PREFACE

The goal of this book is to rigorously present the fundamental concepts of mathematical analysis in the clearest, simplest way, within the context of illuminating examples and stimulating exercises. I hope that the student will assimilate a precise understanding of the subject together with an appreciation of its coherence and significance. The full book is suitable for a year-long course; the first nine chapters are suitable for a one-semester course on functions of a single variable.

I cannot overemphasize the importance of the exercises. To achieve a genuine understanding of the material, it is necessary that the student do many exercises. The exercises are designed to be challenging and to stimulate the student to carefully reread the relevant sections in order to properly assimilate the material. Many of the problems foreshadow future developments. The student should read the book with pencil and paper in hand and actively engage the material. A good way to do this is to try to prove results before reading the proofs.

Mathematical analysis has been seminal in the development of many branches of science. Indeed, the importance of the applications of the computational algorithms that are a part of the subject often leads to courses in which familiarity with implementing these algorithms is emphasized at the expense of the ideas that underlie the subject. While these techniques are very important, without a genuine understanding of the concepts that are at the heart of these algorithms, it is possible to make only limited use of these computational possibilities. I have tried to emphasize the unity of the subject. Mathematical analysis is not a collection of isolated fact and techniques, but is, instead, a coherent body of knowledge. Beyond the intrinsic importance of the actual subject, the study of mathematical analysis instills habits of thought that are essential for a proper understanding of many areas of pure and applied mathematics.

In addition to the absolutely essential topics, other important topics have been arranged in such a way that selections can be made without disturbing the coherence of the course. Chapters and sections containing material that is not subsequently referred to are labeled by asterisks.

At the beginning of this course it is necessary to establish the properties of real numbers on which the subsequent proofs will be built. It has been my experience that in order to cover, within the allotted time, a substantial amount of analysis, it is not possible to provide a detailed construction of the real numbers starting with a serious treatment of set theory. I have chosen to codify the properties of the real numbers as three groups of axioms. In the Preliminaries, the arithmetic and order properties are codified in the Field and Positivity Axioms: a detailed discussion of the consequences of these axioms, which certainly are familiar to the student, is provided in Appendix A. The least familiar of these axioms, the Completeness Axiom, is presented in the first section of the first chapter, Section 1.1.

The first four chapters contain material that is essential. In Chapter 2 the properties of convergent sequences are established. Monotonicity, linearity, sum, and product properties of convergent sequences are proved. Three important consequences of the Completeness Axiom are proved: The Monotone Convergence Theorem, the Nested Interval Theorem, and the Sequential Compactness Theorem. Chapter 2 lays the foundation for the later study of continuity, limits and integration which are approached through the concept of convergent sequences. In Chapter 3 continuous functions and limits are studied. Chapter 4 is devoted to the study of differentiation.

Chapter 5 is optional. The student will be familiar with the properties of the logarithmic and trigonometric functions and their inverses, although, most probably, they will not have seen a rigorous analysis of these functions. In Chapter 5, the natural logarithm, the sine, and the cosine functions are introduced as the (unique) solutions of particular differential equations; on the provisional assumption that these equations have solutions; an analytic derivation of the properties of these functions and their inverses is provided. Later, after the differentiability properties of functions defined by integrals and by power series have been established, it is proven that these differential equations do indeed have solutions, and so the provisional assumptions of Chapter 5 are removed. I consider Chapter 5 to be an opportunity to develop an appreciation of the manner in which the basic theory of the first four chapters can be used in the study of properties of solutions of differential equations. Not all of the chapter need be covered and certainly the viewpoint that the basic properties of the elementary functions are already familiar to the student and therefore the chapter can be skipped is defensible.

Chapter 6 is devoted to essential material on integration. The fundamental properties of the Riemann integral are developed exploiting the properties of convergent real sequences through an integrability criterion called the Archimedes–Riemann Theorem. Chapter 7 contains further topics in integration that are optional: later developments are independent of the material in Chapter 7.

The study of the approximation of functions by Taylor polynomials is the subject of Chapter 8. In Chapter 9, we consider a sequence of functions that converges to a limit function and study the way in which the limit function inherits properties possessed by the functions that are the terms of the sequence; the distinction between pointwise and uniform convergence is emphasized. Depending on the time available and the focus of a course, selections can be made in Chapters 8 and 9; the only topic in these chapters that is needed later is the several variable version of the second-order Taylor Approximation Theorem. I always cover the first three sections of Chapter 8 and one or two of the particular jewels of analysis such as the Weierstrass Approximation Theorem, the example on an infinitely differentiable function that is not analytic, or the example of nowhere differentiable continuous functions.

The study of functions of several variables begins in Chapter 10 with the study of Euclidean space $\mathbb{R}^n$. The scalar product and the norm are introduced. There is no class of subsets of $\mathbb{R}^n$ that plays the same distinguished role with regard to functions of several variables as do intervals with regard to functions of a single variable. For this reason, the general concepts of open and closed subsets of $\mathbb{R}^n$ are introduced and their elementary properties examined. In Chapter 11, we study the manner in which the results about sequences of numbers and functions of a single variable extend to sequences of points in $\mathbb{R}^n$, to functions defined on subsets of Euclidean space, and to mappings between such spaces. The concepts of sequential compactness, compactness, pathwise connectedness and connectedness are examined for sets in $\mathbb{R}^n$ in the context of the special properties possessed by functions that have as their domains such sets. Chapters 10 and 11 are extensions to functions of several variables of the material covered in Chapters 1, 2, and 3 for functions of one variable.

Chapter 12, on metric spaces, is optional. The student will have already seen important specific realizations of the general theory, namely the concept of uniform convergence for sequences of functions and the study of subsets of Euclidean space, and with these examples in mind can better appreciate the general theory. The Contraction Mapping Principle is proved and used to establish the fundamental existence result on the solvability of nonlinear scalar differential equations for a function of one variable. This serves as a powerful example of the use of brief, general theory to furnish concrete information about specific problems. None of the subsequent material depends on Chapter 12.

The material related to differentiation of functions of several variables is covered in Chapters 13 and 14. The central point of these chapters is that a function of several variables that has continuous partial derivatives has directional derivatives in all directions, the Mean Value Theorem holds, and therefore the function has good local approximation properties.

The study of mappings between Euclidean spaces that have continuously differentiable component functions is studied in Chapter 15. At each point in the domain of a continuously differentiable mapping there is defined the derivative matrix, together with the corresponding linear mapping called the differential. Approximation by linear mapping is studied and the chapter concludes with the Chain Rule for mappings. Here, and at other points in the book, it is necessary to understand some linear algebra. As one solution of the problem of establishing what a student can be expected to know, the entire Section 15.1 is devoted to the correspondence between linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices. As for the other topics that involve linear algebra, in Appendix B basic topics in linear algebra are described, and using the cross product of two vectors full proofs are provided for the case of vectors and linear mappings in $\mathbb{R}^3$: in particular, the relation between the determinant and volume is established.

The Inverse Function Theorem and the Implicit Function Theorem are the focus of Chapters 16 and 17, respectively. I have made special effort to clearly present these theorems and related materials, such as the minimization principle for studying nonlinear systems of equations, not as isolated technical results but as part of the theme of understanding what properties a mapping can be expected to inherit from its linearization. These two theorems are surely the clearest expression of the way that a nonlinear object (a mapping or a system of equations) inherits properties from a linear approximation. In a course in which there is very limited time and it is decided that a significant part of integration for functions of several variables must be covered, the material in Chapters 16 and 17, except for the Inverse Function Theorem in the plane, can be deferred and the course can proceed directly from Chapter 15 to Chapter 18.

The theory of integration of functions of several variables occupies the last three chapters of the book. In Chapter 18, the integral is first defined for bounded functions defined on generalized rectangles. Most of the results for functions of a single variable carry over without change of proof. The Archimedes–Riemann Theorem is proved as the principal criterion for integrability. We prove that a bounded function defined on a generalized rectangle is integrable if its set of discontinuities has Jordan content 0. Then integration for bounded functions defined on bounded subsets of $\mathbb{R}^n$ is considered, in terms of extensions of such functions to generalized rectangles containing the original domain. Familiar properties of the integral of a function of a single variable (linearity, monotonicity, additivity over domains, and so forth) are established for the integral of functions of several variables. In Chapter 19, Fubini's Theorem on iterated integration is proved and the Change of Variables Theorem for the integral of functions of several variables is proved. In Chapter 20, the book concludes with the study of line and surface integrals. Our goal is to clearly present a description and proof of the way in which the First Fundamental Theorem of Calculus (Integrating Derivatives) for functions of a single variable can be lifted from the line to the plane (Green's Formula) and then how Green's Formula can be lifted from the plane to three-space (Stokes's Formula). I have resisted the temptation to present the general theory of integration of manifolds. In order to make the analytical ideas transparent, rather than present the most general results, emphasis has been placed on a careful treatment of parameterized paths and parameterized surfaces, so that the essentially technical issues associated with patching of surfaces are not present.

Comments on the New Edition

The first edition was thoroughly scrutinized in the light of almost ten years of experience and much comment from users. More than two hundred new exercises were added, of varying levels of difficulty. A multitude of small changes have been made in the exposition to make the material more accessible to the student. Moreover, quite substantial changes were made that need to be taken into consideration in developing a syllabus for a course. Some comment on these changes is in order.

Chapter 1: Sections 1.1 and 1.2 have been rewritten. Auxiliary material has been pruned or placed in the exercises. The Dedekind Gap Theorem is no longer present since it is no longer needed in the development of the integral. The theorem that any interval of the form $[c, c+1)$ contains exactly one integer has become the basis of the proof of the density of the rationals.

Chapter 2: Lemma 2.9, which we call the Comparison Lemma, has replaced the Squeezing Principal of the first edition as a frequently used tool to establish convergence of a sequence. Proofs of the product and quotient properties of convergent sequences have been simplified. The material from the preceding Chapter 2 has been regathered differently among the Sections 2.1, 2.2, 2.3, and 2.4 and some additional topics and examples have been included. A new optional section, Section 2.5, on compactness has been added with a novel proof of the Heine-Borel Theorem. The theorem formerly called the Bolzano–Weierstrass Theorem is now consistently called the Sequential Compactness Theorem.

Chapter 3: Section 3.4 is now an independent brief section on uniform continuity in which a novel sequential definition of uniform continuity is used. Section 3.5 is now a brief independent section in which the sequential definitions of continuity at a point and uniform continuity are reconciled with the corresponding ϵ-δ criteria.

Section 3.6 is a significantly altered version of Section 3.4 of the first edition in which the main results regarding continuity of inverse functions are derived from the fact that a monotone function whose image is an interval must be continuous. A more careful treatment of rational power functions is provided.

Chapter 4: Section 4.2 on the differentiation of inverse functions and compositions has been amplified and clarified. A better motivated proof of the Mean Value Theorem is provided that is an easily recognized model for the later Cauchy Mean Value Theorem. The Darboux Theorem regarding the intermediate value property possessed by the derivative of a differentiable function and what was called the Fundamental Differential Equation are not present in this edition. Material has been inserted in the section on the Fundamental Theorem of Calculus (Differentiating Integrals) which emphasizes the points previously made in now absent Section 4.5 of the first edition.

Chapter 5: The basic material remains the same as in the first edition but many more details have been added and the material has been divided into more easily digestible subsections.

Chapters 6 and 7: The material in these two chapters on integration is very different from the corresponding material in the first edition. First, the essential material, including both Fundamental Theorems of Calculus, has been gathered together in the single Chapter 6 while the auxiliary material is now in Chapter 7. More importantly, the basis of the development of the integral is different. In the first edition, a function was defined to be integrable provided that there was exactly one number that lay between each lower and upper Darboux sum and then the Dedekind Gap Theorem was used to establish an integrability criterion. We now immediately introduce the concept of lower and upper integrals and define a function to be integrable provided that the upper integral equals the lower integral. We define the concept of an Archimedean sequence of partitions for a bounded function on a closed bounded interval and prove a basic integrability criterion we call the Archimedes–Riemann Theorem. This accessible sequential convergence criterion together with the results we have established for sequences provides a well motivated method to establish the basic properties of the integral. Finally, the gap of a partition P is now denoted by gap P rather that $\|P\|$.

Chapter 8: A number of smaller changes have been made: for instance, a crude initial estimate of e is now obtained by a transparent comparison with Darboux sums rather than the previous subtle change of variables computation and the treatment of Euler's constant is clarified. The proofs of Newton's Binomial Theorem and the Weierstrass Approximation Theorem are simplified.

Chapter 9: Section 9.3 on the manner in which continuity, differentiability and integrability is inherited by the limit of a sequence of functions has been sharpened. The discussion in Section 9.6 of the example of a continuous, nowhere differentiable function has been greatly simplified by the introduction of the geometric concept of a tent function of base length 2ℓ.

Chapter 10: In this and succeeding chapters the distance between two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is now denoted by $\text{dist}(\mathbf{u}, \mathbf{v})$ rather than $d(\mathbf{u}, \mathbf{v})$, what was called a "symmetric neighborhood" of a point is now called "an open ball about" a point and the notation changes from $\mathcal{N}_r(\mathbf{u})$ to $\mathcal{B}_r(\mathbf{u})$. The material in Chapter 10 is essentially unchanged.

Chapter 11: Uniform continuity is now defined in terms of differences of sequences as it was for functions of a single variable. What was Section 11.3 in the first edition is now split into two sections. Section 11.3 is devoted to pathwise connectedness and the intermediate value property while Section 11.4 is devoted to connectedness and the intermediate value property. Both sections are labeled as optional since the only use made later regarding connectedness is that the image of a generalized interval under a continuous function is an interval. A very short independent proof of this fact can be provided when it is needed.

Chapter 12: This chapter remains essentially unchanged and is still optional.

Chapters 13 and 14: In these and succeeding chapters for a function of several variables f at the point $\mathbf{u}$ in its domain the classic notation $\nabla f(\mathbf{u})$ is now consistently used for the derivative vector and $\nabla^2 f(\mathbf{u})$ is used for the Hessian. The exposition in Section 14.1 has been abbreviated and the succeeding two sections labeled as optional.

Chapters 15, 16, and 17: The exposition has been tightened and clarified in a number of places but the material remains essentially the same as in the first edition.

Chapter 18: The development of the integral has been substantially changed in order to parallel the new treatment of integration of functions of a single variable in Chapter 6. This has led to considerable simplification and clarification.

Chapter 19 and Chapter 20: These contain material that was in Chapter 19 of the first edition. The material has not been substantially changed.

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Patrick M. Fitzpatrick