Preface

The first time I cracked open a text on ordinary differential equations (ODE), it was instant love. I don't remember the exact title of the book, or the author's name, but I do remember that the book was thinner than the ones I have seen selected for use in introductory courses in recent times. It was a book a student could read from cover to cover, while taking a course in the subject. I began to write course notes, with the aim of producing a text more to my liking. After a few years of this, the current book emerged.

This book has four chapters. I use Chapter 1 and parts of Chapters 2 and 3 for a first semester introduction to differential equations, and I use the rest of Chapters 2 and 3 together with Chapter 4 for the second semester.

Chapter 1 deals with single differential equations, first equations of order 1,

$$\frac{dx}{dt} = f(t, x), \tag{0.0.1}$$

then equations of order 2,

$$\frac{d^2x}{dt^2} = f(t, x, x'). \tag{0.0.2}$$

We have a brief discussion of higher order equations. For second order equations, we concentrate on the case

$$\frac{d^2x}{dt^2} = f(x, x'), \tag{0.0.3}$$

which can be reduced to a first order equation for $v = x'$, as a function of x. Newton's law $F = ma$ for motion of a particle on a line gives such equations. We also specialize (0.0.2) to the linear case,

$$x'' + bx' + cx = f(t), \tag{0.0.4}$$

and discuss techniques for solving such equations.

While the study of single equations is the place to start, the subject of differential equations is and always has been mainly about *systems* of equations. This study requires a healthy dose of linear algebra. For a number of good reasons, it is not desirable to require a course in linear algebra as a prerequisite (or even a corequisite) for a course in differential equations, but rather the course includes some basic instruction in linear algebra. Chapter 2 provides the needed minicourse in linear algebra. We differ from most introductions to differential equations in providing complete proofs of the relevant results, including material on determinants, eigenvalues and eigenvectors of a linear transformation, and also generalized eigenvectors.

Chapter 3 deals with linear systems of differential equations. We start with the $n \times n$ system

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0 \in \mathbb{C}^n, \tag{0.0.5}$$

where A is an $n \times n$ matrix, and define the matrix exponential e^{tA}, which produces the solution

$$x(t) = e^{tA} x_0. \tag{0.0.6}$$

Material from Chapter 2 plays a central role in analyzing this matrix exponential. We proceed from (0.0.5) to the inhomogeneous system

$$\frac{dx}{dt} = Ax + f(t), \quad x(0) = x_0. \tag{0.0.7}$$

We also study variable coefficient equations

$$\frac{dx}{dt} = A(t)x + f(t). \tag{0.0.8}$$

In particular, we study power series expansions for the solution, when $A(t)$ and $f(t)$ are given by convergent power series. We also consider expansions when (0.0.8) has a "regular singular point." These power series topics are usually introduced in the context of a single second order equation, before the study of systems. Indeed, in Chapter 1, §1.15 touches on this, and §1.16 goes into some detail in the important special case of Bessel's equation. We have saved the general study for Chapter 3, both to speed the introduction to systems and because the presentation in the system context is both more compact and more general than in the context of a single, second order equation.

Chapter 4 crowns the text, with a study of nonlinear systems of differential equations. These can have the form

$$\frac{dx}{dt} = F(t, x), \quad x(t_0) = y, \tag{0.0.9}$$

which resembles (0.0.1), except that now $x(t)$ and $F(t, x)$ take values in $\mathbb{R}^n$. We begin with general existence and uniqueness results. For this, we convert (0.0.9) to the integral equation

$$x(t) = y + \int_{t_0}^{t} F(s, x(s))\, ds. \tag{0.0.10}$$

and use the Picard iteration to produce the solution, for $|t - t_0|$ subject to certain limitations, as a limit of a certain sequence of approximate solutions. This is followed by results on how large the interval of existence can be taken. Next, we look into results on the smooth dependence of solutions $x = x(t, y)$ to (0.0.9) on the initial data y. An important role is played by the linearization of (0.0.9).

From here we proceed to some qualitative studies of solutions, particularly in the autonomous case, $F(t, x) = F(x)$, in which we interpret F as a vector field, and the solution as the flow Φ^t generated by this vector field. One useful tool is the phase portrait, which depicts the behavior of solution curves (also called orbits) for nonlinear $n \times n$ systems. From the point of view of visualization, the portraits work particularly well for $n = 2$, and are also quite useful for $n = 3$.

We study a variety of problems from mathematical physics, including the planetary motion problem for two bodies interacting by the gravitational force, whose solution by Newton was a seminal inspiration to the field of ODE. We also bring in further advances in the study of equations of physics, due to Euler and Lagrange, involving the variational method. This theory impacts both physical and geometrical applications of ODE, the latter including equations for geodesics on surfaces in $\mathbb{R}^n$.

By this point we are looking at nonlinear systems whose solutions are not necessarily amenable to formulas. In addition to qualitative studies of the nature of these solutions, numerical studies arise as an important tool. This is taken up in §4.11. We introduce difference schemes, with emphasis on the Runge-Kutta scheme, as a very useful computational tool.

In Sections 4.13–4.14 we turn to some problems arising in mathematical biology. This is followed with some results on systems with chaotic dynamics, which arise in dimension ≥ 3. This chapter closes with a number of appendices, some providing useful background in calculus, and others taking up further topics in nonlinear systems of ODE.

We follow this introduction with a record of some standard notation that will be in use throughout the text.