Preface

Explorations in Analysis, Topology, and Dynamics is an introduction to the basic notions of real analysis and related topics, and to the practice of mathematics as it is done in upper-level courses and beyond. It is intended for students who have seen some calculus before (perhaps in high school), are motivated to learn the subject at a deeper level, and are generally interested in having a rich intellectual experience in mathematics. This text is written in the style of inquiry based learning. It is organized around and reliant upon students' engagement with the material. We offer investigations to explore, definitions to write, theorem statements to prove, and exercises and problems to solve. Investigations, Definitions, Theorems/Proofs, Exercises, and Problems that are meant to be completed by the reader are marked by an asterisk $*$. In some cases we provide structure or direction for a proof, though students are almost always responsible for assembling the ideas and/or using the available tools to create a complete proof themselves. For select problems and proofs that may require more guidance, we include a hint (indicated by the symbol $\star$). In support of encouraging students to try their own ideas for a solution or proof without restricting them to the suggested method, we have put these hints in Appendix D to make their use optional.

The core mathematical content of this text consists of the underpinnings of calculus of a single variable. We start with the standard set of axioms for an ordered field and the Axiom of Completeness in the form that any sequence of nested closed intervals in $\mathbb{R}$ has a non-empty intersection. Three mathematical tools form a common theme throughout the book: bisection, iteration, and nested intervals. The first time these tools are used in tandem is for the Monotone Convergence Theorem, whose proof we provide as a kind of blueprint for how bisection, iteration, and nested intervals can be leveraged together. For many of the other big theorems, such as the Bolzano–Weierstrass Theorem and the Intermediate Value Theorem, the text provides some guidance but asks the student to complete the proof. Note that these proofs are not intended to be completed by individual students, but rather in groups and with the entire class as a resource in making headway.

The first four chapters of this text grew out of our experiences teaching Math 176 at the University of Michigan. This is an inquiry based learning (IBL) course designed for students in the honors college and sponsored by the Center for Inquiry Based Learning at UM. As in other IBL courses at Michigan, class time was centered around worksheets. These consisted of a selection of the problems contained in the text that students worked on in groups of three or four. Students took turns presenting their work in class, followed by a class discussion of the proposed solution or proof: what it did well, whether it was complete and correct, its structure and relationship to other ideas, etc. The role of the instructor in the classroom was to introduce the material, facilitate group work, and guide class discussion during student presentations. Outside of the classroom, the instructor's work included preparing the worksheets, largely guided by the progress or direction that the discussion took in the previous class. The chapter on dynamical systems is based on material from an intensive two-week-long summer course, part of the Michigan Math and Science Scholars program for high school students at UM. The final chapter, which circles back to ideas from the first chapter, is partly based on portions of Math 486, a content course for pre-service high school mathematics teachers. We have blended and added to these materials in a way that we hope will be useful in diverse settings.

We envision the role of this book as twofold. First, it provides material for classroom use (e.g., worksheets), and second, it tells a mathematical narrative. We hope that this book will help students see a bigger picture than they are able to see when they work on only one worksheet at a time. Of course the instructor shares the responsibility for developing this broader narrative.

Content. Chapter 1 is an introduction to real numbers and sequences. Chapter 2 applies these tools to develop notions of closed, open, and compact sets and continuous functions. Chapter 3 explores highlights (from a theoretical mathematician's point of view) of differential calculus; there are some computational problems, but the emphasis is still on definitions, theorems, and proofs. Chapter 4 addresses highlights of integral calculus. The material in Chapter 5, on dynamical systems, can be used either as diversions throughout the text or on its own. It follows a similar style to the rest of the text, both in how the student interacts with it and with respect to the (bi)section, iteration, and nested intervals theme. In Chapter 6 we circle back to the notion of real numbers and study how fractional parts of real numbers can be represented (in a given base or by continued fractions). We think of generating these expansions as a dynamical process, by iterating certain algorithms.

A traditional first course in real analysis would cover much of the content of Chapters 1, 2, 3, and 4 from a more formal perspective. This text is designed to progress more slowly than a traditional real analysis course might, in part because we are asking students to gradually deepen the rigor of their ideas and arguments. One very tangible example of this is students' involvement in the process of creating definitions. The text allows a fair amount of leeway for a particular instructor and class to determine the level of rigor expected, based on the goals for the particular course.

We have included much more material than one can expect to cover in one term. In Math 176 we have been able to get through the core parts of Chapter 1, Chapter 2 through §2.4, and Chapters 3 and 4, excluding much of the additional material sections. Our intention with this text is to offer the instructor plenty of material for use in a course tailored to their particular class. It is our hope that some of the material from later chapters will serve to enrich the discussion and expose students to what we believe is beautiful mathematics.

On rigor. One of the aims of the book is to introduce students to rigorous mathematical language and argumentation. This is of course not an easy task! Our approach has been to proceed gradually, and to use the opportunities provided by student discussion in class to introduce the basic logical elements as they are needed. At first, a mixture of quantifiers, other logical symbols, and English language is used. For example, in the first definition of convergence of a sequence (a_k), we are happy to arrive at:

$$\lim a_k = L \text{iff}$$
$$\text{for all } \epsilon > 0, \qquad |a_k - L| < \epsilon \text{ for all sufficiently large } k.$$

As the students work with this definition, the need for specifying the precise meaning of "all sufficiently large k" arises naturally. Only then is the second quantifier "there exists K such that…" introduced.

In our experience, the level of rigor expected in the classroom evolves (increases) with time. Often, and especially at first, the instructor can set the level of rigor by asking students if they find the argument convincing, or by asking directly for more details in the explanations in a way that addresses the logical structure of the argument. It can also be very useful for the instructor to occasionally summarize or reformulate a student's proof for the class, thereby modeling what a proof is and how to write it. When the need for clarification arises, mini-lectures on proof and proving can be very useful. Though the instructor is ultimately responsible for deciding whether a proof or an argument is acceptable or needs further work, it has been our experience that, in the course of the semester, the students themselves will increasingly come to demand that precise language and argumentation be used. This philosophy that the class will supply the requirement for rigor, rather than it being demanded by an external authority, is in line with the following claim made by Hyman Bass that even professional mathematicians do not publish complete proofs in the strictest sense. Rather:

> Proving a claim is, for a mathematician, an act of producing, for an audience of peer experts, an argument to convince them that a proof of the claim exists.[1]

[1] *Mathematics and Teaching*, Notices of the AMS Vol. 62, No. 6, pg. 630. See in particular the paragraph on "The Proof vs Proving Paradox".

To the student. This text is an invitation to practice mathematics—not the kind of practice in which you repeat a procedure that you read about or were shown; rather, you will be practicing the discipline of mathematics. A mathematician asks questions, investigates new areas, tries out ideas (that often don't work), writes down progress, revises their work, and builds on their expanding knowledge. This book is structured to guide you in that process.

Explorations in Analysis, Topology, and Dynamics covers the foundational topics of calculus from a theoretical mathematician's point of view. While there are many fascinating and useful applications of calculus, this text is a study of the definitions, the tools, and their relationships that create the backbone of calculus and extend to related areas of mathematics. In pursuit of that, it emphasizes:

(1) The central importance of *ideas*. We will consider technical competence a byproduct of a thorough investigation of ideas, rather than a basic skill with which to start forming ideas.

(2) The central importance of *questions*. An important role of a mathematician is to ask good questions. This text will ask many questions for you to work on—these are meant to form the basis for your own inquiries. That is, you should make these questions your own by working on them and refining and deepening your ideas about them—and by asking further questions (of yourself and of your classmates). This is a style of *inquiry based learning*—learning that starts with your inquiries (either self-formed or by making a question your own) and is consolidated by your own investigations.

(3) The central importance of *the process of investigation*. In particular, we wish to investigate ideas in the style of mathematicians in order to create and refine deductive, quantitative arguments. While we hope you will be proud of the end product, the main goal is to develop some of the logical and quantitative competencies involved in taking ideas/intuition/questions and refining them into deductive-reasoning-validated products.

This book is designed to be implemented with a guide—someone with mathematical experience and sophistication who will be able to help you navigate the process of inquiry and investigation. The open-ended nature of mathematical inquiry can be both exhilarating and overwhelming, and forming and recognizing "good" questions is an acquired skill (one you will be working on acquiring!). A guide will help you strike a balance between focus and free exploration, recognize when to work through roadblocks and when to abandon an unproductive line of inquiry, and how to learn from your work.

Ultimately, we hope that you will find the ideas you encounter and generate interesting, beautiful, and fun!

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