Preface

This text was produced for the second part of a two-part sequence on advanced calculus, whose aim is to provide a firm logical foundation for analysis, for students who have had three semesters of calculus and a course in linear algebra. The first part treats analysis in one variable, and the text [**49**] was written to cover that material. The text at hand treats analysis in several variables. These two texts can be used as companions, but they are written so that they can be used independently, if desired.

Chapter 1 treats background needed for multivariable analysis. The first section gives a brief treatment of one-variable calculus, including the Riemann integral and the fundamental theorem of calculus. This section distills material developed in more detail in the companion text [**49**]. We have included it here to facilitate the independent use of this text. Subsequent sections in Chapter 1 present the basic linear algebra background of use for the rest of this text. They include material on n-dimensional Euclidean spaces and other vector spaces, on linear transformations on such spaces, and on determinants of such linear transformations.

Chapter 2 develops multidimensional differential calculus on domains in n-dimensional Euclidean space $\mathbb{R}^n$. The first section defines the derivative of a differentiable map $F : \mathcal{O} \to \mathbb{R}^m$, at a point $x \in \mathcal{O}$, for $\mathcal{O}$ open in $\mathbb{R}^n$, as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, and establishes basic properties, such as the chain rule. The next section deals with the inverse function theorem, giving a condition for such a map to have a differentiable inverse, when $n = m$. The third section treats $n \times n$ systems of differential equations, bringing in the concepts of vector fields and flows on an open set $\mathcal{O} \in \mathbb{R}^n$. While the emphasis here is on differential calculus, we do make use of integral calculus in one variable, as exposed in Chapter 1.

Chapter 3 treats multidimensional integral calculus. We define the Riemann integral for a class of functions on $\mathbb{R}^n$ and establish basic properties, including a change of variable formula. We then study smooth m-dimensional surfaces in $\mathbb{R}^n$, and extend the Riemann integral to a class of functions on such surfaces. Going further, we abstract the notion of surface to that of a manifold, and study a class of manifolds known as Riemannian manifolds. These possess an object known as a *metric tensor*. We also define the Riemann integral for a class of functions on such manifolds. The change of variable formula is instrumental in this extension of the integral.

In Chapter 4 we introduce a further class of objects that can be defined on surfaces, *differential forms*. A k-form can be integrated over a k-dimensional surface, endowed with an extra structure, an *orientation*. Again the change of variable formula plays a role in establishing this. Important operations on differential forms include products and the exterior derivative. A key result of Chapter 4 is a general Stokes formula, an important integral identity that can be seen as a multidimensional version of the fundamental theorem of calculus. In §4.4 we specialize this general Stokes formula to classical cases, known as theorems of Gauss, Green, and Stokes.

A concluding section of Chapter 4 makes use of material on differential forms to give another proof of the change of variable formula for the integral, much different from the proof given in Chapter 3.

Chapter 5 is devoted to several applications of the material on the Gauss-Green-Stokes theorems from Chapter 4. In §5.1 we use Green's theorem to derive fundamental properties of holomorphic functions of a complex variable. Sprinkled throughout earlier sections are some allusions to functions of complex variables, particularly in some of the exercises in §§2.1–2.2. Readers with no previous exposure to complex variables might wish to return to these exercises after getting through §5.1. In this section, we also discuss some results on the closely related study of harmonic functions. One result is Liouville's theorem, stating that a bounded harmonic function on all of $\mathbb{R}^n$ must be constant. When specialized to holomorphic functions on $\mathbb{C} = \mathbb{R}^2$, this yields a proof of the fundamental theorem of algebra.

In §5.2 we define the notion of smoothly homotopic maps and consider the behavior of closed differential forms under pullback by smoothly homotopic maps. This material is then applied in §5.3, which introduces degree theory and derives some interesting consequences. Key results include the Brouwer fixed point theorem, the Jordan-Brouwer separation theorem (in the smooth case), and the study of critical points of a vector field tangent to a compact surface, and connections with the Euler characteristic. We also show how degree theory yields another proof of the fundamental theorem of algebra.

Chapter 6 applies results of Chapters 2–5 to the study of the geometry of surfaces (and more generally of Riemannian manifolds). Section 6.1 studies geodesics, which are locally length-minimizing curves. Section 6.2 studies curvature. Several varieties of curvature arise, including Gauss curvature and Riemann curvature, and it is of great interest to understand the relations between them. Section 6.3 ties the curvature study of §6.2 to material on degree theory from §5.3, in a result known as the Gauss-Bonnet theorem.

Section 6.4 studies smooth matrix groups, which are smooth surfaces in $M(n, \mathbb{F})$ that are also groups. These carry left and right invariant metric tensors, with important consequences for the application of such groups to other aspects of analysis, including results presented in §7.4.

Chapter 7 is devoted to an introduction to multidimensional Fourier analysis. Section 7.1 treats Fourier series on the n-dimensional torus $\mathbb{T}^n$, and §7.2 treats the Fourier transform for functions on $\mathbb{R}^n$. Section 7.3 introduces a topic that ties the first two together, known as Poisson's summation formula. We apply this formula to establish a classical result of Riemann, his functional equation for the Riemann zeta function.

The material in §§7.1–7.3 bears on topics rather different from the geometrical material emphasized in the latter part of Chapter 3 and in Chapters 4–6. In fact, this part of Chapter 7 could be tackled right after one gets through §3.1. On the other hand, the last three sections of Chapter 7 make strong contact with this geometrical material. Section 7.4 treats Fourier analysis on the sphere S^{n-1}, which involves expanding a function on S^{n-1} in terms of eigenfunctions of the Laplace operator Δ_S, arising from the Riemannian metric on S^{n-1}. This study of course includes integrating functions over S^{n-1}. It also brings in the matrix group $SO(n)$, introduced in Chapter 3, which acts on each eigenspace V_k of Δ_S, and its subgroup $SO(n-1)$, and makes use of integrals over $SO(n-1)$. Section 7.4 also makes use of the Gauss-Green-Stokes formula and applications to harmonic functions, from §§4.4 and 5.1. We believe the reader will gain a good appreciation of the utility of unifying geometrical concepts with those aspects of Fourier analysis developed in the first part of Chapter 7.

We complement §7.4 with a brief discussion of Fourier series on compact matrix groups, in §7.5.

Section 7.6 deals with the purely geometric problem of showing that, among smoothly bounded planar domains $\Omega \subset \mathbb{R}^2$ with fixed area, the disks have the smallest perimeter. This is the two-dimensional isoperimetric inequality. Its placement here is due to the fact that its proof is an application of Fourier series.

The text ends with a collection of appendices, some giving further background material, others providing complements to results of the main text. Appendix A.1 covers some basic notions of metric spaces and compactness used from time to time throughout the text, such as in the study of the Riemann integral and in the proof of the fundamental existence theorem for ODE. As is the case with §1.1, Appendix A.1 distills material developed at a more leisurely pace in [**49**], again serving to make this text independent of the first one.

Appendices A.2 and A.3 complement results on linear algebra presented in Chapter 1 with some further results. Appendix A.2 treats a general class of inner product spaces, both finite and infinite dimensional. Treatments in the latter case are relevant to results on Fourier analysis in Chapter 7. Appendix A.3 treats eigenvalues and eigenvectors of linear transformations on finite-dimensional vector spaces, providing results useful in various places, from §2.1 to §6.6.

Appendix A.4 discusses the remainder term in the power series of a function. Appendix A.5 deals with the Weierstrass theorem on approximating a continuous function by polynomials, and an extension, known as the Stone-Weierstrass theorem, a useful tool in analysis, with applications in §§5.3, 7.1, and 7.4. Appendix A.6 builds on material on harmonic functions presented in Chapters 5 and 7. Results range from a removable singularity theorem to extensions of Liouville's theorem. Appendix A.7 introduces de Rham cohomology, as an extension of degree theory, developed in Chapter 5.

We point out some distinctive features of this treatment of advanced calculus.

1) *Applications of the Gauss-Green-Stokes formulas.* These formulas form a high point in any advanced calculus course, but we do not want them to be seen as the culmination of the course. Their significance arises from their many applications. The first application we treat is to the theory of functions of a complex variable, including the Cauchy integral theorem and basic consequences. This basically constitutes a mini-course in complex analysis. (A much more extensive treatment can be found in [**51**].) We also derive applications to the study of harmonic functions, in n variables, a study that is closely related to complex analysis when $n = 2$.

   We also apply differential forms and the Stokes formula to results of a topological flavor, involving a set of tools known as degree theory. We start with a result known as the Brouwer fixed point theorem. We give a short proof, as a direct application of the Stokes formula, thus making this theorem a precursor to degree theory rather than an application.

2) *The unity of analysis and geometry.* This starts with calculus on surfaces, computing surface areas and surface integrals, given in terms of the metric tensors these surfaces inherit, but it proceeds much further. There is the question of finding geodesics, shortest paths, described by certain differential equations, whose coefficients arise from the metric tensor. Another issue is what makes a curved surface curved. One particular measure is called the Gauss curvature. There are formulas for the integrated Gauss curvature, which in turn make contact with degree theory. Such matters are examples of connections uniting analysis and geometry, and is pursued in the text.

   Other connections arise in the treatment of Fourier analysis. In addition to Fourier analysis on Euclidean space, the text treats Fourier analysis on spheres. Matrix groups, such as rotation groups $SO(n)$, make an appearance, both as tools for studying Fourier analysis on spheres and as further sources of problems in Fourier analysis, thereby expanding the theater in which we bring to bear techniques of advanced calculus developed here.

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