a plane drawing with all interior regions convex.*

Tutte showed that if the exterior region is fixed and the interior edges are springs, then the equilibrium position of this system has every region convex. He proved this using linear algebra. Thomassen [**1980**] found a graph theoretic proof.

There is another way to produce planar graphs with straight line drawings.

Definition 5.44. A **coin graph** has vertices that are the centers of nonoverlapping circles and edges between centers of tangent circles. A **penny graph** is a coin graph with circles all having the same radius.



A coin graph must be planar. Surprisingly, the converse is also true.

Theorem 5.45 (**Circle Packing Theorem**—Koebe [**1936**])**.** *A graph is a coin graph if and only if it is planar.*

This theorem was originally proved using complex analysis. Fary's Theorem follows immediately from the Circle Packing Theorem, and Steinitz's Theorem can be proved using it. Penny graphs do not have a nice characterization, and it is an NP-complete problem to determine whether a graph is a penny graph.

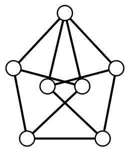
Since every planar graph has a straight edge drawing, we may seek additional properties. Heiko Harborth [**1987**] conjectured that every planar graph has a straight edge drawing with edges having integer length. Note that this is equivalent to every edge having rational length, since such a drawing could be scaled by the least common denominator of the edge lengths. This conjecture remains open, but the following more limited result is known.

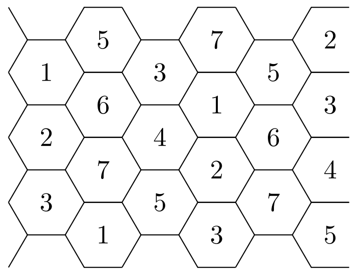
Theorem 5.46 (Geelen/Guo/McKinnon [**2008**])**.** *Let G be a planar graph that can be constructed so that, when added, each vertex has degree at most 2, or has degree 3, with two neighbors adjacent. Then G has a straight edge drawing with edges having integer length.*

The proof of this theorem uses number theory. The graphs in the hypothesis include cubic graphs and 3-trees, but not all 3-degenerate graphs.

Definition 5.47. A **unit distance graph** is a graph whose vertices are points in the plane and whose edges all have length 1. A **matchstick graph** is a unit distance graph with a plane drawing.

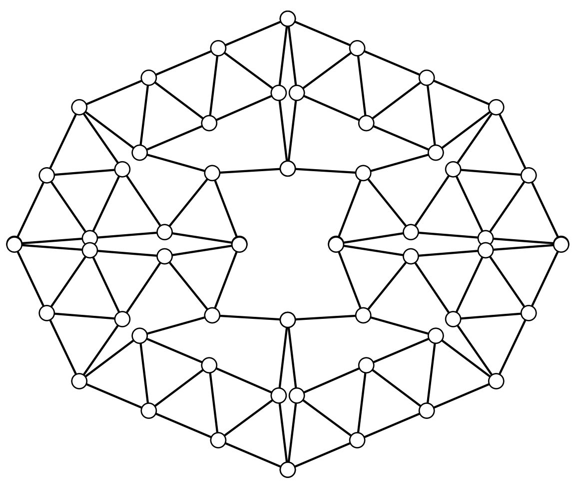
One natural question is how large the chromatic number of a unit distance graph can be. No wheel with even order (including K_4) is a unit distance graph. However, the Moser spindle is. There is a 7-coloring of the points of the plane using a tessellation by regular hexagons with diameter slightly less than 1. This shows that at most seven colors are required for any unit distance graph.





This problem was first discussed by Nelson in 1950 and first published by Hadwiger [**1961**]. For the next 57 years, 4 and 7 were the best known bounds for the maximum chromatic number of a unit distance graph. Then in 2018, biologist Aubrey de Grey [**2018**] found a 5-chromatic unit distance graph with order 1581. This graph contains many copies of the Moser spindle. Other researchers quickly went to work on the problem. Marijn Huele [**2018**] found several successively smaller examples of 5-chromatic unit distance graphs with order around 500.

Every penny graph is a matchstick graph. Penny graphs are 3-degenerate, but this is not true for all matchstick graphs. The **Harborth Graph** (Harborth [**1986**]) is a 4-regular matchstick graph. A precise description due to Gerbracht [**2011**] also shows that it is rigid.



5.4.5. Rigid Graphs. Consider a graph in the plane with edges that are line segments with fixed length that are hinged at the vertices (the angles at the vertices may vary).

Definition 5.48. A graph is **rigid** if when its vertices are placed in general position in the plane (fixing the lengths of the edges), there is no movement of the graph in the plane preserving the edge lengths that does not also preserve all distances between vertices. A graph is **flexible** if it is not rigid. A graph is a **Laman graph** if and only if $m = 2n - 3$ and each nontrivial subgraph with order n' has size $m' \leq 2n' - 3$.

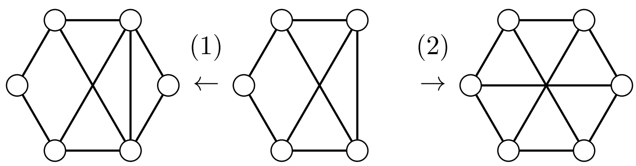
Any rigid graph must be 2-connected, since multiple components could be moved independently, and multiple blocks could be rotated at a cut-vertex. We will show that Laman graphs are exactly the rigid graphs. Laman graphs include all maximal 2-degenerate graphs, and also some with 3-cores.

Henneberg found an operation characterization of rigid graphs, which is proved using linear algebra.

Theorem 5.49 (Henneberg [**1911**]). *A graph is a minimal rigid graph if and only if it can be constructed by starting with K_2 and iterating the following two operations (**Henneberg operations**).*

(1) *Add a vertex of degree* 2.

(2) *Add a vertex of degree* 3 *adjacent to two vertices that are neighbors, and delete the edge between them.*



Laman proved a characterization of rigid graphs involving their sizes.

Theorem 5.50 (Laman [**1970**]). *A graph has a Henneberg construction if and only if it is a Laman graph.*

Proof. (⇒) Assume G has a Henneberg construction. Certainly K_2 is a Laman graph and both operations increase n by 1 and m by 2 in G. If a vertex is added to a subgraph of G, its size is increased by at most 2. Thus the operations preserve Laman graphs.

(⇐) (Haas et al. [**2005**]) If $n = 2$, then K_2 is the only Laman graph. Assume that any Laman graph with order less than $n > 2$ has a Henneberg construction. If G is a Laman graph with order n, then $m = 2n - 3$, and it has a vertex of degree at most 3. If any vertex v has degree 0 or 1, then $G - v$ is not a Laman graph. If v has degree 2, then $G - v$ is a Laman graph since its size is $2(n-1) - 3$ and all its subgraphs are subgraphs of G.

If v has degree 3, let $N(v) = \{v_1, v_2, v_3\}$. Let $H = G - v$, which has order $n - 1$, but only $2(n-1) - 4$ edges. We must add one edge joining one of the three pairs of vertices in $N(v)$. Consider the rigid components of H: maximal subsets of some k vertices spanning $2k - 3$ edges. Now v_1, v_2, and v_3 cannot belong to the same rigid component; otherwise the size restriction would be violated in G on the subset consisting of this component and v. Two rigid components share at most one vertex; otherwise their union would be a larger Laman subgraph. Say v_1 and v_2 are not in a common rigid component. Then adding $e = v_1v_2$ doesn't violate the size restriction on any subset, and it converts H to a Laman graph H'. Then H' and hence G has a Henneberg construction. □

These ideas can be generalized, but there is no complete characterization of graphs that are rigid in three dimensions. Henneberg [**1911**] listed four operations that have been shown to produce all minimally rigid three-dimensional graphs starting from K_3, but it is unknown if they only produce rigid graphs.

The natural generalization of Laman graphs would be graphs with $m = 3n - 6$ and each subgraph with order $n' \geq 3$ having size $m' \leq 3n' - 6$. Each minimally rigid three-dimensional graph must satisfy these conditions, but there are graphs satisfying these conditions that are not rigid in three dimensions. See Tay/Whiteley [**1985**] for a survey of what is known on rigid graphs.

5.5. Genus of Graphs

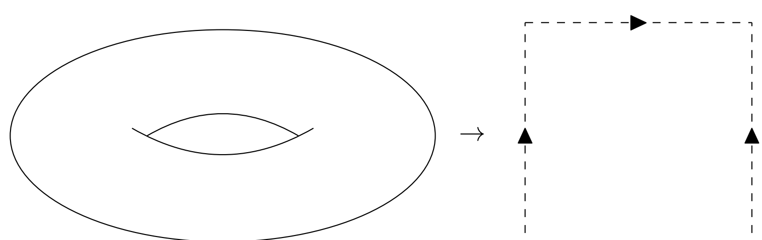
An electrical network can be modeled using a graph, with edges representing wires and vertices representing their intersections. A circuit board or computer chip has an electrical network on a flat surface (one or two-sided). If two wires cross, the network will short-circuit. Thus it is desirable for the network to be planar, so that it can be laid out with no crossings.

However, this may not be possible. In this case, we could build bridges to eliminate crossings. Alternatively, we could drill holes in the in the circuit board and run some wires through them to the other side. It turns out that these approaches are equivalent.

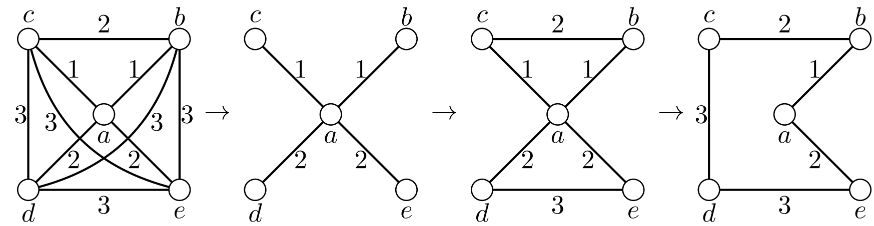
We have seen that drawing graphs in the plane and on a sphere is equivalent. However, there are other topological surfaces that are different. We could attach handles to a sphere or drill holes in it. Topologists consider two surfaces to be equivalent (homeomorphic) if one can be continuously deformed into another. That is, a surface can be stretched or shrunk but cannot be cut or pasted. In this way, we see that attaching a handle or drilling a hole amounts to the same operation.

Definition 5.51. A **torus** is a surface with one handle (or hole). A graph that can be drawn on a torus with no crossings is **toroidal**.

A torus can be drawn to look like a doughnut in space. However, this is not very helpful if we want to draw graphs on it. Instead, imagine cutting the torus twice, once in each direction, to obtain a rectangular figure. Opposite sides of this rectangle are identified, so that edges can leave one side of the rectangle and return on the other. The four corners of the rectangle are the same point.



Any planar graph can be drawn without crossings on the torus, and many nonplanar graphs can as well. These include K_5 and $K_{3,3}$. We have seen that 3-connected planar graphs have an essentially unique drawing in the plane, but this is not the case for drawings on the torus. The lengths of regions may vary in



Theorem 6.27 (Christofides [**1976**]). *Let G be a complete graph with a weight function $l(e)$ that satisfies the triangle inequality. Let $l(C)$ be the weight of the cycle C produced by Christofides' Algorithm. Then $C(G) \leq l(C) \leq \frac{3}{2}C(G)$.*

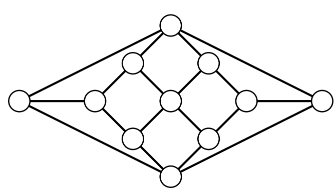
Proof. The first bound follows since Christofides' Algorithm produces a Hamiltonian cycle. Let C, W, T, and M be the cycle, trail, tree, and matching produced by the algorithm. Now the $2k$ odd vertices of T split a minimum spanning cycle into two sets of disjoint paths, one of which has at most half its weight. Thus $l(M) \leq \frac{1}{2}C(G)$. The triangle inequality guarantees that taking a more direct route cannot increase the weight. Thus

$$l(C) \leq l(W) = l(T) + l(M) \leq C(G) + \frac{1}{2}C(G) = \frac{3}{2}C(G). \quad \square$$

Perhaps surprisingly, this is essentially the best known bound on the TSP that is efficient to compute. Note that $\frac{3}{2}C(G)$ is the worst case for the upper bound; Christofides' Algorithm will often produce a closer bound.

6.3. Hamiltonian Planar Graphs

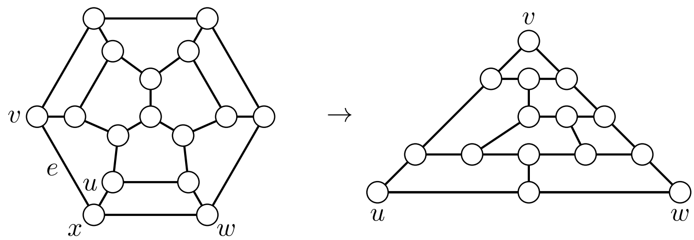
Any graph representing a convex polyhedron is planar and 3-connected. Since Hamiltonian graphs were introduced by showing that the dodecahedron is Hamiltonian, it is natural to ask whether other planar 3-connected graphs must be Hamiltonian. The **Herschel graph**, which is not regular, is not Hamiltonian.



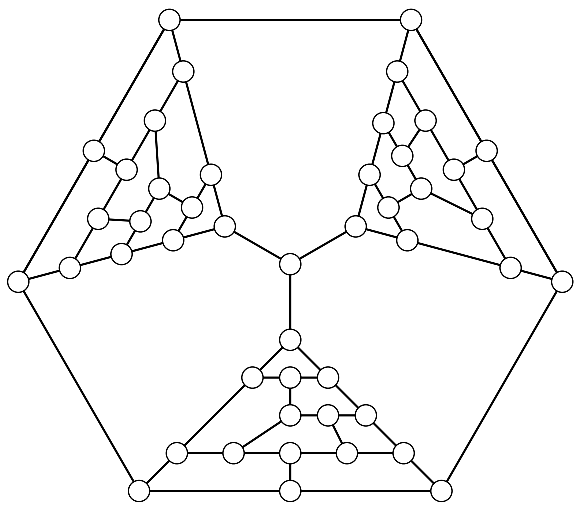
However, this problem is harder when restricted to cubic graphs. Peter Tait believed that all 3-connected cubic planar graphs are Hamiltonian. This came to be known as **Tait's Conjecture** (Tait [**1884**]). This conjecture is closely related to edge coloring and the Four Color Theorem, and it is explored in Section 7.4. Before long, other mathematicians noticed that this claim had not been proven. However, it was not until 1946 that William Tutte [**1946**] produced a counterexample.

To understand this example, first consider the graph T_{16} below left. It is Hamiltonian; indeed it contains an edge e that must be in any Hamiltonian cycle (Exercise 5). It is the smallest cubic 3-connected planar graph with this property (Holton/McKay [**1988**]). Deleting vertex x from it results in a graph called **Tutte's**

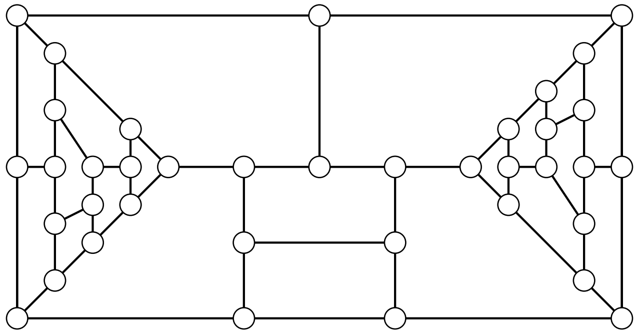
fragment (below right). It has no spanning $u-w$ path, since if it did, the original graph would have a Hamiltonian cycle not containing e.



Tutte's fragment does have spanning $u-v$ and $v-w$ paths, so when it is a subgraph of a cubic graph, any Hamiltonian cycle must use the other edge incident with v. Replacing three vertices of K_4 with Tutte's fragment as shown below produces the **Tutte graph**. Any Hamiltonian cycle would have to use the three edges incident with the center vertex, so no Hamiltonian cycle exists.



Holton and McKay [**1988**] showed that the smallest order of a 3-connected cubic planar non-Hamiltonian graph is 38. There are six such graphs that are formed by replacing two vertices of $C_5 \square K_2$ with copies of Tutte's fragment. One is shown below.

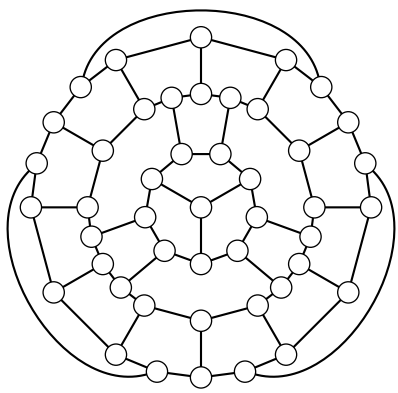


There is a necessary condition for planar Hamiltonian graphs.

Theorem 6.28 (**Grinberg's Theorem**—Grinberg [**1968**])**.** *Let G be a plane graph with Hamiltonian cycle C with r_i regions of length i inside C, and r'_i regions of length i outside C. Then $\sum_{i=3}^{n} (i-2)(r_i - r'_i) = 0$.*

Proof. Consider the graph H induced by C and its interior edges. Euler's Formula says $n - m + r = 2$, so $2n - 4 = 2m - 2r$. Summing the lengths of the interior regions and exterior region shows $2m(H) = \sum i \cdot r_i + n$. Then $2n - 4 = n + \sum i \cdot r_i - 2(\sum r_i + 1)$, so $n - 2 = \sum (i-2) r_i$. Similarly, considering the outside regions shows $n - 2 = \sum (i-2) r'_i$. Equating these expressions shows $\sum (i-2)(r_i - r'_i) = 0$. □

Example. Grinberg constructed the **Grinberg graph**, which has order 46, 21 regions of length 5, 3 of length 8, and one of length 9. If it is Hamiltonian, then taking the equation in Grinberg's Theorem mod 3 gives $7(r_9 - r'_9) = 0$, so $-7 = 0$. Thus the Grinberg graph is non-Hamiltonian.



Grinberg's Theorem is most useful for non-Hamiltonian graphs with many (but not all) regions of length $5, 8, 11, \ldots$. It is more useful for constructing counterexamples than for testing a given graph.

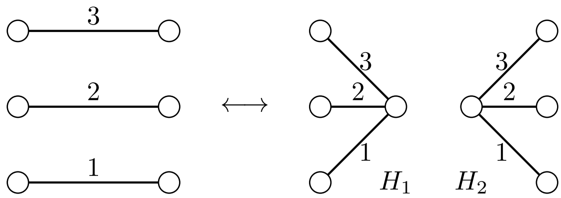
While a planar graph being 3-connected is not enough to guarantee that it is Hamiltonian, being 4-connected does suffice. In fact, it guarantees even more.

Theorem 6.29 (Tutte [**1956**], Thomassen [**1983**])**.** *Every planar 4-connected graph is Hamiltonian-connected.*

6.4. Tournaments

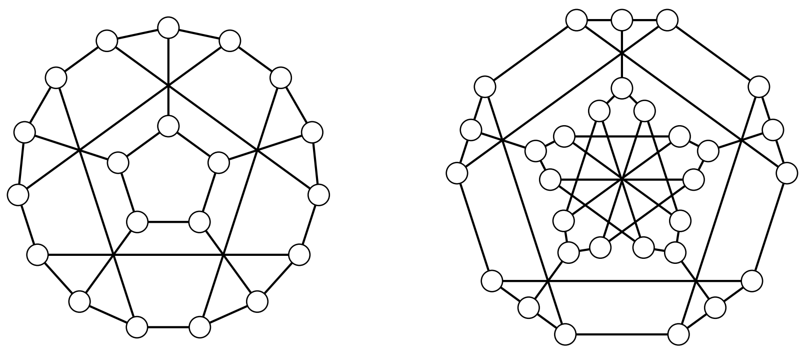
In a round-robin sports tournament, each team plays each other team exactly once. The results of the tournament can be expressed using a digraph with a directed edge from the winner to the loser of each game.

Definition 6.30. Let D be a digraph containing vertices u and v. The **outdegree** $d^+(v)$ is the number of edges from v. The **indegree** $d^-(v)$ is the number of edges to v. A **directed** $u - v$ **path** is a path in the underlying graph so that each vertex except v has outdegree 1. A **directed cycle** is a cycle in the underlying



Repeatedly applying the operations above must produce a cubic graph that is 3-connected with no trivial 3–edge cut and girth at least 5. Each operation preserves planarity in both directions, so it is a planar snark. □

The Four Color Theorem (via Tait's Theorem) implies that any bridgeless cubic planar graph is class 1. Thus any bridgeless cubic class 2 graph (hence any snark) is nonplanar. The operations in this proof show how to construct any class 2 bridgeless cubic graph from a snark. Thus snarks are the interesting cases.



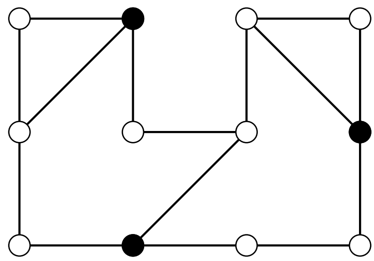
Until 1946, the Petersen graph was the only known snark. By 1975, only three more were discovered. After that, several infinite classes were discovered, along with operations to construct snarks from smaller snarks. A **flower snark** and the **double star snark** are shown above. They and many other snarks resemble the Petersen graph.

Conjecture 7.41 (**Tutte's Conjecture**—Tutte [**1966**]). *Any snark contains a subdivision of the Petersen graph.*

The Four Color Theorem implies that any snark is nonplanar. Tutte's Conjecture is a stronger claim, since the Petersen graph is a specific nonplanar graph. In 1999, Robertson, Sanders, Seymour, and Thomas announced a proof of this conjecture. However, as of 2020 components of the proof are still being published.

7.5. Domination

Example. A museum wants to position guards so that each room is observable. (Alternately, the "guards" could be security cameras.) Not every room needs a guard, but each room without a guard must be adjacent to one with a guard. What is the smallest number of guards that the museum can hire to observe all rooms? Let vertices represent rooms, with edges between pairs of rooms that observe each other. We need a set of vertices so that each vertex not in the set is adjacent to one that is. Given the following graph, we see that the black vertices are such a set. It appears that no smaller set is possible.

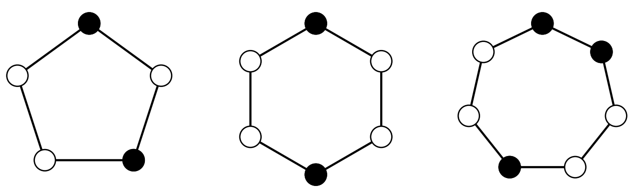


Example. The government builds transmitters in some towns in a rural area. Some towns are within transmission range of other towns (and vice versa) and others are not. How can transmitters be located so that every town is within range of some transmitter and the smallest number of transmitters is used? Let vertices represent towns, with edges between towns in transmission range. Again we need a set of vertices so that each vertex not in the set is adjacent to one that is.

Definition 7.42. A **dominating set** of a graph G is a set S of vertices so that every vertex not in S is adjacent to a vertex in S. A vertex in S is said to **dominate** itself and its neighbors. The **domination number** $\gamma(G)$ is the minimum size of a dominating set of G.

Note that $\gamma(G)$ is also the notation for the genus of a graph. Context should make it clear which meaning is intended.

Example. We show $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$. Number the vertices 0 through $n-1$, and let $S = \{v_0, v_3, v_6, \dots\}$. Then S is a dominating set. Each vertex can dominate at most three vertices, so a smaller set would fail to dominate some vertex.



The domination number is not easy to calculate in general, so we consider several bounds. Certainly $1 \le \gamma(G) \le n$. The extremal graphs for the lower bound are those with a vertex of degree $n-1$. Empty graphs are the extremal graphs for the upper bound.

Considering vertex degrees lead to better bounds. Since any isolated vertex must be in any dominating set of a graph, consider excluding graphs with isolated vertices.

Proposition 7.43 (Ore [**1962**]). *If G is a graph with no isolated vertices, then $\gamma(G) \le \frac{n}{2}$.*

Proof. Consider a spanning forest of G, which is bipartite. Each partite set dominates the other. Choosing the smallest set shows $\gamma(G) \le \frac{n}{2}$. $\square$

This bound is sharp. The extremal graphs are explored in the Exercises. The hypothesis of Proposition 7.43 could be stated as $\delta(G) \geq 1$. A larger minimum degree will reduce the upper bound.

Theorem 7.44. *Let G be a graph.*

If $\delta(G) \geq 1$, then $\gamma(G) \leq \frac{1}{2}n$ (Ore [**1962**]).

If $\delta(G) \geq 2$, then $\gamma(G) \leq \frac{2}{5}n$ (if $n > 7$) (McQuaig/Shepherd [**1989**]).

If $\delta(G) \geq 3$, then $\gamma(G) \leq \frac{3}{8}n$ (Reed [**1996**]).

If $\delta(G) \geq 4$, then $\gamma(G) \leq \frac{4}{11}n$ (Sohn/Xudong [**2009**]).

If $\delta(G) \geq 5$, then $\gamma(G) \leq \frac{5}{14}n$ (Xing, Sun, and Chen [**2006**]).

If $\delta(G) \geq 6$, then $\gamma(G) \leq \frac{6}{17}n$ (Jianxiang et al. [**2008**]).

If $\delta(G) \geq k$, then $\gamma(G) \leq \left[1 - k\left(\frac{1}{k+1}\right)^{1+\frac{1}{k}}\right] n$ (Caro/Roditty [**1985**, **1990**]).

The bounds for $1 \leq k \leq 6$ all satisfy $\gamma(G) \leq \frac{k}{3k-1}n$. For $\delta(G) \geq 7$, the final bound is superior to this formula. The bounds for $1 \leq k \leq 3$ are known to be sharp.

Vertex degrees also produce lower bounds.

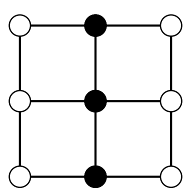
Definition 7.45. Let G be a graph with degrees $d_1 \geq d_2 \geq \cdots \geq d_n$. The **Slater number** of G, $\mathrm{sl}(G)$, is the smallest integer k so that $\sum_{i=1}^{k} d_i \geq n - k$.

Proposition 7.46. *A graph G has $\gamma(G) \geq \mathrm{sl}(G)$. In particular, $\gamma(G) \geq \frac{n}{1+\Delta(G)}$.*

Proof. If S is a minimum dominating set of G, then $v \in S$ dominates $1 + d(v)$ vertices. Thus $n \leq \sum_{v\in S}(1 + d(v)) \leq \sum_{i=1}^{|S|}(1 + d_i) \leq \gamma(G) \cdot (1 + \Delta(G))$. The third expression gives the first result. The last expression gives the latter result. □

This result does not take the structure of a graph into account. It may not be exact, since the vertices with large degree may have many common neighbors.

Example. Consider $G_{3,3}$, with degree sequence 433332222. We find $\gamma(G) \geq \mathrm{sl}(G) = 2$, since $4+3 = 9-2$. (The other bound gives $\gamma(G) \geq \frac{9}{1+4} = 1.8$.) However, this is not exact since the degree 4 vertex is adjacent to all degree 3 vertices. Picking the degree 4 vertex and a neighbor leaves two undominated vertices. In fact, $\gamma(G_{3,3}) = 3$, as seen below.



To show that $\gamma(G) = k$, we must show

(1) $\gamma(G) \leq k$. Find a minimum dominating set, or use an upper bound.

(2) $\gamma(G) \geq k$. Use a lower bound, or find a contradiction to show that $\gamma(G) < k$ is impossible.

The latter step is often very difficult, even for well-known classes of graphs. Indeed, determining the domination number of a graph is NP-complete.

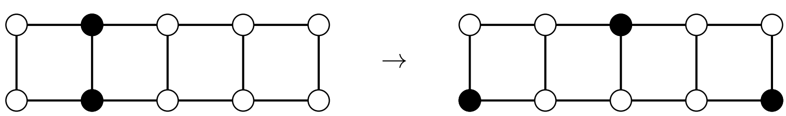
A greedy approach to finding a dominating set is to add a vertex of maximum degree and iterate this step on the graph induced by the undominated vertices. This usually does not produce a minimum dominating set, but the result is not too far from optimal.

Determining the domination number of grids has received considerable attention. Note that any grid has $\Delta(G) \leq 4$, so $\gamma(G_{r,s}) \geq \frac{n}{5}$. For large grids, this is close but not exact due to the boundary vertices. A complete classification of the values of $\gamma(G_{r,s})$, which has 23 separate cases, is given in (Goncalves et al. [**2011**]), with references. The following result illustrates the techniques used to find the domination number.

Proposition 7.47 (Jacobsen/Kinch [**1983**]). *We have* $\gamma(G_{2,s}) = \left\lceil \frac{s+1}{2} \right\rceil$.

Proof. Denote the vertices of $G_{2,s}$ as (i, j), $1 \leq i \leq s$, $1 \leq j \leq 2$. Let $S = \{(1,1), (3,2), (5,1), \dots\}$ except when s is even; we add one more vertex to cover the final corner. Then S is a dominating set with $|S| = \left\lceil \frac{s+1}{2} \right\rceil$. Each vertex dominates at most four vertices, but the two on the ends dominate fewer.

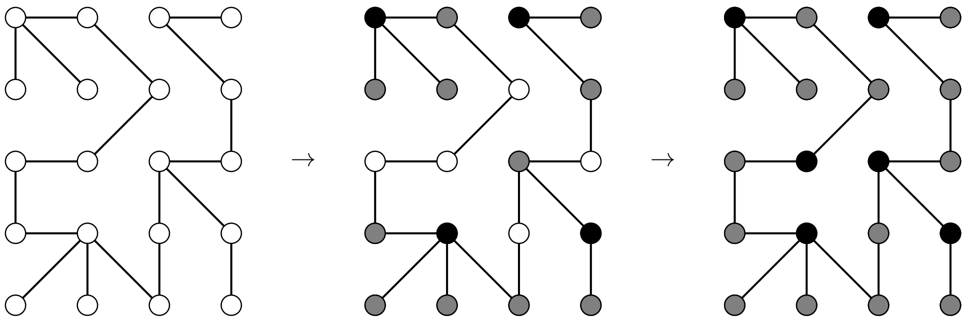
We claim that some minimum dominating set contains $(1,1)$ (or $(1,2)$). If not, then it contains $(2,1)$ and $(2,2)$, which together dominate six vertices. Replacing them with $(1,1)$ and $(3,2)$ dominates seven vertices, including the same six as before. Thus any dominating set of k vertices dominates at most $4k-2$ vertices. Thus $n = 2s \leq 4k-2$, so $k = \left\lceil \frac{2s+2}{4} \right\rceil$. □



There is an algorithm to determine the domination number of a tree exactly, but it does not lead to a convenient formula. We note that when $n \geq 3$, there is no need to use a leaf in the dominating set, since any such leaf could be replaced by its neighbor.

Algorithm 7.48. *Begin with a copy of a tree T, and a set S that will be the dominating set, and a set D of dominated vertices. If T is a star with all leaves in D, add the center to S and stop. Else add all neighbors of leaves to S and add all neighbors of vertices in S to D. Then delete all leaves and vertices in S, and iteratively delete all leaves in D. While any vertices remain, repeat these steps again on each component.*

Example. Consider the tree at left. The vertices added to S are black and the vertices added to D are gray. The first iteration is shown in the middle, and the second (final) iteration is shown at right.



When a graph has a 1-shell (trees rooted on the 2-core), the same approach to finding a minimum dominating set can be found. A 2-monocore graph can be constructed by adding ears and cycles (Theorem 3.34), which can also be used to construct a dominating set.

There are many, many variations of domination. Perhaps a museum does not fully trust its security guards, so it wants each guard to be observed by another guard. Perhaps the transmitters in a rural area need to be able to relay a message between any two transmitters. These variations impose additional restrictions on the dominating set, so $\gamma(G)$ is a lower bound for each of them.

Definition 7.49. An **independent dominating set** of a graph G is a dominating set that is independent. The **independent domination number** $i(G)$ is the minimum size of an independent dominating set of G.

A **connected dominating set** of a connected graph is a dominating set that is connected. The **connected domination number** $\gamma_c(G)$ is the minimum size of a connected dominating set of G.

A **total dominating set** of a graph with no isolated vertices is a dominating set that requires each vertex in the set to be dominated by another vertex. The **total domination number** $\gamma_t(G)$ is the minimum size of a total dominating set of G.

We briefly note a few results on these parameters. The following result on the independent domination number implies that $\gamma(G) \leq i(G) \leq \alpha(G)$.

Proposition 7.50. *A vertex set is an independent dominating set if and only if it is a maximal independent set.*

Proof. An independent set S is maximal if and only if every vertex not in S has a neighbor in S, in which case S is a dominating set. □

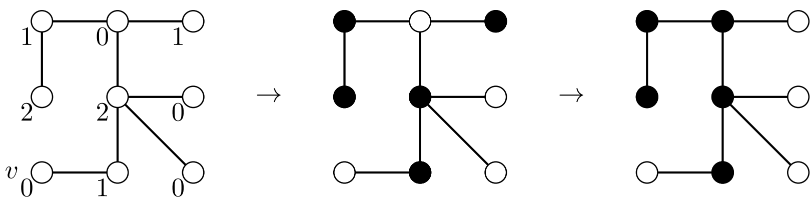
Theorem 7.51 (Allan/Laskar [**1978**]). *If G is a claw-free graph, then $\gamma(G) = i(G)$.*

Proof (Goddard/Henning [**2013**]). Let G be a claw-free graph. Let S be a minimum dominating set so that $G[S]$ has minimum size. Suppose S is not independent. Then there exist adjacent vertices u and v in S. Let $P_v = \{w \in V(G) \,|N(w) \cap S = \{v\}\}$ be the **private neighbors** of v. By the minimality of S, the set P_v is nonempty. Since G is claw-free, the set P_v is a clique. Thus for any $v' \in P_v$, the set $S' = S - \{v\} \cup \{v'\}$ is a minimum dominating set such that $G[S']$ has fewer edges that $G[S]$, a contradiction. □

For total domination, we again have upper bounds based on minimum degree.

Theorem 7.52 (Cockayne/Dawes/Hedetniemi [**1980**]). *Let G be a connected graph with $n \geq 3$. Then $\gamma_t(G) \leq \frac{2}{3}n$.*

Proof (Bickle [**2013**]). Let T be a spanning tree of G, and let v be a leaf of T. Label each vertex of T with its distance from v mod 3. This produces three sets that partition the vertices of G. Then some set contains at least one third of the vertices of G, and the union S of the other two contains at most two thirds of the vertices. Each internal vertex of T is adjacent to a vertex in each of the other sets. If S contains an isolated leaf, replace it with its neighbor. Then S is a total dominating set. □



The extremal graphs for this bound are characterized in the Exercises.

Theorem 7.53 (Henning/Yeo [**2007**]). *Let G be a graph without isolated vertices.*

If $\delta(G) \geq 1$, then $\gamma_t(G) \leq \frac{2}{3}n$, provided G is connected with $n \geq 3$.

If $\delta(G) \geq 2$, then $\gamma_t(G) \leq \frac{4}{7}n$, ($G$ connected and not C_3, C_5, C_6, or C_{10}).

If $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{1}{2}n$.

If $\delta(G) \geq 4$, then $\gamma_t(G) \leq \frac{3}{7}n$.

If $\delta(G) \geq k$, then $\gamma_t(G) \leq \frac{1+\ln k}{k}n$.

Domination is the second most popular research topic in graph theory, after coloring. It is explored in a book (Haynes/Hedetniemi/Slater [**1998**]).

Related Terms: corona, brush, roman domination, domatic number, fractional domination number, Vizing's Conjecture, irredundance number, upper irredundance number, upper domination number, weak product.

Exercises

Section 7.1:

(1) The graph below left represents applicants for various jobs. Determine whether applicants can be matched to each job.