Contents

Preface xiii

Notation xvii

Chapter 1. Prologue: Compositions of an integer 1
- 1.1. Counting compositions 2
- 1.2. The Fibonacci numbers from a combinatorial perspective 3
- 1.3. Weak compositions 5
- 1.4. Compositions with arbitrarily restricted parts 5
- 1.5. The fundamental theorem of composition enumeration 6
- 1.6. Basic tools for manipulating finite sums 7
- Exercises 8

Chapter 2. Sets, functions, and relations 11
- 2.1. Functions 12
- 2.2. Finite sets 16
- 2.3. Cartesian products and their subsets 19
- 2.4. Counting surjections: A recursive formula 21
- 2.5. The domain partition induced by a function 22
- 2.6. The pigeonhole principle for functions 24
- 2.7. Relations 25
- 2.8. The matrix representation of a relation 27
- 2.9. Equivalence relations and partitions 27
- References 27
- Exercises 28
- Project 30

Chapter 3. Binomial coefficients 31
3.1. Subsets of a finite set 31
3.2. Distributions, words, and lattice paths 34
3.3. Binomial inversion and the sieve formula 36
3.4. Problème des ménages 39
3.5. An inversion formula for set functions 41
3.6. *The Bonferroni inequalities 43
References 45
Exercises 45

Chapter 4. Multinomial coefficients and ordered partitions 49
4.1. Multinomial coefficients and ordered partitions 49
4.2. Ordered partitions and preferential rankings 51
4.3. Generating functions for ordered partitions 52
Reference 54
Exercises 54

Chapter 5. Graphs and trees 57
5.1. Graphs 57
5.2. Connected graphs 58
5.3. Trees 59
5.4. *Spanning trees 62
5.5. *Ramsey theory 63
5.6. *The probabilistic method 65
References 66
Exercises 66
Project 67

Chapter 6. Partitions: Stirling, Lah, and cycle numbers 69
6.1. Stirling numbers of the second kind 69
6.2. Restricted growth functions 72
6.3. The numbers $\sigma(n,k)$ and $S(n,k)$ as connection constants 73
6.4. Stirling numbers of the first kind 75
6.5. Ordered occupancy: Lah numbers 76
6.6. Restricted ordered occupancy: Cycle numbers 78
6.7. Balls and boxes: The twenty-fold way 82
References 82
Exercises 83
Projects 85

Chapter 7. Intermission: Some unifying themes 87
7.1. The exponential formula 87
7.2. Comtet's theorem 90
7.3. Lancaster's theorem 92
References 93
Exercises 93
Project 94

Chapter 8. Combinatorics and number theory 97
8.1. Arithmetic functions 97
8.2. Circular words 100
8.3. Partitions of an integer 101
8.4. *Power sums 109
8.5. p-orders and Legendre's theorem 112
8.6. Lucas's congruence for binomial coefficients 114
8.7. *Restricted sums of binomial coefficients 115
References 116
Exercises 117
Project 118

Chapter 9. Differences and sums 119
9.1. Finite difference operators 119
9.2. Polynomial interpolation 121
9.3. The fundamental theorem of the finite difference calculus 122
9.4. The snake oil method 123
9.5. * The harmonic numbers 126
9.6. Linear homogeneous difference equations with constant coefficients 127
9.7. Constructing visibly real-valued solutions to difference equations with obviously real-valued solutions 131
9.8. The fundamental theorem of rational generating functions 132
9.9. Inefficient recursive formulae 134
9.10. Periodic functions and polynomial functions 135
9.11. A nonlinear recursive formula: The Catalan numbers 136
References 139
Exercises 140
Project 141

Chapter 10. Enumeration under group action 143
10.1. Permutation groups and orbits 143

10.2. Pólya's first theorem 145
10.3. The pattern inventory: Pólya's second theorem 148
10.4. Counting isomorphism classes of graphs 150
10.5. G-classes of proper subsets of colorings / group actions 154
10.6. De Bruijn's generalization of Pólya theory 155
10.7. Equivalence classes of boolean functions 157
References 159
Exercises 160

Chapter 11. Finite vector spaces 163
11.1. Vector spaces over finite fields 163
11.2. Linear spans and linear independence 164
11.3. Counting subspaces 165
11.4. The q-binomial coefficients are Comtet numbers 167
11.5. q-binomial inversion 169
11.6. The q-Vandermonde identity 171
11.7. q-multinomial coefficients of the first kind 172
11.8. q-multinomial coefficients of the second kind 173
11.9. The distribution polynomials of statistics on discrete structures 175
11.10. Knuth's analysis 180
11.11. The Galois numbers 182
References 183
Exercises 184
Projects 185

Chapter 12. Ordered sets 187
12.1. Total orders and their generalizations 187
12.2. *Quasi-orders and topologies 188
12.3. *Weak orders and ordered partitions 189
12.4. *Strict orders 191
12.5. Partial orders: basic terminology and notation 192
12.6. Chains and antichains 194
12.7. Matchings and systems of distinct representatives 198
12.8. *Unimodality and logarithmic concavity 200
12.9. Rank functions and Sperner posets 203
12.10. Lattices 203
References 205
Exercises 206
Projects 207

Chapter 13. Formal power series 209
13.1. Semigroup algebras 209
13.2. The Cauchy algebra 210
13.3. Formal power series and polynomials over $\mathbb{C}$ 211
13.4. Infinite sums in $\mathbb{C}^{\mathbb{N}}$ 215
13.5. Summation interchange 217
13.6. Formal derivatives 219
13.7. The formal logarithm 221
13.8. The formal exponential function 222
References 223
Exercises 223
Projects 224

Chapter 14. Incidence algebra: The grand unified theory of enumerative combinatorics 227
14.1. The incidence algebra of a locally finite poset 227
14.2. Infinite sums in $\mathbb{C}^{\mathrm{Int}(P)}$ 229
14.3. The zeta function and the enumeration of chains 231
14.4. The chi function and the enumeration of maximal chains 232
14.5. The Möbius function 233
14.6. Möbius inversion formulas 235
14.7. The Möbius functions of four classical posets 237
14.8. Graded posets and the Jordan–Dedekind chain condition 239
14.9. Binomial posets 240
14.10. The reduced incidence algebra of a binomial poset 243
14.11. Modular binomial lattices 247
References 249
Exercises 249
Projects 250

Appendix A. Analysis review 251
A.1. Infinite series 251
A.2. Power series 252
A.3. Double sequences and series 253
References 254

Appendix B. Topology review 255
B.1. Topological spaces and their bases 255
B.2. Metric topologies 256

B.3. Separation axioms 257
B.4. Product topologies 257
B.5. The topology of pointwise convergence 257
References 259

Appendix C. Abstract algebra review 261
C.1. Algebraic structures with one composition 261
C.2. Algebraic structures with two compositions 262
C.3. *R*-algebraic structures 263
C.4. Substructures 264
C.5. Isomorphic structures 265
References 266

Index 267