Preface

This book offers an introduction to enumerative combinatorics for advanced undergraduates as well as beginning graduate students. One of the attractive features of combinatorics, however, is the modest mathematical background required for its study, and I have occasionally had students as young as sophomores master this material. For much of the book, the only prerequisites are knowledge of elementary linear algebra and familiarity with power series (the latter reviewed in Appendix A). While a prior course covering logic and sets, functions and relations, etc., would also be helpful, those who have not had such a course or simply require a review will find an adequate account of these subjects in Chapter 2. Passing references to abstract algebraic structures (groups, rings, fields, algebras) are mostly shorthand for lists of properties. Appendix C includes a glossary of the basic terminology of abstract algebra, along with a few of its elementary theorems, which should suffice for reading most of the text (the exceptions being Chapter 10 and parts of Chapters 13 and 14). Concepts from point set topology also appear, but only in section 12.2, and in Chapters 13 and 14, where alternatives to a topological treatment are presented. In any case, there is an outline of all pertinent topological results in Appendix B. Sections devoted to special supplementary topics are designated with the asterisk symbol "*".

The material in this book and its style of presentation are deeply indebted to an approach to combinatorics developed by Gian-Carlo Rota and his students, especially Richard Stanley. In particular, bijective proofs (which establish the numerical identity $a = b$ by counting a set S in two different ways, or by exhibiting sets A and B with respective cardinalities a and b, along with a bijection from A to B) are preferred, whenever possible, to verification by symbol manipulation. I also make frequent use of the notion of an m-to-one surjection, which provides a clear and rigorous way of establishing that a set has a/m members. Although combinatorics texts typically follow the (entirely reasonable) practice of adopting the addition and multiplication rules as axioms, I have included careful derivations of these rules for interested readers, based on the beautiful exposition of Seth Warner in Chapter III of his *Modern Algebra I*

(Prentice-Hall, 1965). I have aimed for as concise an exposition as possible, relying on the substantial exercises to elaborate material in the body of the text. At the same time, I have included illustrative examples and solved problems when necessary to illuminate the more difficult topics in the text. Many chapters include collections of related, somewhat more challenging, exercises, labeled "Projects", which can be assigned as take-home examinations or as honors problems.

Chapters 1–9 cover the standard tools of enumeration (recursion, generating functions, and the sieve formula) as well as the fundamental discrete structures (sets and multisets, words and permutations, ordered and unordered partitions of a set, graphs and trees, and combinatorial number theory) most frequently encountered in combinatorics. Chapter 10 covers the Pólya–de Bruijn theory of enumeration under group action, an important tool for counting equivalence classes of discrete structures. In an unusual feature in a book of this type, Chapter 11 gives a thorough account of the structure of vector spaces over finite fields ("q-theory"), both for its intriguing parallels with the combinatorics of finite sets and as a prime example of a class of posets that play an important role later in the book. Chapter 12 deals with ordered structures, and in addition to the classical theory of posets (Dilworth's chain and antichain decomposition theorems, Sperner theory, matchings and systems of distinct representatives, etc.), it contains supplementary material on other generalizations of total orders, such as quasi-orders and weak orders, as well as a discussion of their asymmetric counterparts, the strict orders. Many combinatorics texts finesse convergence questions for generating functions with the comment that this can all be made rigorous by conceptualizing such functions as formal power series. While there is nothing wrong with this practice if one is in a hurry to get on to other subjects, I have decided to devote Chapter 13 to a detailed account of formal power series, based on the beautiful paper by Ivan Niven in the 1969 *American Mathematical Monthly*.

Until the 1960s, many, if not most, mathematicians regarded combinatorics as a more or less unorganized collection of problems and their solutions. In decrying this attitude in his 1968 monograph, *Combinatorial Identities*, John Riordan nicely expressed it as the view that "…the challenge of verification provides the chief interest of the identity; once verified, it drops into an undifferentiated void". Chapter 14 offers what I hope is a convincing refutation of this view, as embodied in Rota's theory of incidence algebras over locally finite posets and the Doubilet–Rota–Stanley theory of binomial posets, arguably the most attractive unification to date of the essential core of enumerative combinatorics. In particular, the latter theory provides a satisfying explanation of why various types of enumeration problems are inevitably associated with particular types of generating functions and, when specialized to the case of modular binomial lattices, an explanation of why the formulas of q-theory so frequently reduce to formulas about chains "when $q = 0$" and about sets "when $q = 1$".

There is more material in this text than can be taught in a typical one-semester undergraduate course. At a minimum, such a course might cover the unstarred sections of Chapters 1–9, with substantial class time devoted to the discussion of exercises. Time permitting, I would also cover sections 11.1–11.4, as well as the unstarred sections of Chapter 12. Chapters 9–13 are independent of each other, which allows instructors considerable flexibility in designing variations on the above. Those wishing to cover

Chapter 14 (in, say, an honors undergraduate or masters level graduate course) will want to ensure that students have read in advance sections 11.1–11.5, the unstarred sections of Chapter 12, and sections 13.1-13.6.

My longtime friend and colleague, Shashikant Mulay, has been generous beyond measure with advice during the writing of this book, as well as many other projects. He is responsible in particular for the formulation and proofs of Theorems 13.5.1 and B.5.6. Two of my former doctoral students have also made important contributions. Mark Shattuck, now an accomplished combinatorist, has given the book a meticulous proofreading, checking the proofs of all of the theorems, as well as suggesting the proof of Theorem 11.11.1 and several exercises. Reid Davis was responsible for calling my initial attention to the important unifying role of modular binomial lattices, only later discovering identical ideas in earlier work of Doubilet, Rota, and Stanley. The thoughtful proposals of three anonymous referees for revisions and additions to my original manuscript have greatly improved the final text. Steve Kennedy and Jennifer Wright Sharp have been all that one could hope for as editors. To all of these individuals I offer my sincere thanks.