Chapter 1

Single differential equations

This first chapter is devoted to differential equations for a single unknown function, with emphasis on equations of the first and second order, i.e.,

$$\frac{dx}{dt} = f(t, x), \tag{1.0.1}$$

and

$$\frac{d^2x}{dt^2} = f\Big(t, x, \frac{dx}{dt}\Big). \tag{1.0.2}$$

Section 1.1 looks at the simplest case of (1.0.1), namely

$$\frac{dx}{dt} = x. \tag{1.0.3}$$

We construct the solution $x(t)$ to (1.0.3) such that $x(0) = 1$ as a power series, defining the exponential function

$$x(t) = e^t. \tag{1.0.4}$$

More generally, $x(t) = e^{ct}$ solves $dx/dt = cx$, with $x(0) = 1$. This holds for all real c and also for *complex* c. Taking $c = i$ and investigating basic properties of $x(t) = e^{it}$, we establish Euler's formula,

$$e^{it} = \cos t + i \sin t, \tag{1.0.5}$$

which in turn leads to a self-contained exposition of basic results on the trigonometric functions.

Section 1.2 treats first order linear equations, of the form

$$\frac{dx}{dt} + a(t)x = b(t), \quad x(t_0) = x_0, \tag{1.0.6}$$

and produces solutions in terms of the exponential function and integrals. Section 1.3 considers some nonlinear first order equations, particularly equations for which *separation of variables* allows one to produce a solution, in terms of various integrals.

We differ from many introductions in not lingering on the topic of first order equations. For example, we do not treat exact equations and integrating factors in this chapter. We consider it more important to get on to the study of second order equations. In any case, exact equations do get their due, in §4.4 of Chapter 4.

In §1.4 we take up second order differential equations. We concentrate there on two special classes, each allowing for a reduction to first order equations. In §1.5 we consider differential equations arising from some physical problems for motion in one space dimension, making use of Newton's law $F = ma$. The equations that arise in this context are amenable to methods of §1.4. In §1.5 we restate these methods in terms that celebrate the physical quantities of kinetic and potential energy, and the conservation of total energy. Section 1.6 deals with the classical pendulum, a close relative of motion on a line. In §1.7 we discuss motion in the presence of resistance, including the pendulum with resistance.

Formulas from §1.6 give rise to complicated integrals, and problems of §1.7 have additional complications. These complications arise because of nonlinearities in the equations. In §1.8 we discuss *linearization* of these equations. The associated linear differential equations are amenable to explicit analysis.

Sections 1.9–1.15 are devoted to linear second order differential equations, starting with constant coefficient equations

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \tag{1.0.7}$$

first with $f \equiv 0$ in §1.9, then allowing f to be nonzero. In §1.10 we consider certain special forms of $f(t)$, including

$$e^{\kappa t}, \quad \sin \sigma t, \quad \cos \sigma t, \quad t^k, \tag{1.0.8}$$

treating these cases by the *method of undetermined coefficients.* We discuss implications of results here, when $f(t) = A \sin \sigma t$, for the forced, linearized pendulum, in §1.11. Sections 1.12–1.13 treat other physical problems leading to equations of the form (1.0.7), namely spring motion problems and models of certain simple electrical circuits, called RLC circuits. In §1.14 we bring up another method, *variation of parameters*, which applies to general functions f in (1.0.7).

Section 1.15 gives some results on variable coefficient second order linear differential equations. Tools brought to bear on these equations include power series representations, extending the power series attack used on (1.0.3), and the Wronskian, first introduced in the constant-coefficient context in §1.12. In §1.16 we concentrate on a particularly important second-order ODE with variable coefficients, Bessel's equation, further pushing power series techniques and the use of the Wronskian. In §1.17 we discuss differential equations of order ≥ 3. In §1.18 we introduce the Laplace transform as a tool to treat nonhomogeneous differential equations, such as (1.0.7) and higher order variants. Material introduced in §§1.15–1.18 will be covered, on a much more general level, in Chapter 3.

We end this chapter with three appendices. Appendix 1.A explains how Bessel functions arise in the search for solutions to some basic partial differential equations. Appendix 1.B has some basic material on Euler's gamma function, of use in §1.16. Appendix 1.C establishes that convergent power series can be differentiated term by term. We also derive the power series of $f(t) = (1 - t)^{-r}$.

1.1. The exponential and trigonometric functions

We construct the exponential function to solve the differential equation

$$\frac{dx}{dt} = x, \quad x(0) = 1. \tag{1.1.1}$$

We seek a solution as a power series

$$x(t) = \sum_{k=0}^{\infty} a_k t^k. \tag{1.1.2}$$

If such a power series converges for t in an interval in $\mathbb{R}$, it can be differentiated term-by-term. (See (1.1.45)–(1.1.50) below, and also §1.C, for more on this.) In such a case,

$$\begin{aligned} x'(t) &= \sum_{k=1}^{\infty} k a_k t^{k-1} \\ &= \sum_{\ell=0}^{\infty} (\ell+1) a_{\ell+1} t^{\ell}, \end{aligned} \tag{1.1.3}$$

so for (1.1.1) to hold we need

$$a_0 = 1, \quad a_{k+1} = \frac{a_k}{k+1}, \tag{1.1.4}$$

i.e., $a_k = 1/k!$, where $k! = k(k-1)\cdots 2\cdot 1$. Thus (1.1.1) is solved by

$$x(t) = e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k, \quad t \in \mathbb{R}. \tag{1.1.5}$$

This defines the exponential function e^t.

More generally, we can define

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k, \quad z \in \mathbb{C}. \tag{1.1.6}$$

The issue of convergence for complex power series is essentially the same as for real power series. Given $z = x + iy,\ x, y \in \mathbb{R}$, we have $|z| = \sqrt{x^2+y^2}$. If also $w \in \mathbb{C}$, then $|z+w| \le |z| + |w|$ and $|zw| = |z|\cdot|w|$. Hence

$$\Bigl|\sum_{k=m}^{m+n} \frac{1}{k!} z^k\Bigr| \le \sum_{k=m}^{m+n} \frac{1}{k!} |z|^k.$$

The ratio test then shows that the series (1.1.6) is absolutely convergent for all $z \in \mathbb{C}$, and uniformly convergent for $|z| \le R$, for each $R < \infty$. Note that

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k}{k!} t^k \tag{1.1.7}$$

solves

$$\frac{d}{dt} e^{at} = a e^{at}, \tag{1.1.8}$$

and this works for each $a \in \mathbb{C}$.

We claim that e^{at} is the only solution to

$$\frac{dy}{dt} = ay, \quad y(0) = 1. \tag{1.1.9}$$

To see this, compute the derivative of $e^{-at}y(t)$:

$$\frac{d}{dt}\big(e^{-at}y(t)\big) = -ae^{-at}y(t) + e^{-at}ay(t) = 0, \tag{1.1.10}$$

where we use the product rule, (1.1.8) (with a replaced by $-a$) and (1.1.9). Thus $e^{-at}y(t)$ is independent of t. Evaluating at $t = 0$ gives

$$e^{-at}y(t) = 1, \quad \forall\, t \in \mathbb{R}, \tag{1.1.11}$$

whenever $y(t)$ solves (1.1.9). Since e^{at} solves (1.1.9), we have $e^{-at}e^{at} = 1$, hence

$$e^{-at} = \frac{1}{e^{at}}, \quad \forall\, t \in \mathbb{R},\ a \in \mathbb{C}. \tag{1.1.12}$$

Thus multiplying both sides of (1.1.11) by e^{at} gives the asserted uniqueness:

$$y(t) = e^{at}, \quad \forall\, t \in \mathbb{R}. \tag{1.1.13}$$

We can draw further useful conclusions from applying d/dt to products of exponential functions. In fact, let $a, b \in \mathbb{C}$. Then

$$\begin{aligned} &\frac{d}{dt}\Big(e^{-at}e^{-bt}e^{(a+b)t}\Big) \\ &= -ae^{-at}e^{-bt}e^{(a+b)t} - be^{-at}e^{-bt}e^{(a+b)t} + (a+b)e^{-at}e^{-bt}e^{(a+b)t} \\ &= 0, \end{aligned} \tag{1.1.14}$$

so again we are differentiating a function that is independent of t. Evaluation at $t = 0$ gives

$$e^{-at}e^{-bt}e^{(a+b)t} = 1, \quad \forall\, t \in \mathbb{R}. \tag{1.1.15}$$

Using (1.1.12), we get

$$e^{(a+b)t} = e^{at}e^{bt}, \quad \forall\, t \in \mathbb{R},\ a, b \in \mathbb{C}, \tag{1.1.16}$$

or, setting $t = 1$,

$$e^{a+b} = e^a e^b, \quad \forall\, a, b \in \mathbb{C}. \tag{1.1.17}$$

We next record some properties of $\exp(t) = e^t$ for real t. The power series (1.1.5) clearly gives $e^t > 0$ for $t \geq 0$. Since $e^{-t} = 1/e^t$, we see that $e^t > 0$ for all $t \in \mathbb{R}$. Since $de^t/dt = e^t > 0$, the function is monotone increasing in t, and since $d^2e^t/dt^2 = e^t > 0$, this function is convex. Note that

$$e^1 = 1 + 1 + \frac{1}{2} + \cdots > 2, \tag{1.1.18}$$

so $e^k > 2^k \nearrow +\infty$ as $k \to +\infty$. Hence

$$\lim_{t \to +\infty} e^t = +\infty. \tag{1.1.19}$$

Since $e^{-t} = 1/e^t$,

$$\lim_{t \to -\infty} e^t = 0. \tag{1.1.20}$$

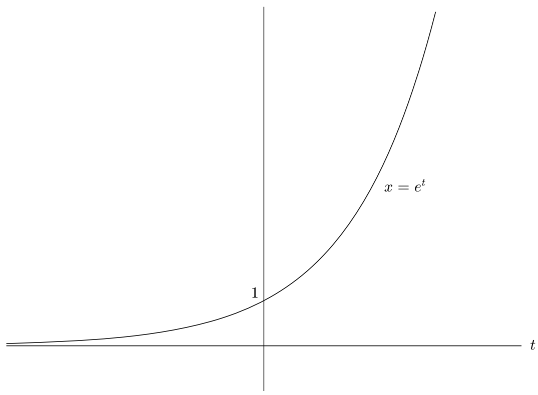

Figure 1.1.1. Exponential function

As a consequence,

$$\exp : \mathbb{R} \longrightarrow (0, \infty) \tag{1.1.21}$$

is smooth and one-to-one and onto, with positive derivative, so the inverse function theorem of one-variable calculus applies. There is a smooth inverse

$$L : (0, \infty) \longrightarrow \mathbb{R}. \tag{1.1.22}$$

We call this inverse the natural logarithm:

$$\log x = L(x). \tag{1.1.23}$$

See Figures 1.1.1 and 1.1.2 for graphs of $x = e^t$ and $t = \log x$.

Applying d/dt to

$$L(e^t) = t \tag{1.1.24}$$

gives

$$L'(e^t)e^t = 1, \quad \text{hence} \quad L'(e^t) = \frac{1}{e^t}, \tag{1.1.25}$$

i.e.,

$$\frac{d}{dx} \log x = \frac{1}{x}. \tag{1.1.26}$$

Since $\log 1 = 0$, we get

$$\log x = \int_1^x \frac{dy}{y}. \tag{1.1.27}$$

An immediate consequence of (1.1.17) (for $a, b \in \mathbb{R}$) is the identity

$$\log xy = \log x + \log y, \quad x, y \in (0, \infty). \tag{1.1.28}$$

We move on to a study of e^z for purely imaginary z, i.e., of

$$\gamma(t) = e^{it}, \quad t \in \mathbb{R}. \tag{1.1.29}$$

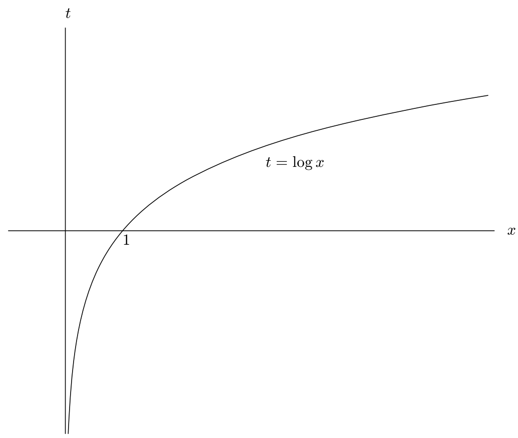

Figure 1.1.2. The logarithm

This traces out a curve in the complex plane, and we want to understand which curve it is. Let us set

$$e^{it} = c(t) + is(t), \tag{1.1.30}$$

with $c(t)$ and $s(t)$ real valued. First we calculate $|e^{it}|^2 = c(t)^2 + s(t)^2$. For $x, y \in \mathbb{R}$,

$$z = x + iy \Longrightarrow \overline{z} = x - iy \Longrightarrow z\overline{z} = x^2 + y^2 = |z|^2. \tag{1.1.31}$$

It is elementary that

$$\begin{gathered} z, w \in \mathbb{C} \Longrightarrow \overline{zw} = \overline{z}\,\overline{w} \Longrightarrow \overline{z^n} = \overline{z}^n, \\ \text{and } \ \overline{z+w} = \overline{z} + \overline{w}. \end{gathered} \tag{1.1.32}$$

Hence

$$\overline{e^z} = \sum_{k=0}^{\infty} \frac{\overline{z}^k}{k!} = e^{\overline{z}}. \tag{1.1.33}$$

In particular,

$$t \in \mathbb{R} \Longrightarrow |e^{it}|^2 = e^{it}e^{-it} = 1. \tag{1.1.34}$$

Hence $t \mapsto \gamma(t) = e^{it}$ has image in the unit circle centered at the origin in $\mathbb{C}$. Also

$$\gamma'(t) = ie^{it} \Longrightarrow |\gamma'(t)| \equiv 1, \tag{1.1.35}$$

so $\gamma(t)$ moves at unit speed on the unit circle. We have

$$\gamma(0) = 1, \quad \gamma'(0) = i. \tag{1.1.36}$$

Thus, for t between 0 and the circumference of the unit circle, the arc from $\gamma(0)$ to $\gamma(t)$ is an arc on the unit circle, pictured in Figure 1.1.3, of length

$$\ell(t) = \int_0^t |\gamma'(s)|\, ds = t. \tag{1.1.37}$$

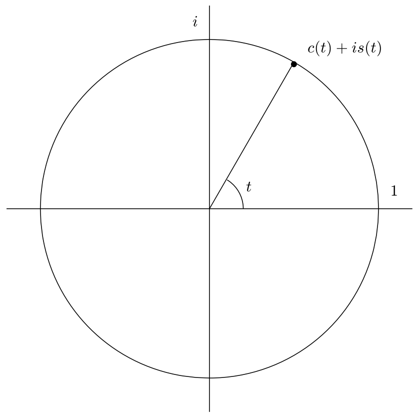

Figure 1.1.3. Behind Euler's formula

Standard definitions from trigonometry say that the line segments from 0 to 1 and from 0 to $\gamma(t)$ meet at angle whose measurement in radians is equal to the length of the arc of the unit circle from 1 to $\gamma(t)$, i.e., to $\ell(t)$. The cosine of this angle is defined to be the x-coordinate of $\gamma(t)$ and the sine of the angle is defined to be the y-coordinate of $\gamma(t)$. Hence the computation (1.1.37) gives

$$c(t) = \cos t, \quad s(t) = \sin t. \tag{1.1.38}$$

Thus (1.1.30) becomes

$$e^{it} = \cos t + i \sin t, \tag{1.1.39}$$

which is Euler's formula. The identity

$$\frac{d}{dt} e^{it} = ie^{it}, \tag{1.1.40}$$

applied to (1.1.39), yields

$$\frac{d}{dt} \cos t = -\sin t, \quad \frac{d}{dt} \sin t = \cos t. \tag{1.1.41}$$

We can use (1.1.17) to derive formulas for sin and cos of the sum of two angles. Indeed, comparing

$$e^{i(s+t)} = \cos(s+t) + i \sin(s+t) \tag{1.1.42}$$

with

$$e^{is} e^{it} = (\cos s + i \sin s)(\cos t + i \sin t) \tag{1.1.43}$$

gives

$$\begin{aligned} \cos(s+t) &= (\cos s)(\cos t) - (\sin s)(\sin t), \\ \sin(s+t) &= (\sin s)(\cos t) + (\cos s)(\sin t). \end{aligned} \tag{1.1.44}$$

Returning to basics, we recall that the calculations done so far in this section were all predicated on the fact that the power series (1.1.7) can be differentiated term by term. This is a special case of a general result about convergent power series, established in §1.C. However, making use of the special structure of (1.1.7), we include a direct demonstration here. To begin, look at

$$E_n^a(t) = \sum_{k=0}^{n} \frac{a^k}{k!} t^k, \tag{1.1.45}$$

which satisfies

$$\begin{aligned} \frac{d}{dt} E_n^a(t) &= \sum_{k=1}^{n} \frac{a^k}{(k-1)!} t^{k-1} \\ &= \sum_{\ell=0}^{n-1} \frac{a^{\ell+1}}{\ell!} t^\ell \\ &= a E_{n-1}^a(t). \end{aligned} \tag{1.1.46}$$

Integration gives

$$a \int_0^t E_{n-1}^a(s)\, ds = E_n^a(t) - 1. \tag{1.1.47}$$

Now we have

$$E_{n-1}^a(s) \longrightarrow e^{as}, \quad E_n^a(t) \longrightarrow e^{at}, \tag{1.1.48}$$

uniformly on finite intervals, as $n \to \infty$, and then the integral estimate

$$\Bigl| \int_0^t (E(s) - F(s))\, ds \Bigr| \le |t| \max_{0 \le s \le t} |E(s) - F(s)|$$

implies

$$\int_0^t E_{n-1}^a(s)\, ds \longrightarrow \int_0^t e^{as}\, ds, \tag{1.1.49}$$

as $n \to \infty$. Consequently, we can pass to the limit $n \to \infty$ in (1.1.47) and get

$$a \int_0^t e^{as}\, ds = e^{at} - 1. \tag{1.1.50}$$

Applying d/dt to the left side of (1.1.50) gives ae^{at}, by the fundamental theorem of calculus. Hence this must be the derivative of the right side of (1.1.50), and this gives (1.1.8).

Having the integral formula (1.1.50), we proceed to obtain formulas for $\int t^n e^{at}\, dt$. In fact, from (1.1.46), (1.1.8), and the product rule, we obtain

$$\begin{aligned} \frac{d}{dt}\bigl(e^{-at} E_n^a(t)\bigr) &= -ae^{-at} E_n^a(t) + ae^{-at} E_{n-1}^a(t) \\ &= -\frac{a^{n+1}}{n!} t^n e^{-at}. \end{aligned} \tag{1.1.51}$$

Then the fundamental theorem of calculus gives

$$\begin{aligned}\int t^n e^{-at}\,dt &= -\frac{n!}{a^{n+1}} E_n^a(t) e^{-at} + C \\ &= -\frac{n!}{a^{n+1}}\Big(1 + at + \frac{a^2t^2}{2!} + \cdots + \frac{a^n t^n}{n!}\Big) e^{-at} + C.\end{aligned} \tag{1.1.52}$$

We have an analogous formula for $\int t^n e^{at}\,dt$, by replacing a by $-a$.

Exercises

1. As noted, if $z = x+iy,\ x, y \in \mathbb{R}$, then $|z| = \sqrt{x^2+y^2}$ is equivalent to $|z|^2 = z\,\overline{z}$. Use this to show that if also $w \in \mathbb{C}$,

$$|zw| = |z| \cdot |w|.$$

Note that

$$\begin{aligned}|z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\ &= |z|^2 + |w|^2 + w\overline{z} + z\overline{w} \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}\, z\overline{w}.\end{aligned}$$

Show that $\mathrm{Re}(z\overline{w}) \le |zw|$, and use this in concert with an expansion of $(|z|+|w|)^2$ and the first identity above to deduce that

$$|z+w| \le |z| + |w|.$$

2. Define π to be the smallest positive number such that $e^{\pi i} = -1$. Show that

$$e^{\pi i/2} = i, \quad e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

Hint. See Figure 1.1.4, showing $a = e^{\pi i/3}$.

3. Show that

$$\cos^2 t + \sin^2 t = 1,$$

and

$$1 + \tan^2 t = \sec^2 t,$$

where

$$\tan t = \frac{\sin t}{\cos t}, \quad \sec t = \frac{1}{\cos t}.$$

4. Show that

$$\begin{aligned}\frac{d}{dt}\tan t &= \sec^2 t = 1 + \tan^2 t, \\ \frac{d}{dt}\sec t &= \sec t\ \tan t.\end{aligned}$$

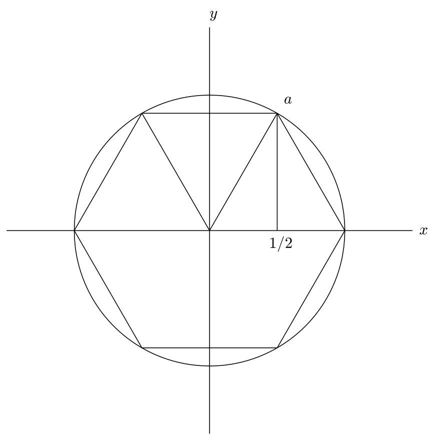

Figure 1.1.4. Hexagon

5. Evaluate
$$\int_0^y \frac{dx}{1+x^2}.$$
Hint. Set $x = \tan t$.

6. Evaluate
$$\int_0^y \frac{dx}{\sqrt{1-x^2}}.$$
Hint. Set $x = \sin t$.

7. Show that
$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$
Hint. Show that $\sin \pi/6 = 1/2$. Use Exercise 2 and the identity $e^{\pi i/6} = e^{\pi i/2} e^{-\pi i/3}$.

8. Set
$$\cosh t = \frac{1}{2}(e^t + e^{-t}), \quad \sinh t = \frac{1}{2}(e^t - e^{-t}).$$
Show that
$$\frac{d}{dt} \cosh t = \sinh t, \quad \frac{d}{dt} \sinh t = \cosh t,$$
and
$$\cosh^2 t - \sinh^2 t = 1.$$

9. Evaluate
$$\int_0^y \frac{dx}{\sqrt{1+x^2}}.$$

Hint. Set $x = \sinh t$.

10. Evaluate

$$\int_0^y \sqrt{1+x^2}\,dx.$$

11. Using Exercise 4, verify that

$$\begin{aligned} \frac{d}{dt}(\sec t + \tan t) &= \sec t(\sec t + \tan t), \\ \frac{d}{dt}(\sec t\ \tan t) &= \sec^3 t + \sec t\ \tan^2 t, \\ &= 2\sec^3 t - \sec t. \end{aligned}$$

12. Next verify that

$$\begin{aligned} \frac{d}{dt}\log|\sec t| &= \tan t, \\ \frac{d}{dt}\log|\sec t + \tan t| &= \sec t. \end{aligned}$$

13. Now verify that

$$\begin{aligned} \int \tan t\,dt &= \log|\sec t|, \\ \int \sec t\,dt &= \log|\sec t + \tan t|, \\ 2\int \sec^3 t\,dt &= \sec t\ \tan t + \int \sec t\,dt. \end{aligned}$$

(Here, we omit the arbitrary additive constants.)

14. Here is another approach to the evaluation of $\int \sec t\,dt$. We evaluate

$$I(u) = \int_0^u \frac{dv}{\sqrt{1+v^2}}$$

in two ways.
(a) Using $v = \sinh y$, show that

$$I(u) = \int_0^{\sinh^{-1} u} dy = \sinh^{-1} u.$$

(b) Using $v = \tan t$, show that

$$I(u) = \int_0^{\tan^{-1} u} \sec t\,dt.$$

Deduce that

$$\int_0^x \sec t\,dt = \sinh^{-1}(\tan x), \quad \text{for } |x| < \frac{\pi}{2}.$$

Deduce from the formula above that also

$$\cosh\Big(\int_0^x \sec t\,dt\Big) = \sec x,$$

and hence that

$$\exp\Bigl(\int_0^x \sec t\,dt\Bigr) = \sec x + \tan x.$$

Compare these formulas with the analogue in Exercise 13.

15. For $E_n^a(t)$ as in (1.1.45), $k \geq 1$, $0 < T < \infty$, show that

$$\max_{|t|\leq T} |E_{n+k}^a(t) - E_n^a(t)| \leq \frac{|aT|^{n+1}}{(n+1)!}\Bigl(1 + \frac{|aT|}{n+2} + \frac{|aT|^2}{(n+2)(n+3)} + \cdots\Bigr), \tag{1.1.53}$$

and that this is

$$\leq 2\frac{|aT|^{n+1}}{(n+1)!}, \quad \text{for} \quad n+2 > 2|aT|. \tag{1.1.54}$$

Deduce that

$$\max_{|t|\leq T} |e^{at} - E_n^a(t)| \tag{1.1.55}$$

satisfies (1.1.54). Show that, for each a, T, (1.1.54) tends to 0 as $n \to \infty$, yielding the assertion made about convergence in (1.1.48).

16. Show that

$$\Bigl|\int_0^t e^{as}\,ds - \int_0^t E_n^a(s)\,ds\Bigr| \leq |t| \max_{|s|\leq|t|} |e^{as} - E_n^a(s)|,$$

and observe how this, together with Exercise 15, yields (1.1.49).

17. Show that

$$|t| < 1 \Rightarrow \log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots. \tag{1.1.56}$$

Hint. Rewrite (1.1.27) as

$$\log(1+t) = \int_0^t \frac{ds}{1+s},$$

expand

$$\frac{1}{1+s} = 1 - s + s^2 - s^3 + \cdots, \quad |s| < 1,$$

and integrate term by term.

18. Use (1.1.52) with $a = -i$ to produce formulas for

$$\int t^n \cos t\,dt \quad \text{and} \quad \int t^n \sin t\,dt.$$

19. Figure 1.1.5 (a)–(b) shows graphs of the image of

$$\gamma(t) = e^{\alpha t}, \quad 0 \leq t \leq 6\pi,$$

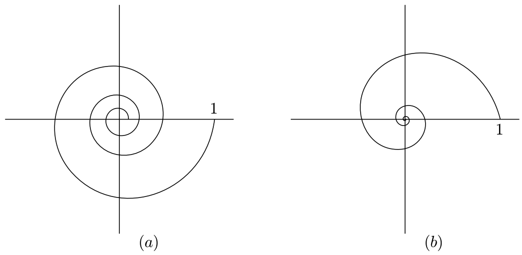

Figure 1.1.5. Spirals

for

$$\alpha = -\frac{1}{4} + i,$$
$$\alpha = -\frac{1}{8} - i.$$

Match each value of α to (a) or (b).

20. Given $t > 0$ and $a \in \mathbb{C}$, we define t^a by

$$t^a = e^{a \log t}.$$

Show that, for $t > 0$,

$$\frac{d}{dt} t^a = a t^{a-1}.$$

1.2. First order linear equations

Here we tackle first order linear equations. These are equations of the form

$$\frac{dx}{dt} + a(t)x = b(t), \quad x(t_0) = x_0, \tag{1.2.1}$$

given functions $a(t)$ and $b(t)$, continuous on some interval containing t_0. As a warm-up, we first treat

$$\frac{dx}{dt} + ax = b, \quad x(0) = x_0, \tag{1.2.2}$$

with a and b constants. One key to solving (1.2.2) is the identity

$$\frac{d}{dt}(e^{at}x) = e^{at}\Bigl(\frac{dx}{dt} + ax\Bigr), \tag{1.2.3}$$

which follows by applying the product formula and (1.1.8). Thus, multiplying both sides of (1.2.2) by e^{at} gives

$$\frac{d}{dt}(e^{at}x) = e^{at}b, \tag{1.2.4}$$

and then integrating both sides from 0 to t gives

$$e^{at}x(t) = x_0 + \int_0^t e^{as} b \, ds. \tag{1.2.5}$$

We can carry out the integral, using (1.1.45), and get

$$e^{at}x(t) = x_0 + \frac{e^{at}-1}{a}b, \tag{1.2.6}$$

and finally division by e^{at} yields

$$\begin{aligned} x(t) &= e^{-at}x_0 + \frac{b}{a}(1-e^{-at}) \\ &= \frac{b}{a} + e^{-at}\Bigl(x_0 - \frac{b}{a}\Bigr). \end{aligned} \tag{1.2.7}$$

In order to tackle (1.2.1), we need a replacement for (1.2.3). To get it, note that if $A(t)$ is differentiable, the chain rule plus (1.1.8) gives

$$\frac{d}{dt}e^{A(t)} = e^{A(t)}A'(t). \tag{1.2.8}$$

Hence

$$\frac{d}{dt}\bigl(e^{A(t)}x\bigr) = e^{A(t)}\Bigl(\frac{dx}{dt} + A'(t)x\Bigr). \tag{1.2.9}$$

Thus we can multiply (1.2.1) by $e^{A(t)}$ and get

$$\frac{d}{dt}\bigl(e^{A(t)}x\bigr) = e^{A(t)}b(t), \tag{1.2.10}$$

provided

$$A'(t) = a(t). \tag{1.2.11}$$

To arrange this, we can set

$$A(t) = \int_{t_0}^{t} a(s)\,ds. \tag{1.2.12}$$

Then we can integrate (1.2.10) from t_0 to t, to get

$$e^{A(t)}x(t) = x_0 + \int_{t_0}^{t} e^{A(s)}b(s)\,ds, \tag{1.2.13}$$

and hence

$$x(t) = e^{-A(t)}x_0 + e^{-A(t)}\int_{t_0}^{t} e^{A(s)}b(s)\,ds. \tag{1.2.14}$$

For example, consider

$$\frac{dx}{dt} - tx = b(t), \quad x(0) = x_0. \tag{1.2.15}$$

From (1.2.12) we get

$$A(t) = -\frac{t^2}{2}, \tag{1.2.16}$$

and (1.2.10) becomes

$$\frac{d}{dt}(e^{-t^2/2}x) = e^{-t^2/2}b(t), \tag{1.2.17}$$

hence

$$e^{-t^2/2}x(t) = x_0 + \int_0^t e^{-s^2/2} b(s)\, ds. \tag{1.2.18}$$

Let us look at two special cases. First,

$$b(t) = t. \tag{1.2.19}$$

Then the integral in (1.2.18) is

$$\int_0^t e^{-s^2/2} s\, ds = \int_0^{t^2/2} e^{-\sigma}\, d\sigma = 1 - e^{-t^2/2}. \tag{1.2.20}$$

The second case is

$$b(t) = 1. \tag{1.2.21}$$

Then the integral in (1.2.18) is

$$\int_0^t e^{-s^2/2}\, ds. \tag{1.2.22}$$

This is not an elementary function, but it can be related to the special function

$$\operatorname{Erf}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-s^2/2}\, ds. \tag{1.2.23}$$

Namely,

$$\frac{1}{\sqrt{2\pi}} \int_0^t e^{-s^2/2}\, ds = \operatorname{Erf}(t) - \operatorname{Erf}(0). \tag{1.2.24}$$

Note that

$$\operatorname{Erf}(0) = \frac{1}{2} \operatorname{Erf}(\infty) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} I, \tag{1.2.25}$$

where

$$\begin{aligned} I = \int_{-\infty}^{\infty} e^{-s^2/2}\, ds \Rightarrow I^2 &= \int_{\mathbb{R}^2} e^{-|x|^2/2}\, dx \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \\ &= 2\pi \int_0^{\infty} e^{-s}\, ds \\ &= 2\pi. \end{aligned} \tag{1.2.26}$$

Hence we have

$$\operatorname{Erf}(\infty) = 1, \quad \operatorname{Erf}(0) = \frac{1}{2}. \tag{1.2.27}$$

Bernoulli equations

Equations of the form

$$\frac{dx}{dt} + a(t)x = b(t)x^n \tag{1.2.28}$$

are called Bernoulli equations. Such an equation is not linear if $n \neq 1$ or 0, but in these cases one gets a linear equation by the substitution

$$y = x^{1-n}. \tag{1.2.29}$$

In fact, (1.2.29) gives $y' = (1-n)x^{-n}x'$, and plugging in (1.2.28) gives

$$\frac{dy}{dt} = (1-n)[b(t) - a(t)y], \tag{1.2.30}$$

which is linear.

Exercises

Solve the following initial value problems. Do the integrals if you can.

1.
$$\frac{dx}{dt} + \frac{1}{t}x = t^2, \quad x(1) = 0.$$

2.
$$\frac{dx}{dt} + t^2 x = t^2, \quad x(0) = 1.$$

3.
$$\frac{dx}{dt} + x = \cos t, \quad x(0) = 0.$$

4.
$$\frac{dx}{dt} + tx = t^3, \quad x(0) = 1.$$

5.
$$\frac{dx}{dt} + tx = x^3, \quad x(0) = 1.$$

6.
$$\frac{dx}{dt} + (\tan t)x = \cos t, \quad x(0) = 1.$$

7.
$$\frac{dx}{dt} + (\sec t)x = \cos t, \quad x(0) = 1.$$

1.3. Separable equations

A separable differential equation is one for which the method of separation of variables, which we introduce in this section, is applicable. We illustrate this with another approach to the equation (1.2.2), which we rewrite as

$$\frac{dx}{dt} = b - ax, \quad x(0) = x_0. \tag{1.3.1}$$

Separating variables involves moving the x-dependent objects to the left and the t-dependent objects to the right, when possible. In case (1.3.1), this is possible; we have

$$\frac{dx}{b - ax} = dt. \tag{1.3.2}$$

We next integrate both sides. A change of variable allows us to use (1.1.27), to obtain

$$\int \frac{dx}{b - ax} = -\frac{1}{a} \int \frac{dx}{x - b/a} = -\frac{1}{a} \log\Big|x - \frac{b}{a}\Big| + C. \tag{1.3.3}$$

Hence (1.3.2) yields

$$-\frac{1}{a} \log\Big|x - \frac{b}{a}\Big| = t - C, \tag{1.3.4}$$

hence

$$x(t) - \frac{b}{a} = \pm e^{-at+aC} = Ke^{-at}. \tag{1.3.5}$$

Here K is a constant, which can be found by using the initial condition $x(0) = x_0$. We get $x_0 - b/a = K$, so (1.3.5) yields

$$x(t) = \frac{b}{a} + e^{-at}\Big(x_0 - \frac{b}{a}\Big), \tag{1.3.6}$$

consistent with (1.2.7).

Generally, a separable differential equation is one that can be put in the form

$$\frac{dx}{dt} = f(x)g(t), \tag{1.3.7}$$

and then separation of variables gives

$$\frac{dx}{f(x)} = g(t)\, dt, \tag{1.3.8}$$

integrating to

$$\int \frac{dx}{f(x)} = \int g(t)\, dt. \tag{1.3.9}$$

Here is another basic example:

$$\frac{dx}{dt} = x^2, \quad x(0) = 1. \tag{1.3.10}$$

We get

$$\frac{dx}{x^2} = dt, \tag{1.3.11}$$

which integrates to

$$-\frac{1}{x} = t + C, \tag{1.3.12}$$

hence $x = -1/(t + C)$. The initial condition in (1.3.10) gives $C = -1$, so the solution to (1.3.10) is

$$x(t) = \frac{1}{1-t}. \tag{1.3.13}$$

Note that this solution blows up as $t \nearrow 1$.

The hanging cable

Suppose a length of cable, lying in the (x, y)-plane, is fastened at $(-a, 0)$ and at $(a, 0)$, and hangs down freely, in equilibrium, as pictured in Figure 1.3.1. The force of gravity acts in the direction of the negative y-axis. We want the equation of the curve traced out by the cable, which we assume to have length $2L$ (not stretchable) and uniform mass density.

To tackle this problem, we introduce $\theta(x)$, the angle the tangent to the curve at $(x, y(x))$ makes with the x-axis, which is given by

$$\tan\theta(x) = y'(x). \tag{1.3.14}$$

We will derive a differential equation for $\theta(x)$, as follows.

At each point $(x, y(x))$, there is a tension on the cable, of magnitude $T(x)$, and the physical laws governing the behavior of the cable are the following. First, the horizontal component of the tension, given by $T(x)\cos\theta(x)$, is constant. Second, the vertical component of the tension, given by $T(x)\sin\theta(x)$, is proportional to the weight of the cable lying below $y = y(x)$, hence to the length $L(x)$ of the cable, from $(0, y(0))$ to $(x, y(x))$. In other words, we have

$$\begin{aligned} T(x)\cos\theta(x) &= T_0, \\ T(x)\sin\theta(x) &= \kappa L(x), \end{aligned} \tag{1.3.15}$$

where T_0 and κ are certain constants (whose quotient will be specified below). As for $L(x)$, we have

$$\begin{aligned} L(x) &= \int_0^x \sqrt{1 + y'(t)^2}\, dt \\ &= \int_0^x \sec\theta(t)\, dt, \end{aligned} \tag{1.3.16}$$

by (1.3.14) and Exercise 3 of §1.1.

Taking the quotient of the two identities in (1.3.15) yields

$$\tan\theta(x) = \beta \int_0^x \sec\theta(t)\, dt, \quad \beta = \frac{\kappa}{T_0}. \tag{1.3.17}$$

Differentiating (1.3.17) with respect to x and using Exercise 4 of §1.1, we get

$$\sec^2\theta(x)\, \frac{d\theta}{dx} = \beta \sec\theta(x), \tag{1.3.18}$$

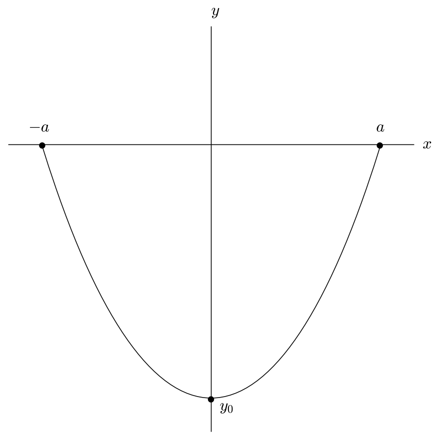

Figure 1.3.1. Catenary

i.e.,

$$\frac{d\theta}{dx} = \beta \cos\theta. \tag{1.3.19}$$

We can separate variables here, to obtain

$$\int \sec\theta \, d\theta = \int \beta \, dx. \tag{1.3.20}$$

Exercise 14 of §1.1 applies to the integral on the left, and we get

$$\sec\theta(x) = \cosh(\beta x + \alpha). \tag{1.3.21}$$

To yield the expected result $\theta(0) = 0$ (see Figure 1.3.1 again), we set $\alpha = 0$.

To get a formula for $y(x)$, use (1.3.14) to write

$$y(x) = y_0 + \int_0^x \tan\theta(t)\, dt, \quad y_0 = y(0). \tag{1.3.22}$$

Now, by Exercises 3 and 8 of §1.1, together with (1.3.21), we have

$$\tan^2\theta(x) = \sec^2\theta(x) - 1 = \cosh^2\beta x - 1 = \sinh^2\beta x, \tag{1.3.23}$$

so (3.22) gives

$$\begin{aligned} y(x) &= y_0 + \int_0^x \sinh\beta t \, dt \\ &= y_0 - \frac{1}{\beta} + \frac{1}{\beta}\cosh\beta x. \end{aligned} \tag{1.3.24}$$

The graph of such a curve is called a *catenary*.

If we are given that the endpoints of the cable are at $(\pm a, 0)$ and that the total length is $2L$ (necessarily $L > a$), we can recover β and y_0 in (1.3.24), as follows.

From (1.3.16) and (1.3.21),

$$L = \int_0^a \cosh \beta t \, dt = \frac{1}{\beta} \sinh \beta a, \tag{1.3.25}$$

so β is uniquely determined by the property that

$$\frac{\sinh \tau}{\tau} = \frac{L}{a}, \quad \beta = \frac{\tau}{a} > 0. \tag{1.3.26}$$

Note that $h(\tau) = (\sinh \tau)/\tau$ is smooth, $h(0) = 1$, $h'(\tau) > 0$ for $\tau > 0$, and $h(\tau) \nearrow +\infty$ as $\tau \nearrow +\infty$. Once one has β, then the identity $y(a) = 0$ gives

$$y_0 = \frac{1}{\beta} - \frac{1}{\beta} \cosh \beta a. \tag{1.3.27}$$

Homogeneous equations, separable in new variables

One can make a change of variable to convert a differential equation of the form

$$\frac{dx}{dt} = f(t,x) \tag{1.3.28}$$

to a separable equation when $f(t,x)$ has the following homogeneity property:

$$f(rt, rx) = f(t,x), \quad \forall\, r \in \mathbb{R} \setminus 0. \tag{1.3.29}$$

In such a case, f has the form

$$f(t,x) = g\Big(\frac{x}{t}\Big). \tag{1.3.30}$$

We can set

$$y = \frac{x}{t}, \tag{1.3.31}$$

so $x = ty,\ x' = ty' + y$, and (1.3.28) turns into

$$\frac{dy}{dt} = \frac{g(y) - y}{t}, \tag{1.3.32}$$

which is separable.

For example, consider

$$\frac{dx}{dt} = \frac{x^2 - t^2}{x^2 + t^2} + \frac{x}{t}. \tag{1.3.33}$$

In this case, (1.3.29) applies, and we can take $g(y) = (y^2-1)/(y^2+1)+y$ in (1.3.30), so with y as in (1.3.31) we have

$$\frac{dy}{dt} = \frac{1}{t} \frac{y^2 - 1}{y^2 + 1}, \tag{1.3.34}$$

which separates to

$$\Big(1 + \frac{2}{y^2 - 1}\Big)\, dy = \frac{dt}{t}. \tag{1.3.35}$$

To integrate the left side of (1.3.35), write

$$\frac{2}{y^2 - 1} = \frac{1}{y+1} - \frac{1}{y-1}, \tag{1.3.36}$$

to get

$$\int \frac{2}{y^2-1}\,dy = \log|y+1| - \log|y-1| = \log\Big|\frac{y+1}{y-1}\Big|, \tag{1.3.37}$$

the latter identity by (1.1.28). Thus the solution to (1.3.33) is given implicitly by

$$\frac{x}{t} + \log\Big|\frac{x+t}{x-t}\Big| = \log|t| + C. \tag{1.3.38}$$

Exercises

Solve the following initial value problems. Do the integrals, if you can.

1.
$$\frac{dx}{dt} = x^2 + 1, \quad x(0) = 0.$$

2.
$$\frac{dx}{dt} = \sqrt{x^2+1}, \quad x(0) = 0.$$

3.
$$\frac{dx}{dt} = \frac{x^2+1}{t^2+1}, \quad x(0) = 1.$$

4.
$$\frac{dx}{dt} = (x^2-1)e^t, \quad x(0) = 2.$$

5.
$$\frac{dx}{dt} = e^{x-t}, \quad x(0) = 0.$$

6.
$$\frac{dx}{dt} = \frac{xt}{x^2+t^2}, \quad x(0) = 1.$$

1.4. Second order equations–reducible cases

Second order differential equations have the form

$$x'' = f(t,x,x'), \quad x(t_0) = x_0, \quad x'(t_0) = v_0. \tag{1.4.1}$$

There are some important cases, with special structure, which reduce to first order equations for

$$v(t) = \frac{dx}{dt}. \tag{1.4.2}$$

One such case is

$$x'' = f(t,x'), \tag{1.4.3}$$

which for v given by (1.4.2) yields

$$\frac{dv}{dt} = f(t, v), \quad v(t_0) = v_0. \tag{1.4.4}$$

Depending on the nature of $f(t, v)$, methods discussed in §§1.2–1.3 might apply to (1.4.4). Once one has $v(t)$, then

$$x(t) = x_0 + \int_{t_0}^{t} v(s)\, ds. \tag{1.4.5}$$

The following is a more significant special case:

$$x'' = f(x, x'). \tag{1.4.6}$$

Direct substitution of v, given by (1.4.2), yields

$$\frac{dv}{dt} = f(x, v), \tag{1.4.7}$$

which is not satisfactory, since (1.4.7) contains too many variables. One route to success is to rewrite the equation as one for v as a function of x, using

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}. \tag{1.4.8}$$

Substitution into (1.4.7) gives the first order equation

$$\frac{dv}{dx} = \frac{f(x, v)}{v}, \quad v(x_0) = v_0. \tag{1.4.9}$$

Again, depending on the nature of $f(x, v)/v$, methods developed in §§2.2–2.3 might apply to (1.4.9).

An important special case of (1.4.6) is

$$x'' = f(x), \tag{1.4.10}$$

in which case (1.4.9) becomes

$$\frac{dv}{dx} = \frac{f(x)}{v}, \tag{1.4.11}$$

which is separable,

$$v\, dv = f(x)\, dx, \tag{1.4.12}$$

hence

$$\frac{1}{2}v^2 = g(x) + C, \quad \int f(x)\, dx = g(x) + C. \tag{1.4.13}$$

Thus

$$\frac{dx}{dt} = v = \pm\sqrt{2g(x) + 2C}, \tag{1.4.14}$$

which in turn is separable,

$$\pm \int \frac{dx}{\sqrt{2g(x) + 2C}} = t + C_2. \tag{1.4.15}$$

The constants C and C_2 are determined by the initial conditions.

Exercises

Use $v = dx/dt$ to transform each of the following equations to first order equations, either for $v = v(t)$ or for $v = v(x)$, as appropriate. Solve these first order equations, if you can.

1.
$$\frac{d^2x}{dt^2} = t\frac{dx}{dt}.$$

2.
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + t.$$

3.
$$\frac{d^2x}{dt^2} = x\frac{dx}{dt}.$$

4.
$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + x.$$

5.
$$\frac{d^2x}{dt^2} = x^2.$$

1.5. Newton's equations for motion in one dimension

Newton's law for motion in one dimension (1D) of a particle of mass m, subject to a force F, is

$$F = ma, \tag{1.5.1}$$

where a is acceleration,

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}, \tag{1.5.2}$$

the rate of change of the velocity

$$v(t) = dx/dt.$$

In general one might have

$$F = F(t, x, x').$$

If F is t-independent, $F = F(x, x')$, which puts us in the setting of (1.4.6).

Frequently, one has $F = F(x)$, which puts us in the setting of (1.4.10). We revisit this setting, bringing in some more concepts from physics. We set

$$F(x) = -V'(x). \tag{1.5.3}$$

$V(x)$, defined up to an additive constant, is called the potential energy. The total energy is the sum of the potential energy and the kinetic energy, $mv^2/2$:

$$E = \frac{1}{2}mv(t)^2 + V(x(t)). \tag{1.5.4}$$

Note that

$$\frac{dE}{dt} = mv(t)v'(t) + V'(x(t))x'(t) \\ = ma(t)v(t) - F(x(t))v(t) \\ = 0, \tag{1.5.5}$$

the last identity by (1.5.1). This identity celebrates energy conservation. Given that x solves

$$m\frac{d^2x}{dt^2} = -V'(x), \quad x(t_0) = x_0,\ x'(t_0) = v_0, \tag{1.5.6}$$

one has from (1.5.5) that for all t,

$$\frac{1}{2}mx'(t)^2 + V(x(t)) = E_0, \tag{1.5.7}$$

where

$$E_0 = \frac{1}{2}mv_0^2 + V(x_0). \tag{1.5.8}$$

The equation (1.5.7) is equivalent to

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\big(E_0 - V(x)\big)}, \tag{1.5.9}$$

which separates to

$$\int \frac{dx}{\sqrt{E_0 - V(x)}} = \pm\sqrt{\frac{2}{m}}t + C, \tag{1.5.10}$$

or, alternatively,

$$\int_{x_0}^{x} \frac{dy}{\sqrt{E_0 - V(y)}} = \pm\sqrt{\frac{2}{m}}(t - t_0). \tag{1.5.11}$$

Note that (1.5.7) and (1.5.10) recover (1.4.13) and (1.4.15).

Projectile problem

Let's look in more detail at a special case, modeling the motion of a projectile of mass m traveling directly away from (or toward) the Earth. In such a case, Newton's law of gravity gives

$$F(x) = -\frac{Km}{x^2}, \quad \text{hence } V(x) = -\frac{Km}{x}, \quad x \in (0, \infty). \tag{1.5.12}$$

In such a case, the conserved energy is

$$E_0 = \frac{m}{2}\Big(v^2 - \frac{2K}{x}\Big) = \frac{m}{2}\mathcal{E}(x, v). \tag{1.5.13}$$

See Figure 1.5.1 for a sketch of level curves of the function $\mathcal{E}(x, v)$. There are three cases to consider:

$$\mathcal{E} = -a^2 < 0, \quad \mathcal{E} = 0, \quad \mathcal{E} = a^2 > 0, \quad \text{i.e.,} \\ E_0 = -\frac{m}{2}a^2 < 0, \quad E_0 = 0, \quad E_0 = \frac{m}{2}a^2 > 0. \tag{1.5.14}$$

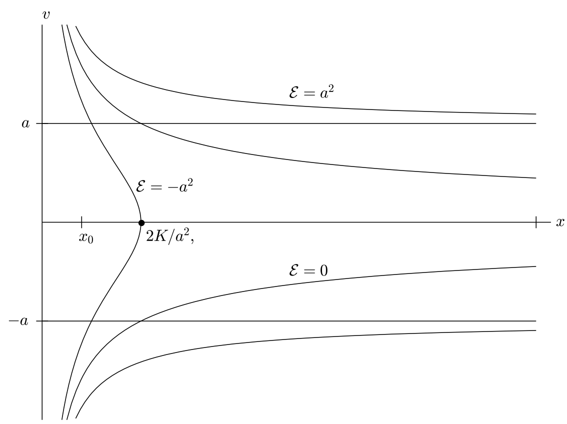

Figure 1.5.1. Projectile paths

In the first case, $x(t)$ has a maximum at $x_{\max} = 2K/a^2$. In the other two cases, $x(t) \to +\infty$ as $t \to +\infty$ (if $v_0 > 0$) or as $t \to -\infty$ (if $v_0 < 0$). Given $x_0 \in (0, \infty)$, the velocity $v_0 \in (0, \infty)$ for which $\mathcal{E}(x_0, v_0) = 0$ is called the *escape velocity.*

We investigate the integral on the left side of (1.5.10), i.e.,

$$\int \frac{dx}{\sqrt{E_0 + Km/x}}, \tag{1.5.15}$$

which in the three cases in (1.5.14) is $\sqrt{2/m}$ times

$$\int \frac{x\,dx}{\sqrt{2Kx - a^2x^2}}, \quad \int \sqrt{\frac{x}{2K}}\,dx, \quad \int \frac{x\,dx}{\sqrt{2Kx + a^2x^2}}, \tag{1.5.16}$$

respectively. The second integral in (1.5.16) is easy; we investigate how to compute the other two, which we rewrite as

$$\frac{1}{a}\int \frac{x\,dx}{\sqrt{2kx - x^2}}, \quad \frac{1}{a}\int \frac{x\,dx}{\sqrt{2kx + x^2}}, \quad k = \frac{K}{a^2}. \tag{1.5.17}$$

We can compute these integrals by completing the square:

$$x^2 - 2kx = (x-k)^2 - k^2, \quad x^2 + 2kx = (x+k)^2 - k^2. \tag{1.5.18}$$

The respective change of variables $y = x - k$ and $y = x + k$ turn the integrals in (1.5.17) into the respective integrals,

$$\int \frac{(y+k)\,dy}{\sqrt{k^2 - y^2}}, \quad \int \frac{(y-k)\,dy}{\sqrt{y^2 - k^2}}. \tag{1.5.19}$$

By inspection,

$$\int \frac{y\,dy}{\sqrt{k^2 - y^2}} = -\sqrt{k^2 - y^2} + C, \quad \int \frac{y\,dy}{\sqrt{y^2 - k^2}} = \sqrt{y^2 - k^2} + C. \tag{1.5.20}$$

The remaining parts of (1.5.19), after a change of variable $y = kz$, become

$$k \int \frac{dz}{\sqrt{1 - z^2}}, \quad k \int \frac{dz}{\sqrt{z^2 - 1}}. \tag{1.5.21}$$

To do these integrals, use

$$\begin{aligned} z = \sin s &\Longrightarrow \int \frac{dz}{\sqrt{1-z^2}} = \int \frac{\cos s}{\cos s}\, ds = s + C, \\ z = \cosh s &\Longrightarrow \int \frac{dz}{\sqrt{z^2-1}} = \int \frac{\sinh s}{\sinh s}\, ds = s + C. \end{aligned} \tag{1.5.22}$$

Exercises

1. Make calculations analogous to (1.5.12)–(1.5.15) for each of the following forces. Examine whether you can do the resulting integrals.

(a)

$$F(x) = -Kx.$$

(b)

$$F(x) = -Kx^2.$$

(c)

$$F(x) = -\frac{K}{x}.$$

(d)

$$F(x) = x - x^3.$$

2. For such forces as given above, in each case find a potential energy $V(x)$ and sketch the level curves in the (x, v)-plane of the energy function

$$E(x, v) = \frac{m}{2} v^2 + V(x).$$

3. Use the substitution

$$x = k^2 \sin^2 \theta$$

to evaluate

$$\int \frac{dx}{\sqrt{\frac{k^2}{x} - 1}},$$

and use

$$x = k^2 \sinh^2 u$$

to evaluate

$$\int \frac{dx}{\sqrt{\frac{k^2}{x} + 1}}.$$

Use these calculations as alternatives for evaluating (1.5.15), for $E_0 < 0$ and $E_0 > 0$, respectively.

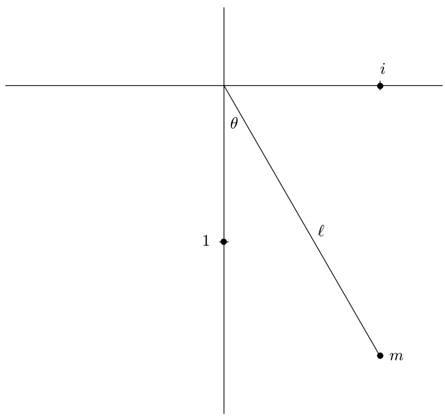

Figure 1.6.1. Pendulum

1.6. The pendulum

We produce a differential equation to describe the motion of a pendulum, which will be modeled by a rigid rod, of length ℓ, suspended at one end. We assume the rod has negligible mass, except for an object of mass m at the other end, as illustrated in Figure 1.6.1. The rod is held at an angle $\theta = \theta_0$ from the downward pointing vertical, and released at time $t = 0$, after which it moves because of the force of gravity. We seek a differential equation for θ as a function of t.

The end with the mass m traces out a path in a plane, which we identify with the complex plane, with the origin at the point where the pendulum is suspended, and the real axis pointing vertically down. We can write the path as

$$z(t) = \ell e^{i\theta(t)}. \tag{1.6.1}$$

The velocity is

$$v(t) = z'(t) = i\ell\theta'(t)e^{i\theta(t)}, \tag{1.6.2}$$

and the acceleration is

$$a(t) = v'(t) = \ell[i\theta''(t) - \theta'(t)^2]e^{i\theta(t)}. \tag{1.6.3}$$

The force of gravity on the mass is mg, where $g = 32$ ft/sec^2, provided the pendulum is located on the surface of the Earth. The total force F on the mass is the sum of the gravitational force and the force the rod exerts on the mass to keep it always at a distance ℓ from the origin. The force the rod exerts is parallel to $e^{i\theta(t)}$, so

$$F(t) = mg + \Phi(t)e^{i\theta(t)}, \tag{1.6.4}$$

for some real valued $\Phi(t)$ (to be determined). We can rewrite mg as

$$mg = mge^{-i\theta(t)}e^{i\theta(t)} = mg[\cos\theta(t) - i\sin\theta(t)]e^{i\theta(t)}, \tag{1.6.5}$$

and hence

$$F(t) = [-img \sin\theta(t) + mg\cos\theta(t) + \Phi(t)]e^{i\theta(t)}. \tag{1.6.6}$$

Newton's law $F = ma$ applied to (1.6.3)–(1.6.6) gives

$$m\ell[i\theta''(t) - \theta'(t)^2] = -img\sin\theta(t) + (mg\cos\theta(t) + \Phi(t)). \tag{1.6.7}$$

Comparing imaginary parts gives

$$m\ell\theta''(t) = -mg\sin\theta(t), \tag{1.6.8}$$

or

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0. \tag{1.6.9}$$

This is the pendulum equation.

The kinetic energy of this pendulum is

$$\frac{1}{2}m|v(t)|^2 = \frac{m\ell^2}{2}\theta'(t)^2, \tag{1.6.10}$$

and its potential energy (up to an additive constant) is given by $-mg$ times the real part of $z(t)$, i.e.,

$$V(\theta) = -mg\ell\cos\theta. \tag{1.6.11}$$

The total energy is hence

$$E = \frac{m\ell^2}{2}\theta'(t)^2 - mg\ell\cos\theta(t). \tag{1.6.12}$$

Note that

$$\begin{aligned} \frac{dE}{dt} &= m\ell^2\theta'(t)\theta''(t) + mg\ell(\sin\theta(t))\theta'(t) \\ &= m\ell^2\theta'(t)\Big(\theta''(t) + \frac{g}{\ell}\sin\theta(t)\Big), \end{aligned} \tag{1.6.13}$$

so the pendulum equation (1.6.9) implies $dE/dt = 0$, i.e., we have conservation of energy. Under the initial condition formulated at the beginning of this section,

$$\theta(0) = \theta_0, \quad \theta'(0) = 0, \tag{1.6.14}$$

we have initial energy

$$E_0 = -mg\ell\cos\theta_0, \tag{1.6.15}$$

and the energy conservation gives

$$\mathcal{E}(\theta, \theta') = \frac{2E_0}{m\ell^2} = A_0, \tag{1.6.16}$$

where

$$\mathcal{E}(\theta, \psi) = \psi^2 - \frac{2g}{\ell}\cos\theta. \tag{1.6.17}$$

Level curves of this function are depicted in Figure 1.6.2. If $\theta(t)$ solves (1.6.9) and $\psi(t) = \theta'(t)$, then $(\theta(t), \psi(t))$ traces out a path on one of these level curves.

Note that

$$\nabla\mathcal{E}(\theta, \psi) = \Big(\frac{2g}{\ell}\sin\theta, 2\psi\Big), \tag{1.6.18}$$

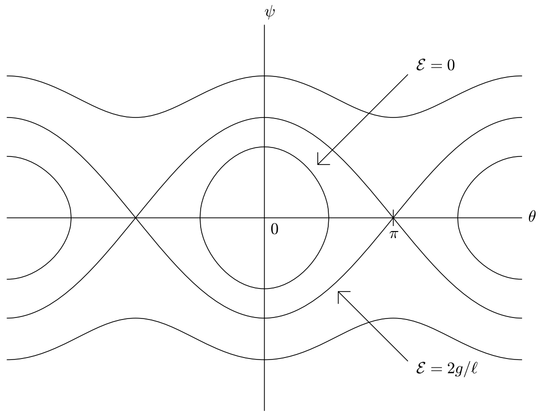

Figure 1.6.2. Level curves of $\mathcal{E}(\theta, \psi) = \psi^2 - (2g/\ell)\cos\theta$

so $\mathcal{E}$ has critical points at $\theta = k\pi$, $\psi = 0$. The matrix of second order partial derivatives of $\mathcal{E}$ is

$$D^2\mathcal{E}(\theta, \psi) = \begin{pmatrix} \frac{2g}{\ell}\cos\theta & 0 \\ 0 & 2 \end{pmatrix}, \tag{1.6.19}$$

so

$$D^2\mathcal{E}(k\pi, 0) = \begin{pmatrix} (-1)^k \frac{2g}{\ell} & 0 \\ 0 & 2 \end{pmatrix}. \tag{1.6.20}$$

We see that at the critical point $(k\pi, 0)$, $\mathcal{E}$ has a local minimum if k is even and a saddle-type behavior if k is odd, as illustrated in Figure 1.6.2.

Note that if the initial condition (1.6.14) holds, then $A_0 = -(2g/\ell)\cos\theta_0$, and hence $A_0 < 2g/\ell$, so the curve traced by $(\theta(t), \psi(t))$ is a closed curve. One might instead have initial data of the form

$$\theta(0) = \theta_0, \quad \theta'(0) = \psi_0, \tag{1.6.21}$$

and one could pick ψ_0 so that $\mathcal{E}(\theta_0, \psi_0) > 2g/\ell$.

We proceed to formulas parallel to (1.5.7)–(1.5.11). Starting from the energy conservation (1.6.16), which we rewrite as

$$\theta'(t)^2 - \frac{2g}{\ell}\cos\theta(t) = A_0, \tag{1.6.22}$$

we have

$$\theta'(t) = \pm\sqrt{\frac{2g}{\ell}}\sqrt{A_1 + \cos\theta}, \quad A_1 = \frac{\ell}{2g}A_0 = \frac{E_0}{mg\ell}, \tag{1.6.23}$$

which separates and integrates to

$$\int \frac{d\theta}{\sqrt{A_1 + \cos\theta}} = \pm\sqrt{\frac{2g}{\ell}}\, t + C. \tag{1.6.24}$$

In the current set-up, where, by (1.6.12), $E_0 \ge -mg\ell$, we have

$$A_1 \ge -1. \tag{1.6.25}$$

Note that to achieve $A_1 = -1$ requires $\theta(0) = 0$ and $\theta'(0) = 0$, in which case (1.6.23) yields the initial value problem

$$\theta'(t) = \pm\sqrt{\frac{2g}{\ell}}\,\sqrt{-1+\cos\theta}, \quad \theta(0) = 0, \tag{1.6.26}$$

with solution

$$\theta(t) \equiv 0. \tag{1.6.27}$$

In this case (1.6.24) has no meaning. Indeed, if $\theta > 0$ and one considers

$$\int_0^\theta \frac{d\varphi}{\sqrt{-1+\cos\varphi}}, \tag{1.6.28}$$

the integrand is imaginary and furthermore it is not integrable. Nevertheless, $\theta(t) \equiv 0$ is a solution to the original problem.

Let us now assume $A_1 > -1$. Write

$$B_1 = A_1 + 1 > 0, \tag{1.6.29}$$

so

$$\begin{aligned} A_1 + \cos\theta &= B_1 - (1 - \cos\theta) \\ &= B_1 - 2\sin^2\frac{\theta}{2}, \end{aligned} \tag{1.6.30}$$

thanks to the identity $\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = 1 - 2\sin^2\varphi$. We can rewrite the left side of (1.6.24) as

$$\begin{aligned} \int \frac{d\theta}{\sqrt{A_1+\cos\theta}} &= \int \frac{d\theta}{\sqrt{B_1 - 2\sin^2\theta/2}} \\ &= \frac{\beta}{\sqrt{2}}\int \frac{d\theta}{\sqrt{1-\beta^2\sin^2\theta/2}}, \end{aligned} \tag{1.6.31}$$

with

$$\beta = \sqrt{\frac{2}{B_1}} > 0. \tag{1.6.32}$$

The last integral in (1.6.31) is known as an elliptic integral when $\beta^2 \ne 1$, i.e., when $A_1 \ne 1$. Material on such integrals can be found in books that treat elliptic function theory, including [**47**].

The case $\beta = 1$ (i.e., $A_1 = 1$, or $E_0 = mg\ell$) does give rise to an elementary integral, namely

$$\begin{aligned} \int \frac{d\theta}{\sqrt{1+\cos\theta}} &= \frac{1}{\sqrt{2}}\int \sec\frac{\theta}{2}\,d\theta \\ &= \sqrt{2}\sinh^{-1}\Bigl(\tan\frac{\theta}{2}\Bigr) + C, \end{aligned} \tag{1.6.33}$$

for $|\theta| < \pi$, the latter identity by Exercise 14 of §1.1.

Further study of the elliptic integral in (1.6.24)

Let us pursue the computations arising from (1.6.24) in more detail, taking the initial condition

$$\theta(0) = 0, \quad \theta'(0) = \psi_0, \quad \psi_0 \in (0, \infty). \tag{1.6.34}$$

Then (1.6.24) yields, for the solution $\theta(t)$,

$$\int_0^{\theta(t)} \frac{d\vartheta}{\sqrt{A_1 + \cos\vartheta}} = \sqrt{\frac{2g}{\ell}}\, t. \tag{1.6.35}$$

In such a case,

$$A_1 = \frac{E_0}{mg\ell} = \frac{\ell}{2g}\psi_0^2 - 1, \quad \text{hence} \ \ B_1 = \frac{\ell}{2g}\psi_0^2. \tag{1.6.36}$$

Then (1.6.31) yields

$$\begin{aligned} \int_0^{\theta(t)} \frac{d\vartheta}{\sqrt{1 - \beta^2 \sin^2 \vartheta/2}} &= \frac{\sqrt{2}}{\beta} \sqrt{\frac{2g}{\ell}}\, t \\ &= \psi_0 t, \end{aligned} \tag{1.6.37}$$

with

$$\beta = \sqrt{\frac{2}{B_1}} = \frac{2}{\psi_0}\sqrt{\frac{g}{\ell}}. \tag{1.6.38}$$

Let us specialize (1.6.37) to

$$\beta = 1, \quad \text{hence} \ \ \psi_0 = 2\sqrt{\frac{g}{\ell}}, \quad \text{so} \ \ E_0 = mg\ell. \tag{1.6.39}$$

By (1.6.33), we get

$$\tan\frac{\theta(t)}{2} = \sinh\frac{\psi_0}{2}\, t = \sinh\sqrt{\frac{g}{\ell}}\, t, \tag{1.6.40}$$

or

$$\theta(t) = 2\tan^{-1}\sinh\sqrt{\frac{g}{\ell}}\, t. \tag{1.6.41}$$

Applying d/dt yields

$$\theta'(t) = \psi(t) = 2\sqrt{\frac{g}{\ell}}\, \frac{1}{\cosh\sqrt{\frac{g}{\ell}}\, t}. \tag{1.6.42}$$

In this case,

$$\theta(t) \to \pm\pi \ \ \text{and} \ \ \psi(t) \to 0, \quad \text{as} \ \ t \to \pm\infty. \tag{1.6.43}$$

The curve $(\theta(t), \psi(t))$ and its mirror image are called *separatrices*. These curves separate bounded periodic solutions from unbounded solution curves.

We turn to the case

$$\beta > 1, \quad \text{hence} \ \ 0 < \psi_0 < 2\sqrt{\frac{g}{\ell}}, \quad \text{so} \ \ E < mg\ell. \tag{1.6.44}$$

In this case,

$$\theta(t) \text{ is periodic, say of period } \Pi(\psi_0), \tag{1.6.45}$$

and we want to find a formula for $\Pi(\psi_0)$. Looking at Figure 1.6.2, we see that

$$\psi(t) = 0 \quad \text{at} \quad t = \frac{1}{4}\Pi(\psi_0). \tag{1.6.46}$$

Comparison with the formula

$$\begin{aligned} \psi(t) = \frac{d\theta}{dt} &= \sqrt{\frac{2g}{\ell}}\sqrt{B_1 - 2\sin^2\frac{\theta}{2}} \\ &= \beta\sqrt{\frac{g}{\ell}}\sqrt{1 - \beta^2\sin^2\frac{\theta}{2}} \end{aligned} \tag{1.6.47}$$

gives

$$\psi = 0 \quad \text{when} \quad \sin^2\frac{\theta}{2} = \frac{B_1}{2}, \tag{1.6.48}$$

and hence

$$\begin{gathered} \frac{1}{4}\Pi(\psi_0) = \sqrt{\frac{\ell}{2g}}\int_0^{\theta_1}\frac{d\vartheta}{\sqrt{B_1 - 2\sin^2\vartheta/2}}, \\ \sin^2\frac{\theta_1}{2} = \frac{B_1}{2} = \frac{1}{\beta^2} = \frac{\ell}{4g}\,\psi_0^2. \end{gathered} \tag{1.6.49}$$

Equivalently,

$$\begin{aligned} \frac{1}{4}\Pi(\psi_0) &= \frac{1}{\psi_0}\int_0^{\theta_1}\frac{d\vartheta}{\sqrt{1 - \beta^2\sin^2\vartheta/2}} \\ &= \frac{2}{\psi_0}\int_0^{\theta_1/2}\frac{d\varphi}{\sqrt{1 - \beta^2\sin^2\varphi}}, \end{aligned} \tag{1.6.50}$$

with θ_1 as in (1.6.49). Making the change of variable $x = \sin\varphi$, we get

$$\frac{1}{4}\Pi(\psi_0) = \frac{2}{\psi_0}\int_0^{1/\beta}\frac{dx}{\sqrt{(1 - x^2)(1 - \beta^2 x^2)}}, \tag{1.6.51}$$

and finally, setting $y = \beta x$ yields

$$\begin{aligned} \frac{1}{4}\Pi(\psi_0) &= \frac{2\alpha}{\psi_0}\int_0^1\frac{dy}{\sqrt{(1 - y^2)(1 - \alpha^2 y^2)}} \\ &= \sqrt{\frac{\ell}{g}}\int_0^1\frac{dy}{\sqrt{(1 - y^2)(1 - \alpha^2 y^2)}}, \end{aligned} \tag{1.6.52}$$

with

$$\alpha = \frac{1}{\beta} = \sqrt{\frac{B_1}{2}} = \frac{1}{2}\sqrt{\frac{\ell}{g}}\,\psi_0, \tag{1.6.53}$$

so $0 < \alpha < 1$. Clearly, $\alpha \to 0$ when $\psi_0 \to 0$, so we have

$$\begin{aligned}\lim_{\psi_0 \to 0} \Pi(\psi_0) &= 4\sqrt{\frac{\ell}{g}} \int_0^1 \frac{dy}{\sqrt{1-y^2}} \\ &= 2\pi\sqrt{\frac{\ell}{g}}.\end{aligned} \tag{1.6.54}$$

This coincides with the period of solutions to

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0, \tag{1.6.55}$$

which we will identify in §1.8 with the linearization of the pendulum equation about the zero solution.

Finally, we examine the case

$$0 < \beta < 1, \quad \text{hence} \quad \psi_0 = \frac{2}{\beta}\sqrt{\frac{g}{\ell}} > 2\sqrt{\frac{g}{\ell}}, \quad \text{so} \quad E > mg\ell. \tag{1.6.56}$$

In such a case, we see from (1.6.47) that $\theta(t)$ is monotone in t. However, it does possess the "periodicity"

$$\theta(t+s) = \theta(t) + 2\pi, \quad \text{with} \quad s = \Pi(\psi_0), \tag{1.6.57}$$

where, when (1.6.56) holds,

$$\begin{aligned}\Pi(\psi_0) &= \sqrt{\frac{\ell}{2g}} \int_0^{2\pi} \frac{d\vartheta}{\sqrt{A_1 + \cos\vartheta}} \\ &= \frac{1}{\psi_0} \int_0^{2\pi} \frac{d\vartheta}{\sqrt{1-\beta^2 \sin^2 \vartheta/2}} \\ &= \frac{2}{\psi_0} \int_0^{\pi} \frac{d\varphi}{\sqrt{1-\beta^2\sin^2\varphi}}.\end{aligned} \tag{1.6.58}$$

Making the change of variable $x = \sin\varphi$, we get

$$\Pi(\psi_0) = \frac{2}{\psi_0} \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\beta^2x^2)}}. \tag{1.6.59}$$

Remark. The integrals in (1.6.52) and (1.6.59) are called *complete elliptic integrals*. One can expand these integrals in convergent power series in α^2 and β^2, respectively, using the formula

$$\begin{aligned}&\frac{1}{\sqrt{1-u}} = \sum_{k=0}^{\infty} a_k u^k, \quad \text{for} \quad |u| < 1, \quad \text{with} \\ &a_0 = 1, \quad a_k = \Big(1-\frac{1}{2}\Big)\Big(2-\frac{1}{2}\Big)\cdots\Big(k-\frac{1}{2}\Big)\end{aligned} \tag{1.6.60}$$

(see Apendix 1.C), with $u = \alpha^2 x^2$ in (1.6.52) and $u = \beta^2 x^2$ in (1.6.59), and then integrating term by term. The coefficients in the resulting power series involve

$$\begin{aligned}\int_0^1 \frac{x^{2k}}{\sqrt{1-x^2}}\,dx &= \int_0^{\pi/2} \sin^{2k}\varphi\,d\varphi \\ &= \frac{1}{4}\Big(\frac{1}{2i}\Big)^{2k} \int_0^{2\pi} \big(e^{i\varphi} - e^{-i\varphi}\big)^{2k}\,d\varphi \\ &= \pi 2^{-2k-1}\binom{2k}{k}.\end{aligned} \tag{1.6.61}$$

One can also express these complete elliptic integrals in terms of a function known as the Gauss *arithmetic-geometric mean* (cf. [**47**], Chapter 6, §4).

Exercises

1. Let E be given by (6.8). Show that if $\theta(t)$ solves (6.6) and $|\theta(t)| < \pi/2$ for all t, then $E < 0$.

2. Show that the level set in Figure 1.6.2 where $\mathcal{E} = 2g/\ell$ (i.e., $E = mg\ell$) is given by

$$\psi = \pm 2\sqrt{\frac{g}{\ell}}\cos\frac{\theta}{2}.$$

3. By (1.6.3), the component of acceleration parallel to $e^{i\theta}$ is $-\ell\theta'(t)^2 e^{i\theta(t)}$. Compute the component of the gravitational force parallel to $e^{i\theta(t)}$, and deduce that the force the rod exerts on the mass to keep it always at a distance ℓ from the origin is $\Phi e^{i\theta(t)}$, with

$$\Phi = -m\ell\theta'(t)^2 - mg\cos\theta.$$

Deduce that, with E as in (1.6.12),

$$\Phi(t) = \frac{E}{\ell} - \frac{3m\ell}{2}\theta'(t)^2.$$

4. Apply the change of variable $s = \sin\varphi$ to the last integral in (1.6.31), i.e., to

$$\int \frac{d\varphi}{\sqrt{1-\beta^2\sin^2\varphi}}.$$

Show that the integral becomes

$$\int \frac{ds}{\sqrt{(1-s^2)(1-\beta^2 s^2)}}.$$

Specialize to $\beta = 1$ and obtain an alternative derivation of the formula for

$$\int \sec\varphi\,d\varphi,$$

given in Exercise 13 of §1.1.

5. Suppose the mass at the end of the pendulum has a charge q_1 and there is a charge q_2 fixed at $(x, y) = (2\ell, 0)$. Then the force $F(t)$ is modified to

$$F(t) = mg - Kq_1q_2 \frac{2\ell - \ell e^{i\theta(t)}}{|2\ell - \ell e^{i\theta(t)}|^3} + \Phi(t)e^{i\theta(t)},$$

where K is a positive constant. Use this to produce a modification of the pendulum equation.

1.7. Motion with resistance

In many real cases, the force acting on a moving object is the sum of a force associated with a potential and a resistance, typically depending on the velocity and acting to slow the motion down. For example, the motion of a ball of mass m falling through the air near the surface of the Earth can be modeled by the differential equation

$$m\frac{d^2x}{dt^2} = mg - \alpha\frac{dx}{dt}, \tag{1.7.1}$$

where the x-axis points down toward the Earth. Here $g = 32$ ft/sec^2 and α is an experimentally determined constant, depending on the size of the ball, and measures air resistance. We can rewrite (1.7.1) as an equation for $v = dx/dt$,

$$\frac{dv}{dt} = g - \frac{\alpha}{m}v, \tag{1.7.2}$$

an equation that is both linear and separable. Unless v is small, the formula $-\alpha v$ for the force of air resistance is not so accurate, and a more accurate equation might be

$$\frac{dv}{dt} = g - \frac{\alpha}{m}v - \frac{\beta}{m}v^3. \tag{1.7.3}$$

This is not linear, but it is separable. For v close to the speed of sound in air, even this model loses validity.

If the ball is falling from the stratosphere toward the surface of the Earth, the variation in air density, hence in air resistance, must be taken into account. One might replace the model (1.7.1) by

$$m\frac{d^2x}{dt^2} = mg - \alpha(x)\frac{dx}{dt}. \tag{1.7.4}$$

The method of (1.4.6)–(1.4.9) is applicable here, yielding for $v = dx/dt$ the equation

$$\frac{dv}{dx} = \frac{mg}{v} - \alpha(x). \tag{1.7.5}$$

This, however, is not typically amenable to a solution in terms of elementary functions.

Another example of motion with resistance arises in the pendulum. Between air resistance and friction where the rod is attached, the pendulum equation (1.6.9) might be modified to the following damped pendulum equation:

$$\frac{d^2\theta}{dt^2} + \frac{\alpha}{m}\frac{d\theta}{dt} + \frac{g}{\ell}\sin\theta = 0, \tag{1.7.6}$$

for some positive constant α. Again the method of (1.4.6)–(1.4.9) is applicable, and it yields for $\psi = d\theta/dt$ the equation

$$\frac{d\psi}{d\theta} = -\frac{\alpha}{m} - \frac{g}{\ell}\frac{\sin\theta}{\psi}. \tag{1.7.7}$$

However, this equation is not particularly tractable, and does not yield much insight into the behavior of solutions to (1.7.6).

Exercises

1. Suppose $v(t)$ solves (1.7.2) and $v(0) = 0$. Show that

$$\lim_{t\to+\infty} v(t) = \frac{mg}{\alpha},$$

and

$$v(t) < \frac{mg}{\alpha}, \quad \forall\, t \in [0, \infty).$$

What does it mean to call mg/α the *terminal velocity*?

2. Do the analogue of Exercise 1 when $v(t)$ solves (1.7.3) and $v(0) = 0$.

3. In the setting of Exercise 1, what happens if, instead of $v(0) = 0$, we have

$$v(0) = v_0 > \frac{mg}{\alpha}?$$

4. Apply the method of separation of variables to (1.7.3). Note that

$$g - \frac{\alpha}{m}v - \frac{\beta}{m}v^3 = p(v)$$

has three complex roots (at least one of which must be real). For what values of α, β, and m does $p(v)$ have one real root and for what values does it have three real roots? How does this bear on the behavior of

$$\int \frac{dv}{p(v)}?$$

5. More general models for motion with resistance involve the following modification of (1.5.6):

$$m\frac{d^2x}{dt^2} = -V'(x) - \alpha\frac{dx}{dt}.$$

Parallel to (1.5.4), set

$$E(t) = \frac{1}{2}m\Big(\frac{dx}{dt}\Big)^2 + V(x(t)).$$

Show that

$$\frac{dE}{dt} \le 0.$$

One says energy is *dissipated*, due to the resistance.

1.8. Linearization

As we have seen, some equations, such as the pendulum equation (1.6.9), which we rewrite here as

$$\frac{d^2x}{dt^2} + \frac{g}{\ell}\sin x = 0, \tag{1.8.1}$$

can be "solved" in terms of an integral, in this case (1.6.24), i.e.,

$$\int \frac{dx}{\sqrt{A_1 + \cos x}} = \pm\sqrt{\frac{2g}{\ell}}\,t + C. \tag{1.8.2}$$

However, the integral is a complicated special function. By contrast, other equations, such as the damped pendulum equation (1.7.6), which we rewrite

$$\frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{g}{\ell}\sin x = 0, \tag{1.8.3}$$

are not even amenable to solutions as "explicit" as (1.8.2). In such cases one might nevertheless gain valuable insight into solutions that are small perturbations of some known particular solution to (1.8.1) or (1.8.3), or more generally

$$x''(t) = f(t, x(t), x'(t)). \tag{1.8.4}$$

In case (1.8.1) and (1.8.3), $x(t) \equiv 0$ is a solution. More generally, one might have a known solution $y(t)$ of (1.8.4); i.e., $y(t)$ is known and satisfies

$$y''(t) = f(t, y(t), y'(t)). \tag{1.8.5}$$

Now take $x(t) = y(t) + \varepsilon u(t)$. We derive an equation for $u(t)$ so that $x(t)$ satisfies (1.8.4), at least up to $O(\varepsilon^2)$, i.e.,

$$y''(t) + \varepsilon u''(t) = f\big(t, y(t) + \varepsilon u(t), y'(t) + \varepsilon u'(t)\big) + O(\varepsilon^2). \tag{1.8.6}$$

To get this equation, write, with $f = f(t, x, v)$,

$$f(t, y+\varepsilon u, y'+\varepsilon u') = f(t, y, y') + \varepsilon\Big(\frac{\partial f}{\partial x}(t, y, y')u + \frac{\partial f}{\partial v}(t, y, y')u'\Big) + O(\varepsilon^2), \tag{1.8.7}$$

the first order Taylor polynomial approximation. Plugging this into (1.8.6) and using (1.8.5), we see that (1.8.6) holds provided $u(t)$ satisfies the equation

$$u''(t) = A(t)u(t) + B(t)u'(t), \tag{1.8.8}$$

where

$$A(t) = \frac{\partial f}{\partial x}\big(t, y(t), y'(t)\big), \quad B(t) = \frac{\partial f}{\partial v}\big(t, y(t), y'(t)\big). \tag{1.8.9}$$

The equation (1.8.8) is a linear equation, called the *linearization* of (1.8.4) about the solution $y(t)$.

In case (1.8.1), $f(t,x,v) = -(g/\ell)\sin x$, and the linearization about $y(t) = 0$ of this equation is

$$\frac{d^2u}{dt^2} + \frac{g}{\ell}u = 0. \tag{1.8.10}$$

In case (1.8.3), $f(t,x,v) = (\alpha/m)v + (g/\ell)\sin x$, and the linearization about $y(t) = 0$ of this equation is

$$\frac{d^2u}{dt^2} + \frac{\alpha}{m}\frac{du}{dt} + \frac{g}{\ell}u = 0. \tag{1.8.11}$$

To take another example, consider

$$x''(t) = tx(t) - x(t)^2. \tag{1.8.12}$$

One solution is

$$y(t) = t. \tag{1.8.13}$$

In this case we have (1.8.4) with $f(t,x,v) = tx - x^2$, hence $f_x(t,x,v) = t - 2x$ and $f_v(t,x,v) = 0$. Then $f_x(t,y,y') = f_x(t,t,1) = -t$, and the linearization of (1.8.12) about $y(t) = t$ is

$$u''(t) + tu(t) = 0. \tag{1.8.14}$$

Exercises

Compute the linearizations of the following equations, about the given solution $y(t)$.

1.
$$x'' + \cosh x - \cosh 1 = 0, \quad y(t) = 1.$$

2.
$$x'' + \cosh x - \cosh t = 0, \quad y(t) = t.$$

3.
$$x'' + x' \sin x = 0, \quad y(t) = 0.$$

4.
$$x'' + x' \sin x = 0, \quad y(t) = \frac{\pi}{2}.$$

5.
$$x'' + \sin x = 0, \quad y(t) = \pi.$$

1.9. Second order constant-coefficient linear equations–homogeneous

Here we look into solving differential equations of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0, \tag{1.9.1}$$

with constants a, b, and c. We assume $a \neq 0$. We impose an initial condition, such as

$$x(0) = \alpha, \quad x'(0) = \beta. \tag{1.9.2}$$

We look for solutions in the form

$$x(t) = e^{rt}, \tag{1.9.3}$$

for some constant r, which worked so well for first order equations in §1.1. By results derived there, if $x(t)$ has the form (1.9.3), then $x'(t) = re^{rt}$ and $x''(t) = r^2e^{rt}$, so substitution into the left side of (1.9.1) gives

$$(ar^2 + br + c)e^{rt}, \tag{1.9.4}$$

which vanishes if and only if r satisfies the equation

$$ar^2 + br + c = 0. \tag{1.9.5}$$

The polynomial $p(r) = ar^2+br+c$ is called the characteristic polynomial associated with the differential equation (1.9.1). Its roots are given by

$$r_\pm = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}. \tag{1.9.6}$$

There are two cases to consider:

$$\begin{aligned} &\text{(I)} \qquad \mathrm{b}^2 - 4\mathrm{ac} \neq 0, \\ &\text{(II)} \qquad \mathrm{b}^2 - 4\mathrm{ac} = 0. \end{aligned}$$

In Case I, the equation (1.9.5) has two distinct roots, and we get two distinct solutions to (1.9.1), e^{r_+t} and e^{r_-t}. It is easy to see that whenever $x_1(t)$ and $x_2(t)$ solve (1.9.1), so does $C_1x_1(t) + C_2x_2(t)$, for arbitrary constants C_1 and C_2. Hence

$$x(t) = C_+e^{r_+t} + C_-e^{r_-t} \tag{1.9.7}$$

solves (1.9.1), for all constants C_+ and C_-.

Having this, we can find a solution to (1.9.1) with initial data (1.9.2) as follows. Taking $x(t)$ as in (1.9.7), so $x'(t) = C_+r_+e^{r_+t} + C_-r_-e^{r_-t}$, we set $t = 0$ to obtain

$$x(0) = C_+ + C_-, \quad x'(0) = r_+C_+ + r_-C_-, \tag{1.9.8}$$

so (1.9.2) holds if and only if C_+ and C_- satisfy

$$\begin{aligned} C_+ + C_- &= \alpha, \\ r_+C_+ + r_-C_- &= \beta. \end{aligned} \tag{1.9.9}$$

This set of two linear equations for C_+ and C_- has a unique solution if and only if $r_+ \neq r_-$. In fact, the first equation in (1.9.9) gives

$$r_-C_+ + r_-C_- = r_-\alpha, \tag{1.9.10}$$

and subtracting this from the second equation in (1.9.9) yields

$$C_+ = \frac{\beta - \alpha r_-}{r_+ - r_-}, \tag{1.9.11}$$

and then the first equation in (1.9.9) yields

$$C_- = \alpha - C_+ = \frac{\alpha r_+ - \beta}{r_+ - r_-}. \tag{1.9.12}$$

In Case II, $r = -b/2a$ is a double root of the characteristic polynomial, and we have the solution $x(t) = e^{rt}$ to (1.9.1). We claim there is another solution to (1.9.1) that is not simply a constant multiple of this one. We look for a second solution in the form

$$x(t) = u(t)e^{rt}, \tag{1.9.13}$$

hoping to get a simpler differential equation for $u(t)$. Note that then $x' = (u' + ru)e^{rt}$ and $x'' = (u'' + 2ru' + r^2u)e^{rt}$, and hence

$$\begin{aligned} ax'' + bx' + cx &= \Big\{a(u'' + 2ru' + r^2u) + b(u' + ru) + cu\Big\}e^{rt} \\ &= \Big\{au'' + (2ar + b)u' + (ar^2 + br + c)u\Big\}e^{rt} \\ &= au''e^{rt}, \end{aligned} \tag{1.9.14}$$

given that (1.9.5) holds with $r = -b/2a$. Thus the vanishing of (1.9.14) is equivalent to $u''(t) = 0$, i.e., to $u(t) = C_1 + C_2t$. Hence another solution to (1.9.1) in this case is te^{rt}, and, in place of (1.9.7), we have solutions

$$x(t) = C_1e^{rt} + C_2te^{rt}, \tag{1.9.15}$$

for all constants C_1 and C_2.

We can then find a solution to (1.9.1) with initial data (1.9.2) as follows. Taking $x(t)$ as in (1.9.15), so $x'(t) = C_1re^{rt} + C_2rte^{rt} + C_2e^{rt}$, we set $t = 0$ to obtain

$$x(0) = C_1, \quad x'(0) = C_1 + C_2, \tag{1.9.16}$$

so (1.9.2) is satisfied if and only if C_1 and C_2 satisfy

$$C_1 = \alpha, \quad C_1 + C_2 = \beta, \tag{1.9.17}$$

i.e., if and only if

$$C_1 = \alpha, \quad C_2 = \beta - \alpha. \tag{1.9.18}$$

We claim that the constructions given above provide *all* of the solutions to (1.9.1), in the two respective cases. To see this, let $x(t)$ be any solution to (1.9.1), let $r = r_+$ (which equals r_- in Case II), and consider $u(t) = e^{-rt}x(t)$, as in (1.9.13). The computation (1.9.14) holds if $r_+ = r_-$, and if $r_+ \neq r_-$ we get

$$ax'' + bx' + cx = \Big\{au'' + (2ar + b)u'\Big\}e^{rt}. \tag{1.9.19}$$

As we have seen, when $r_+ = r_-$ this forces $u''(t) \equiv 0$, which hence forces $u(t)$ to have the form $C_1 + C_2t$ for some constants C_j, and hence $x(t) = C_1e^{rt} + C_2te^{rt}$. When $r_+ \neq r_-$, vanishing of (1.9.19) forces

$$av' + (2ar + b)v = 0, \quad \text{with} \ \ v = u', \tag{1.9.20}$$

which, by results of §1.1, forces

$$\begin{aligned} v(t) &= K_0e^{-(2r+b/a)t}, \quad \text{hence} \\ u(t) &= K_1 + K_2e^{-(2r+b/a)t}, \end{aligned} \tag{1.9.21}$$

for some constants K_0, K_1, and K_2. This in turn implies

$$x(t) = K_1e^{rt} + K_2e^{-(r+b/a)t}. \tag{1.9.22}$$

But (1.9.6) gives $r_+ + r_- = -b/a$, hence

$$r = r_+ \Longrightarrow -\Big(r + \frac{b}{a}\Big) = r_-, \tag{1.9.23}$$

so (1.9.22) is indeed of the form (1.9.7), with $C_+ = K_1$ and $C_- = K_2$.

The arguments given above show that indeed all solutions to (1.9.1) have the form (1.9.7) or (1.9.15), in Cases I and II, respectively. We say that (1.9.7) (in Case I) and (1.9.15) (in Case II) provide the *general solution* to (1.9.1). This analysis of the general solutions together with the computations giving (1.9.12) and (1.9.18), establish the following.

Theorem 1.9.1. *Given a, b, and c, with $a \neq 0$, and given α and β, the initial value problem* (1.9.1)–(1.9.2) *has a unique solution $x(t)$. In Case I, $x(t)$ has the form* (1.9.7), *and in Case II, it has the form* (1.9.15).

Remark. The uniqueness proof given above uses the same principle as that for the first order equation $dx/dt = ax$ in §1.1, but the details here are more elaborate. In §3.1, we will give a uniqueness proof for first order, constant coefficient linear systems that looks almost exactly like the argument in §1.1.

The results derived above apply whether a, b, and c are real or not. If we assume they are real, then Case I naturally divides into two subcases,

$$\text{(IA)} \qquad \mathrm{b}^2 - 4\mathrm{ac} > 0,$$
$$\text{(IB)} \qquad \mathrm{b}^2 - 4\mathrm{ac} < 0.$$

In Case IA, the roots of the characteristic equation (1.9.5) given by (1.9.6) are real. In Case IB, we have complex roots, of the form

$$r_\pm = r \pm i\sigma, \quad r = -\frac{b}{2a}, \quad \sigma = \frac{1}{2a}\sqrt{4ac - b^2}. \tag{1.9.24}$$

Hence the solutions (1.9.7) have the form

$$x(t) = C_+ x_+(t) + C_- x_-(t), \quad x_\pm(t) = e^{(r\pm i\sigma)t}. \tag{1.9.25}$$

From §1 we have $e^{(r\pm i\sigma)t} = e^{rt}e^{\pm i\sigma t}$, and also

$$e^{\pm i\sigma t} = \cos\sigma t \pm i\sin\sigma t. \tag{1.9.26}$$

Hence

$$x_\pm(t) = e^{rt}(\cos\sigma t \pm i\sin\sigma t). \tag{1.9.27}$$

In particular, the following are also solutions to (1.9.1):

$$\begin{aligned} x_1(t) &= \frac{1}{2}\big(x_+(t) + x_-(t)\big) = e^{rt}\cos\sigma t, \\ x_2(t) &= \frac{1}{2i}\big(x_+(t) - x_-(t)\big) = e^{rt}\sin\sigma t. \end{aligned} \tag{1.9.28}$$

We can hence rewrite (1.9.25) as $x(t) = C_1x_1(t) + C_2x_2(t)$, or equivalently

$$x(t) = C_1e^{rt}\cos\sigma t + C_2e^{rt}\sin\sigma t, \tag{1.9.29}$$

for some constants C_1 and C_2, related to C_+ and C_- by

$$C_1 = C_+ + C_-, \quad C_2 = i(C_+ - C_-). \tag{1.9.30}$$

We can combine these relations with (1.9.11)–(1.9.12) to solve the initial value problem (1.9.1)–(1.9.2).

We now apply the methods just developed to the linearized pendulum and damped pendulum equations (1.8.10) and (1.8.11), i.e.,

$$\frac{d^2u}{dt^2} + \frac{g}{\ell}u = 0, \tag{1.9.31}$$

and

$$\frac{d^2u}{dt^2} + \frac{\alpha}{m}\frac{du}{dt} + \frac{g}{\ell}u = 0. \tag{1.9.32}$$

Here, g, ℓ, α, and m are all > 0. Let us set

$$k = \sqrt{\frac{g}{\ell}}, \quad b = \frac{\alpha}{m}, \tag{1.9.33}$$

so $b > 0, k > 0$, and the equations (1.9.31)–(1.9.32) become

$$\frac{d^2u}{dt^2} + k^2 u = 0, \tag{1.9.34}$$

and

$$\frac{d^2u}{dt^2} + b\frac{du}{dt} + k^2 u = 0. \tag{1.9.35}$$

The characteristic equation for (1.9.34) is $r^2 + k^2 = 0$, with roots $r = \pm ik$. The general solution to (1.9.34) can hence be written either as $u(t) = C_+ e^{ikt} + C_- e^{-ikt}$ or as

$$u(t) = C_1 \cos kt + C_2 \sin kt. \tag{1.9.36}$$

The resulting motion is oscillatory motion, with period $2\pi/k$.

The characteristic equation for (1.9.35) is $r^2 + br + k^2 = 0$, with roots

$$r_\pm = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4k^2}. \tag{1.9.37}$$

There are three cases to consider:

$$\begin{aligned} &(\mathrm{IB}) && \mathrm{b}^2 - 4\mathrm{k}^2 < 0, \\ &(\mathrm{II}) && \mathrm{b}^2 - 4\mathrm{k}^2 = 0, \\ &(\mathrm{IA}) && \mathrm{b}^2 - 4\mathrm{k}^2 > 0. \end{aligned}$$

In Case IB, say $b^2 - 4k^2 = -4\kappa^2$. Then $r_\pm = -(b/2) \pm i\kappa$, and the general solution to (1.9.35) has the form

$$u(t) = C_1 e^{-bt/2} \cos \kappa t + C_2 e^{-bt/2} \sin \kappa t. \tag{1.9.38}$$

These decay exponentially as $t \nearrow +\infty$. This is damped oscillatory motion. The oscillatory factors have period

$$\frac{2\pi}{\kappa} = \frac{2\pi}{\sqrt{k^2 - (b/2)^2}}, \tag{1.9.39}$$

which approaches ∞ as $b \nearrow 2k$.

In Case IA, say $\beta = \sqrt{b^2 - 4k^2}$, so $r_\pm = (-b \pm \beta)/2$. Note that $0 < \beta < b$, so both r_+ and r_- are negative. The general solution to (1.9.35) then has the form

$$u(t) = C_1 e^{(-b+\beta)t/2} + C_2 e^{(-b-\beta)t/2}, \quad -b \pm \beta < 0. \tag{1.9.40}$$

These decay without oscillation as $t \nearrow +\infty$. One says this motion is *overdamped.* In Case II, the characteristic equation for (1.9.35) has the double root $-b/2$, and the general solution to (1.9.35) has the form

$$u(t) = C_1 e^{-bt/2} + C_2 t e^{-bt/2}. \tag{1.9.41}$$

These also decay without oscillation as $t \nearrow +\infty$. One says this motion is *critically damped.*

The nonlinear damped pendulum equation (1.7.6) can also be shown to manifest these damped oscillatory, critically damped, and overdamped behaviors.

Exercises

1. Find the general solution to each of the following equations for $x = x(t)$.

(a)
$$x'' + 25x = 0.$$

(b)
$$x'' - 25x = 0.$$

(c)
$$x'' - 2x' + x = 0.$$

(d)
$$x'' + 2x' + x = 0.$$

(e)
$$x'' + x' + x = 0.$$

2. In each case (a)–(e) of Exercise 1, find the solution satisfying the initial condiiton

$$x(0) = 1, \quad x'(0) = 0.$$

3. In each case (a)–(e) of Exercise 1, find the solution satisfying the initial condition

$$x(0) = 0, \quad x'(0) = 1.$$

4. For $\varepsilon \neq 0$, solve the initial value problem

$$x_\varepsilon'' - 2x_\varepsilon' + (1 - \varepsilon^2)x_\varepsilon = 0, \quad x_\varepsilon(0) = 0, \;\; x_\varepsilon'(0) = 1.$$

Compute the limit

$$x(t) = \lim_{\varepsilon \to 0} x_\varepsilon(t),$$

and show that the limit solves

$$x'' - 2x' + x = 0, \quad x(0) = 0, \;\; x'(0) = 1.$$

1.10. Nonhomogeneous equations I–undetermined coefficients

We study nonhomogeneous, second order, constant coefficient linear equations, that is to say, equations of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \tag{1.10.1}$$

with constants a, b, and c ($a \neq 0$) and a given function $f(t)$. The equation (1.10.1) is called nonhomogeneous whenever $f(t)$ is not identically 0. We might impose initial conditions, like

$$x(0) = \alpha, \quad x'(0) = \beta. \tag{1.10.2}$$

In this section we assume $f(t)$ is a constant multiple of one of the following functions, or perhaps a finite sum of such functions:

$$e^{\kappa t}, \tag{1.10.3}$$

$$\sin \sigma t, \tag{1.10.4}$$

$$\cos \sigma t, \tag{1.10.5}$$

$$t^k. \tag{1.10.6}$$

We discuss a method, called the *method of undetermined coefficients*, to solve (1.10.1) in such cases. In §1.14 we will discuss a method that applies to a broader class of functions f.

We begin with the case (1.10.3). The first strategy is to seek a solution in the form

$$x(t) = Ae^{\kappa t}. \tag{1.10.7}$$

Here A is the undetermined coefficient. The goal will be to determine it. Plugging (1.10.7) into the left side of (1.10.1) gives

$$ax'' + bx' + cx = A(a\kappa^2 + b\kappa + c)e^{\kappa t}. \tag{1.10.8}$$

As long as κ is not a root of the characteristic polynomial $p(r) = ar^2 + br + c$, we get a solution to (1.10.1) in the form (1.10.7), with

$$A = \frac{1}{a\kappa^2 + b\kappa + c}. \tag{1.10.9}$$

In such a case, the equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = Be^{\kappa t} \tag{1.10.10}$$

has a solution,

$$x_p(t) = ABe^{\kappa t}, \tag{1.10.11}$$

with A given by (1.10.9). We say $x_p(t)$ is a *particular* solution to (1.10.10). If $x(t)$ is another solution, then, because the equation is linear, $y(t) = x(t) - x_p(t)$ solves the homogeneous equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{1.10.12}$$

which was treated in §1.9. If, for example, $p(r)$ has distinct roots r_+ and r_-, we know the general solution of (1.10.11) is

$$y(t) = C_+ e^{r_+ t} + C_- e^{r_- t}. \tag{1.10.13}$$

Then the general solution to (1.10.10) is

$$x(t) = \frac{B}{a\kappa^2 + b\kappa + c} e^{\kappa t} + C_+ e^{r_+ t} + C_- e^{r_- t}. \tag{1.10.14}$$

In (1.10.14), a, b, c, B, and κ are given by (1.10.10), and C_+ and C_- are arbitrary constants. If the initial conditions in (1.10.2) are imposed, they will determine C_+ and C_-. If r_+ and r_- are complex, we could rewrite (1.10.13)–(1.10.14), using Euler's formula, as in §1.9.

Formulas (1.10.11)–(1.10.14) hold under the hypothesis that r_+, r_-, and κ are all distinct. If the characteristic polynomial has a double root $r = r_+ = r_-$, distinct from κ, then we replace (1.10.13) by

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}, \tag{1.10.15}$$

and the general solution to (10.10) has the form

$$x(t) = \frac{B}{a\kappa^2 + b\kappa + c} e^{\kappa t} + C_1 e^{rt} + C_2 t e^{rt}. \tag{1.10.16}$$

Again, the initial conditions (1.10.2) would determine C_1 and C_2.

We turn to the case that κ is a root of the characteristic polynomial $p(r)$. In such a case, (1.10.8) vanishes, and there is not a solution to (1.10.1) in the form (1.10.7). This study splits into two cases. First assume $p(r)$ has distinct roots. Say $\kappa = r_+ \neq r_-$. Then (1.10.1) (with $f(t) = e^{\kappa t}$) will have a solution of the form

$$x(t) = A t e^{\kappa t}. \tag{1.10.17}$$

Indeed, a computation parallel to (1.9.14), with $u(t) = At,\ r = \kappa$, gives

$$ax'' + bx' + cx = (2a\kappa + b) A e^{\kappa t}, \tag{1.10.18}$$

since in this case $u'' = 0$ and $a\kappa^2 + b\kappa + c = 0$. Then (1.10.1) holds with $f(t) = e^{\kappa t}$, provided

$$A = \frac{1}{2a\kappa + b}, \tag{1.10.19}$$

and more generally a particular solution to (1.10.10) is given by

$$x_p(t) = ABte^{\kappa t}, \tag{1.10.20}$$

with A given by (1.10.19). As above, the general solution to (1.10.10) then has the form

$$x(t) = x_p(t) + y(t), \tag{1.10.21}$$

where $y(t)$ solves (1.10.12), hence has the form (1.10.13) (given $r_+ \neq r_-$).

To finish the analysis of (1.10.10), it remains to consider the case $\kappa = r_+ = r_-$. Then functions of the form (1.10.15) (with $r = \kappa$) solve (1.10.12), so there is not a solution to (1.10.1) (with $f(t) = e^{\kappa t}$) of the form (1.10.17). Instead, we will find a solution of the form

$$x(t) = At^2 e^{\kappa t}. \tag{1.10.22}$$

In this case, a computation parallel to (1.9.14), with $u(t) = At^2,\ r = \kappa$, gives

$$ax'' + bx' + cx = 2aAe^{\kappa t}, \tag{1.10.23}$$

since in this case $u'' = 2A,\ 2a\kappa + b = 0$, and $a\kappa^2 + b\kappa + c = 0$. Then (1.10.1) holds with $f(t) = e^{\kappa t}$ provided

$$A = \frac{1}{2a}, \tag{1.10.24}$$

and more generally a particular solution to (1.10.10) is given by

$$x_p(t) = ABt^2 e^{\kappa t}, \tag{1.10.25}$$

with A given by (1.10.24). Then the general solution to (1.10.10) has the form (1.10.21), where $y(t)$ solves (1.10.12), hence has the form (1.10.15), with $r = \kappa$. (Recall we are assuming $r_+ = r_-$.)

As a slight extension of (1.10.10), consider the equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = B_1 e^{\kappa_1 t} + B_2 e^{\kappa_2 t}. \tag{1.10.26}$$

This has a solution of the form

$$x_p(t) = x_{p1}(t) + x_{p2}(t), \tag{1.10.27}$$

where $x_{pj}(t)$ are particular solutions of (1.10.10), with B replaced by B_j and κ replaced by κ_j. Then the general solution to (1.10.26) has the form (1.10.21), with $x_p(t)$ given by (1.10.27) and $y(t)$ solving (1.10.12).

We move on to cases of $f(t)$ given by (1.10.4) and (1.10.5), which we combine as follows:

$$a\frac{d^2x}{dt^2} + c\frac{dx}{dt} + cx = b_1 \sin\sigma t + b_2 \cos\sigma t. \tag{1.10.28}$$

Via Euler's formula we can write

$$\begin{gathered} b_1 \sin\sigma t + b_2 \cos\sigma t = B_1 e^{i\sigma t} + B_2 e^{-i\sigma t}, \\ B_1 = \frac{b_1}{2i} + \frac{b_2}{2}, \quad B_2 = -\frac{b_1}{2i} + \frac{b_2}{2}, \end{gathered} \tag{1.10.29}$$

and we are back in the setting (1.10.26), with $\kappa_1 = i\sigma,\ \kappa_2 = -i\sigma$. Thus, for example, if $\pm i\sigma$ are not roots of the characteristic polynomial $p(r) = ar^2 + br + c$, we have a particular solution of the form

$$x_p(t) = A_1 B_1 e^{i\sigma t} + A_2 B_2 e^{-i\sigma t}, \tag{1.10.30}$$

where B_1 and B_2 are as in (1.10.29) and the undetermined coefficients A_1 and A_2 can be obtained by plugging into (1.10.28). As an alternative presentation, we can again use Euler's formula to rewrite (1.10.30) as

$$x_p(t) = a_1 \sin\sigma t + a_2 \cos\sigma t, \tag{1.10.31}$$

where the undetermined coefficients a_1 and a_2 are obtained by plugging into (1.10.28).

If a, b, and c in (1.10.1) are all real, then $p(r)$ will not have purely imaginary roots if $b \neq 0$. If $b = 0$, the roots will be $r_\pm = \pm\sqrt{-c/a}$, which are real if $c/a < 0$ and purely imaginary if $c/a > 0$. In case $r_\pm = \pm i\sigma$, considerations parallel to

(1.10.17)–(1.10.20) apply, with $\kappa = \pm i\sigma$. Again a further application of Euler's formula gives

$$x_p(t) = a_1 t \sin \sigma t + a_2 t \cos \sigma t, \tag{1.10.32}$$

where the coefficients a_1 and a_2 are obtained by plugging into (1.10.28).

We now move to cases of $f(t)$ given by (1.10.6). Take $k = 1$, so we are looking at

$$ax'' + bx' + cx = t. \tag{1.10.33}$$

We try

$$x(t) = At + B, \tag{1.10.34}$$

for which $x' = A$, $x'' = 0$, and the left side of (1.10.33) is $cAt + (B + bA)$. The condition that (1.10.33) hold is

$$cA = 1, \quad B + bA = 0, \tag{1.10.35}$$

solved by

$$A = \frac{1}{c}, \quad B = -\frac{b}{c}, \tag{1.10.36}$$

assuming $c \neq 0$. If $c = 0$, we want to solve

$$av' + bv = t \tag{1.10.37}$$

for $v = dx/dt$. We try

$$v(t) = \alpha t + \beta, \tag{1.10.38}$$

for which $v' = \alpha$ and the left side of (1.10.37) is $a\alpha + b(\alpha t + \beta)$. The condition that (1.10.37) hold is

$$b\alpha = 1, \quad a\alpha + b\beta = 0, \tag{1.10.39}$$

solved by

$$\alpha = \frac{1}{b}, \quad \beta = -\frac{a}{b^2}, \tag{1.10.40}$$

assuming $b \neq 0$. In such a case, we can take

$$x(t) = \frac{\alpha}{2} t^2 + \beta t. \tag{1.10.41}$$

In case $c = b = 0$, (1.10.32) becomes

$$ax'' = t, \tag{1.10.42}$$

with solution

$$x(t) = \frac{1}{6a} t^3. \tag{1.10.43}$$

Analogous considerations apply to (1.10.6) with $k \geq 2$. The method can also be extended to treat $f(t)$ in the form

$$t^k e^{\kappa t}, \quad t^k \sin \sigma t, \quad t^k \cos \sigma t. \tag{1.10.44}$$

We omit details. In such cases, it is just as convenient to use the method developed in §1.14.

See §1.16 for further insight on why the method of undetermined coefficients works for functions $f(t)$ of the form (1.10.3)–(1.10.6), and more generally of the form (1.10.44).

Exercises

1. Find the general solution to each of the following equations for $x = x(t)$.

(a)

$$x'' + 25x = e^{5t}.$$

(b)

$$x'' - 25x = e^{5t}.$$

(c)

$$x'' - 2x + x = \sin t.$$

(d)

$$x'' + 2x' + x = e^{t}.$$

(e)

$$x'' + x' + x = \cos t.$$

2. In each case (a)–(e) of Exercise 1, find the solution satisfying the initial conditions

$$x(0) = 1, \quad x'(0) = 0.$$

3. In each case (a)–(e) of Exercises 1, find the solution satisfying the initial conditions

$$x(0) = 0, \quad x'(0) = 1.$$

4. For $\varepsilon \neq 0$, solve the initial value problem

$$x_\varepsilon'' - 25x_\varepsilon = e^{(5+\varepsilon)t}, \quad x_\varepsilon(0) = 1, \;\; x_\varepsilon'(0) = 0.$$

Compute the limit

$$x(t) = \lim_{\varepsilon \to 0} x_\varepsilon(t),$$

and show that the limit solves

$$x'' - 25x = e^{5t}, \quad x(0) = 1, \;\; x'(0) = 0.$$

1.11. Forced pendulum–resonance

Here we study the following special cases of (1.10.28), modeling the linearized pendulum and damped pendulum, respectively, subjected to an additional periodic force of the form $F_0 \sin \sigma t$. The equations we consider are, respectively,

$$\frac{d^2u}{dt^2} + \frac{g}{\ell}u = F_0 \sin \sigma t, \tag{1.11.1}$$

and

$$\frac{d^2u}{dt^2} + \frac{\alpha}{m}\frac{du}{dt} + \frac{g}{\ell}u = F_0 \sin \sigma t. \tag{1.11.2}$$

The quantities α, m, g, and ℓ are all positive, and we take F_0 and σ to be real. As in (1.9.33), we set

$$k = \sqrt{\frac{g}{\ell}}, \quad b = \frac{\alpha}{m}, \tag{1.11.3}$$

so $b > 0,\ k > 0$, and the equations (1.11.1)–(1.11.2) become

$$\frac{d^2u}{dt^2} + k^2 u = F_0 \sin \sigma t, \tag{1.11.4}$$

and

$$\frac{d^2u}{dt^2} + b\frac{du}{dt} + k^2 u = F_0 \sin \sigma t. \tag{1.11.5}$$

As long as $k \neq \pm\sigma$, we can set $u(t) = a_1 \sin \sigma t$ and the left side of (1.11.4) equals $a_1(k^2 - \sigma^2) \sin \sigma t$, so a solution to (1.11.4) is

$$u_p(t) = \frac{F_0}{k^2 - \sigma^2} \sin \sigma t, \tag{1.11.6}$$

in such a case. Note how the coefficient $F_0/(k^2 - \sigma^2)$ blows up as $\sigma \to \pm k$. If $\sigma = k$, then, as in (1.10.32), we need to seek a solution to (1.11.4) of the form

$$u_p(t) = a_1 t \sin \sigma t + a_2 t \cos \sigma t. \tag{1.11.7}$$

In such a case,

$$u_p'' + k^2 u_p = 2a_1\sigma \cos \sigma t - 2a_2\sigma \sin \sigma t, \tag{1.11.8}$$

so (1.11.4) holds provided

$$-2a_2\sigma = F_0, \quad 2a_1\sigma = 0, \tag{1.11.9}$$

i.e., we have

$$u_p(t) = -\frac{F_0}{2\sigma} t \cos \sigma t. \tag{1.11.10}$$

Note that $u_p(t)$ grows without bound as $|t| \to \infty$ in this case, as opposed to the bounded behavior in t given by (1.11.6) when $\sigma^2 \neq k^2$. We say we have a *resonance* at $\sigma^2 = k^2$.

Moving on to (1.11.5), as in (1.9.37) the characteristic polynomial $p(r) = r^2 + br + k^2$ has roots

$$r_\pm = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4k^2}, \tag{1.11.11}$$

and as long as $b > 0$, $\pm i\sigma \neq r_\pm$. Hence we can seek a solution to (1.11.5) in the form

$$u_p(t) = a_1 \sin \sigma t + a_2 \cos \sigma t. \tag{1.11.12}$$

A computation gives

$$\begin{aligned} u_p'' + bu_p' + k^2 u_p = (&-a_1\sigma^2 - a_2 b\sigma + a_1 k^2) \sin \sigma t \\ &+ (-a_2\sigma^2 + a_1 b\sigma + a_2 k^2) \cos \sigma t, \end{aligned} \tag{1.11.13}$$

so u_p is a solution to (1.11.5) if and only if

$$\begin{aligned} (k^2 - \sigma^2) a_1 - (b\sigma) a_2 &= F_0, \\ (b\sigma) a_1 + (k^2 - \sigma^2) a_2 &= 0. \end{aligned} \tag{1.11.14}$$

Solving for a_1 and a_2 gives

$$\begin{aligned} a_1 &= \frac{k^2 - \sigma^2}{(k^2 - \sigma^2)^2 + (b\sigma)^2} F_0, \\ a_2 &= -\frac{b\sigma}{(k^2 - \sigma^2)^2 + (b\sigma)^2} F_0. \end{aligned} \tag{1.11.15}$$

We can rewrite (1.11.12) as

$$u_p(t) = A \sin(\sigma t + \theta), \tag{1.11.16}$$

for some constants A and θ, using the identity

$$A \sin(\sigma t + \theta) = A(\cos \theta) \sin \sigma t + A(\sin \theta) \cos \sigma t. \tag{1.11.17}$$

It follows that (1.11.16) is equivalent to (1.11.12) provided

$$A \cos \theta = a_1, \quad A \sin \theta = a_2, \tag{1.11.18}$$

i.e., provided

$$a_1 + i a_2 = A e^{i\theta}. \tag{1.11.19}$$

We take $A > 0$ such that

$$A^2 = a_1^2 + a_2^2 = \frac{F_0^2}{(k^2 - \sigma^2)^2 + (b\sigma)^2}. \tag{1.11.20}$$

Thus

$$A = \frac{|F_0|}{\sqrt{(k^2 - \sigma^2)^2 + (b\sigma)^2}} \tag{1.11.21}$$

is the amplitude of the solution (1.11.16).

If b, k, and F_0 are fixed quantities in (1.11.5) and σ is allowed to vary, A in (1.11.21) is maximized at the value of σ for which

$$\beta(\sigma) = (k^2 - \sigma^2)^2 + (b\sigma)^2 \tag{1.11.22}$$

is minimal. We have

$$\begin{aligned} \beta'(\sigma) &= 4\sigma^3 + 2(b^2 - 2k^2)\sigma \\ &= 4\sigma \Big[\sigma^2 - \Big(k^2 - \frac{b^2}{2} \Big) \Big]. \end{aligned} \tag{1.11.23}$$

Note that $\sigma = 0$ is a critical point, and $\beta(0) = k^4$. There are two cases. First,

$$k^2 - \frac{b^2}{2} > 0 \Longrightarrow \beta_{\min} = \beta\Bigl(\pm\sqrt{k^2 - \frac{b^2}{2}}\Bigr) = b^2\Bigl(k^2 - \frac{b^2}{4}\Bigr), \tag{1.11.24}$$

since $k^4 \ge b^2(k^2 - b^2/4)$. (Indeed, taking $\xi = k^2/b^2$, this inequality is equivalent to $\xi^2 \ge \xi - 1/4$, and $\xi^2 - \xi + 1/4 = (\xi - 1/2)^2$.) In the second case,

$$k^2 - \frac{b^2}{2} \le 0 \Longrightarrow \beta_{\min} = \beta(0) = k^4. \tag{1.11.25}$$

In these respective cases, we get

$$A_{\max} = \frac{|F_0|}{b}\Bigl(k^2 - \frac{b^2}{4}\Bigr)^{-1/2}, \tag{1.11.26}$$

and

$$A_{\max} = \frac{|F_0|}{k^2}. \tag{1.11.27}$$

In the first case, i.e., (1.11.24), we say resonance is achieved at $\sigma^2 = k^2 - b^2/2$. Recall from §1.9 that critical damping occurs for $k^2 = b^2/4$, for the unforced pendulum, so in case (1.11.24) the unforced pendulum has damped oscillatory motion.

Exercises

1. Find the general solution to

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + u = 3\sin\sigma t. \tag{1.11.28}$$

2. For the equation in Exercise 1, find the value of σ for which there is resonance.

3. Would the answer to Exercise 2 change if the right side of (1.11.28) were changed to

$$10\sin\sigma t?$$

Explain.

4. Redo Exercises 1–2 with (1.11.28) replaced by each of the following:

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + 3u = \sin\sigma t,$$
$$\frac{d^2u}{dt^2} + 2\frac{du}{dt} + 3u = 2\sin\sigma t.$$

5. Redo Exercise 1 with (1.11.28) replaced by the following:

$$\frac{d^2u}{dt^2} + 2\frac{du}{dt} + u = 3\sin\sigma t.$$

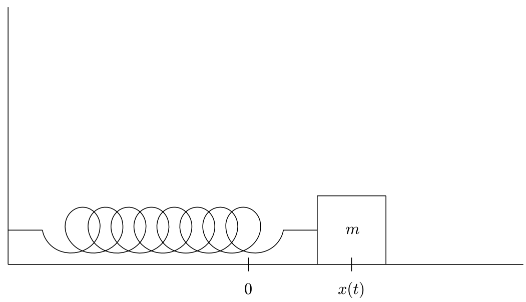

Figure 1.12.1. Mass on a spring

Discuss the issue of resonance in this case.

1.12. Spring motion

We consider the motion of a body of mass m, attached to one end of a spring, as depicted in Figure 1.12.1. The other end of the spring is attached to a rigid wall, and the weight slides along the floor, pushed or pulled by the spring. We assume that the force of the spring is a function of position:

$$F_1 = F_1(x). \tag{1.12.1}$$

We pick the origin to be the position where the spring is relaxed, so $F(0) = 0$. A good approximation, valid for small oscillations, is

$$F_1(x) = -Kx, \tag{1.12.2}$$

with a positive constant K (called the spring constant). This approximation loses accuracy if $|x|$ is large. Sliding along the floor typically produces a frictional force that is a function of the velocity $v = dx/dt$. A good approximation for the frictional force is

$$F_2 = F_2(v) = -av, \tag{1.12.3}$$

where a is a positive constant, called the coefficient of friction. The total force on the mass is $F = F_1 + F_2$, and Newton's law $F = ma$ yields the differential equation

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + Kx = 0. \tag{1.12.4}$$

This has the same form as (1.9.35), i.e.,

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k^2x = 0, \tag{1.12.5}$$

with

$$b = \frac{a}{m}, \quad k^2 = \frac{K}{m}, \tag{1.12.6}$$

both positive, and the analysis of (1.9.35) applies here, including notions of oscillatory damped, critically damped, and overdamped motion.

One can consider systems of several masses, connected via springs. These situations lead to systems of differential equations, studied in Chapter 3.

Exercises

1. Suppose one has a spring system as in Figure 1.12.1. Assume the mass m is 2 kg and the spring constant K is 6 kg/sec^2. There is a frictional force of a kg/sec. Find the values of a for which the spring motion is

(a) damped oscillatory,

(b) critically damped,

(c) overdamped.

2. In the context of Exercise 1, suppose there is also an external force of the form

$$10 \sin \sigma t \quad \text{kg-m/sec}^2.$$

(Assume x is given in meters.) Take

$$a = 2,$$

so (12.4) becomes

$$2\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 6x = 10 \sin \sigma t.$$

Find the value of σ for which there is resonance.

1.13. RLC circuits

Here we derive a differential equation for the current flowing through the circuit depicted in Figure 1.13.1, which consists of a resistor, with resistance R (in ohms), a capacitor, with capacitance C (in farads), and an inductor, with inductance L (in henrys). The circuit is plugged into a source of electricity, providing voltage $E(t)$ (in volts). As stated, we want to find a differential equation for the current $I(t)$ (in amps).

The equation is derived using two types of basic laws. The first type consists of two rules, which are special cases of Kirchhoff's laws:

(A) The sum of the voltage drops across the three circuit elements is $E(t)$.
(B) For each t, the same current $I(t)$ flows through each circuit element.

For more complicated circuits than the one depicted in Figure 1.13.1, these rules take a more elaborate form. We return to this in Chapter 3.

The second type of law specifies the voltage drop across each circuit element:

$$\begin{aligned} &\text{(a)} \qquad \text{Resistor:} \quad \mathrm{V} = IR, \\ &\text{(b)} \qquad \text{Inductor:} \quad \mathrm{V} = L\frac{dI}{dt}, \\ &\text{(c)} \qquad \text{Capacitor:} \quad \mathrm{V} = \frac{Q}{C}. \end{aligned}$$

As stated above, V is measured in volts, I in amps, R in ohms, L in henrys, and C in farads. In addition, Q is the charge on the capacitor, measured in coulombs. The rule (c) is supplemented by the following formula for the current across the capacitor:

$$\text{(c2)} \qquad \mathrm{I} = \frac{\mathrm{dQ}}{\mathrm{dt}}.$$

In (b) and (c2), time is measured in seconds.

In Figure 1.13.1, the circuit elements are numbered. We let $V_j = V_j(t)$ denote the voltage drop across element j. Rules (A), (B), and (a) give

$$V_1 + V_2 + V_3 = E(t), \tag{1.13.1}$$

$$V_1 = RI. \tag{1.13.2}$$

Rules (B), (b), and (c)–(c2) give differential equations:

$$L\frac{dI}{dt} = V_3, \tag{1.13.3}$$

$$C\frac{dV_2}{dt} = I. \tag{1.13.4}$$

Plugging (1.13.2)–(1.13.3) into (1.13.1) gives

$$RI + V_2 + L\frac{dI}{dt} = E(t). \tag{1.13.5}$$

Applying d/dt to (1.13.5) and using (1.13.4) gives

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t). \tag{1.13.6}$$

This is the equation for the RLC circuit in Figure 1.13.1. If we divide by L we get

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{E'(t)}{L}, \tag{1.13.7}$$

which has the same form as the (linearized) damped driven pendulum (1.11.5), with

$$b = \frac{R}{L}, \quad k^2 = \frac{1}{LC}, \tag{1.13.8}$$

except that at this point $E'(t)/L$ is not specified to agree with the right side of (1.11.5). However, indeed, if alternating current powers this circuit, it is reasonable to take

$$E(t) = E_0 \cos \sigma t, \tag{1.13.9}$$

so

$$\frac{1}{L}E'(t) = -\frac{\sigma E_0}{L}\sin \sigma t = F_0 \sin \sigma t. \tag{1.13.10}$$

Then analyses of solutions done in §1.11, including analyses of resonance phenomena, apply in this setting.

Actually, in this setting a different perspective on resonance is in order. The frequency $\sigma/2\pi$ cycles/sec of the alternating current is typically fixed, while one might be able to adjust the capacitance C. Let us assume R and L are also fixed, so b in (1.13.8) is fixed but one might adjust k. Recalling the formulas (1.11.16) and (1.11.21), which in this setting take the form

$$I_p(t) = A\sin(\sigma t + \theta), \quad A = \frac{|F_0|}{\sqrt{(k^2-\sigma^2)^2 + (b\sigma)^2}}, \tag{1.13.11}$$

we see that for fixed b and σ, this amplitude is maximized for k satisfying

$$k^2 = \sigma^2, \tag{1.13.12}$$

i.e., for

$$LC = \frac{1}{\sigma^2}. \tag{1.13.13}$$

More elaborate circuits, containing a larger number of circuit elements, and more loops, are naturally treated in the context of systems of differential equations. See Chapter 3 for more on this.

Remark. Consistent with formulas (a)–(c) and (c2), the units mentioned above are related as follows:

$$\begin{aligned} 1 \text{ amp} &= 1\ \frac{\text{coulomb}}{\text{sec}} \\ 1 \text{ farad} &= 1\ \frac{\text{coulomb}}{\text{volt}} \\ 1 \text{ henry} &= 1\ \frac{\text{volt-sec}}{\text{amp}} \\ 1 \text{ ohm} &= 1\ \frac{\text{volt}}{\text{amp}}. \end{aligned} \tag{1.13.14}$$

To relate these to other physical units, we mention that

$$\begin{aligned} 1 \text{ volt} &= 1 \text{ joule/coulomb} \\ 1 \text{ watt} &= 1 \text{ volt-amp} = 1 \text{ joule/sec} \\ 1 \text{ joule} &= 1 \text{ Newton-meter} \\ 1 \text{ Newton} &= 1 \text{ kg-m/sec}^2. \end{aligned} \tag{1.13.15}$$

The force of gravity at the surface of the Earth on a 1 kg. object is 9.8 Newtons, or, alternatively, 2.2 pounds. In other words, one Newton is about 0.224 pounds. Hence one joule is about 0.735 foot-pounds.

The coulomb is a unit of charge with the following property. If two particles, of charge q_1 and q_2 coulombs, are separated by r meters, the force between them is given by Coulomb's law:

$$F = k\frac{q_1 q_2}{r^2} \text{ Newtons}, \quad k = 8.99 \times 10^9. \tag{1.13.16}$$

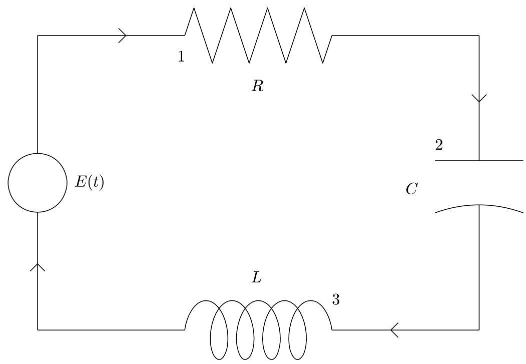

Figure 1.13.1. RLC circuit

Investigations into the nature of electrons have shown that

$$-1 \text{ coulomb} = \text{ charge of } 6.24 \times 10^{18} \text{ electrons.} \tag{1.13.17}$$

In connection with this, we mention that one gram of water contains 3.3×10^{23} electrons.

Exercises

1. Consider a circuit as in Figure 1.13.1. Assume the inductance is 4 henrys and the applied current has the form (1.13.9) with a frequency of 60 hertz, i.e., 60 cycles/sec. Find the value of the capacitance C, in farads, to achieve resonance.

2. Redo Exercise 1, this time with inductance of 10^{-6} henry and applied current of the form (1.13.9) with a frequency of 120 megahertz.

1.14. Nonhomogeneous equations II–variation of parameters

Here we present another approach to solving

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t), \tag{1.14.1}$$

(with constant b and c) called the method of variation of parameters. It works as follows. Let $y_1(t)$ and $y_2(t)$ be a complete set of solutions of the homogeneous equation

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{1.14.2}$$

The method consists of seeking a solution to (1.14.1) in the form

$$x(t) = u_1(t)y_1(t) + u_2(t)y_2(t), \tag{1.14.3}$$

and finding equations for $u_j(t)$ that are simpler then the original equation (1.14.1). We have

$$x' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2. \tag{1.14.4}$$

It will be convenient to arrange that x'' not involve second order derivatives of u_1 and u_2. To achieve this, we impose the condition

$$u_1' y_1 + u_2' y_2 = 0. \tag{1.14.5}$$

Then $x'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$, and using (1.14.2) to replace y_j'' by $-by_j' - cy_j$, we get

$$x'' = u_1' y_1' + u_2' y_2' - (by_1' + cy_1)u_1 - (by_2' + cy_2)u_2, \tag{1.14.6}$$

hence

$$x'' + bx' + cx = y_1' u_1' + y_2' u_2'. \tag{1.14.7}$$

Thus we have a solution to (1.14.1) in the form (1.14.3) provided u_1' and u_2' solve

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0, \\ y_1' u_1' + y_2' u_2' &= f. \end{aligned} \tag{1.14.8}$$

This linear system for u_1' and u_2' has the explicit solution

$$u_1' = -\frac{y_2}{W} f, \quad u_2' = \frac{y_1}{W} f, \tag{1.14.9}$$

where $W(t)$ is the following determinant, called the Wronskian determinant,

$$W = y_1 y_2' - y_2 y_1' = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}. \tag{1.14.10}$$

Determinants will be studied in the next chapter. The reader who has not seen them can take the first identity in (1.14.10) as a definition and ignore the second identity (for now).

Note that if the roots of the characteristic polynomial $p(r) = r^2 + br + c$ are distinct, $r_+ \neq r_-$, we can take

$$y_1 = e^{r_+ t}, \quad y_2 = e^{r_- t}, \tag{1.14.11}$$

and then

$$\begin{aligned} W(t) &= r_- e^{r_+ t} e^{r_- t} - r_+ e^{r_- t} e^{r_+ t} \\ &= (r_- - r_+) e^{(r_+ + r_-)t}, \end{aligned} \tag{1.14.12}$$

which is nowhere vanishing. If there is a double root, $r_+ = r_- = r$, we can take

$$y_1 = e^{rt}, \quad y_2 = te^{rt}, \tag{1.14.13}$$

and then

$$W(t) = e^{rt}(e^{rt} + tre^{rt}) - te^{rt} re^{rt} = e^{2rt}, \tag{1.14.14}$$

which is also nowhere vanishing.

Returning to (1.14.9), we can take

$$\begin{aligned} u_1(t) &= -\int_0^t \frac{y_2(s)}{W(s)} f(s)\,ds + C_1, \\ u_2(t) &= \int_0^t \frac{y_1(s)}{W(s)} f(s)\,ds + C_2, \end{aligned} \tag{1.14.15}$$

so

$$x(t) = C_1 y_1(t) + C_2 y_2(t) + \int_0^t \Big[y_2(t) y_1(s) - y_1(t) y_2(s) \Big] \frac{f(s)}{W(s)}\,ds. \tag{1.14.16}$$

Denote the last term, i.e., the integral, by $x_p(t)$.

Note that when the characteristic polynomial $r^2 + br + c$ has distinct roots $r_+ \neq r_-$ and (1.14.11)–(1.14.12) hold, we get

$$\begin{aligned} x_p(t) &= \frac{1}{r_- - r_+} \int_0^t \Big[e^{r_- t} e^{r_+ s} - e^{r_+ t} e^{r_- s} \Big] \frac{f(s)}{e^{(r_+ + r_-)s}}\,ds \\ &= \frac{1}{r_- - r_+} \int_0^t \big[e^{r_-(t-s)} - e^{r_+(t-s)} \big] f(s)\,ds. \end{aligned} \tag{1.14.17}$$

When the characteristic polynomial has double roots $r_+ = r_- = r$ and (1.14.13)–(1.14.14) hold, we get

$$\begin{aligned} x_p(t) &= \int_0^t \Big[t e^{rt} e^{rs} - e^{rt} s e^{rs} \Big] \frac{f(s)}{e^{2rs}}\,ds \\ &= \int_0^t (t-s) e^{r(t-s)} f(s)\,ds. \end{aligned} \tag{1.14.18}$$

The next section will continue the study of the Wronskian. Further material on the Wronskian and the method of variation of parameters, in a more general context, can be found in Chapter 3.

Exercises

Use the method of variation of parameters to solve each of the following for $x = x(t)$.

1.
$$x'' + x = e^t.$$

2.
$$x'' + x = \sin t.$$

3.
$$x'' + x = t.$$

4.
$$x'' + x = t^2.$$

5.
$$x'' + x = \tan t.$$

1.15. Variable coefficient second order equations

The general, possibly nonlinear, second order differential equation

$$\frac{d^2x}{dt^2} = F\Big(t, x, \frac{dx}{dt}\Big), \tag{1.15.1}$$

has already been mentioned in §1.4. If $F(t, x, v)$ is defined and smooth on a neighborhood of t_0, x_0, v_0, and one imposes an initial condition

$$x(t_0) = x_0, \quad x'(t_0) = v_0, \tag{1.15.2}$$

it is a fundamental result that the initial value problem (1.15.1)–(1.15.2) has a unique solution, at least for t in some interval containing t_0. A more general result of this sort will be proven in Chapter 4.

Linear second order equations have the form

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)x = f(t). \tag{1.15.3}$$

The existence and uniqueness results stated above apply. There are many specific and much-studied examples, such as Bessel's equation,

$$\frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} + \Big(1 - \frac{\nu^2}{t^2}\Big)x = 0, \tag{1.15.4}$$

whose solutions are called Bessel functions, and Airy's equation,

$$\frac{d^2x}{dt^2} - tx = 0, \tag{1.15.5}$$

whose solutions are Airy functions, just to mention two examples. Such functions are important and show up in many contexts. We will take a closer look at Bessel's equation in the next section. Linear variable coefficient equations could arise from RLC circuits in which one has variable capacitors, resistors, and inductors, turning (1.13.6) into

$$L(t)\frac{d^2I}{dt^2} + R(t)\frac{dI}{dt} + \frac{1}{C(t)}I = E'(t). \tag{1.15.6}$$

The most frequent source of such equations as (1.15.4)–(1.15.5) comes from the theory of partial differential equations (PDE). One such indication of how (1.15.4) arises is given in Appendix 1.A. The reader can find out much more about these equations in a text on PDE such as [**45**]. Solutions to these equations cannot generally be given in terms of elementary functions, such as exponential functions, but are further special functions, for which many analytical techniques have been developed.

As with the exponential function, analyzed in §1.1, power series techniques are very useful. We illustrate this by producing a power series

$$x(t) = \sum_{k-0}^{\infty} a_k t^k \tag{1.15.7}$$

for the solution to the Airy equation (1.15.5), with initial data

$$x(0) = 1, \quad x'(0) = 0. \tag{1.15.8}$$

If (1.15.7) is a convergent power series, then

$$x''(t) = \sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k, \tag{1.15.9}$$

while

$$tx(t) = \sum_{k=1}^{\infty} a_{k-1}t^k. \tag{1.15.10}$$

Comparison gives the recursive formula

$$a_{k+3} = \frac{a_k}{(k+3)(k+2)}. \tag{1.15.11}$$

To get started, we note that

$$a_0 = x(0) = 1, \quad a_1 = x'(0) = 0, \quad a_2 = \frac{1}{2}x''(0) = 0. \tag{1.15.12}$$

Thus $a_{3\ell+j} = 0$ for $j = 1, 2$, and we get

$$x(t) = \sum_{\ell=0}^{\infty} \alpha_\ell t^{3\ell}, \tag{1.15.13}$$

where $\alpha_\ell = a_{3\ell}$ is given recursively by

$$\alpha_{\ell+1} = \frac{\alpha_\ell}{(3\ell+3)(3\ell+2)}, \quad \alpha_0 = 1. \tag{1.15.14}$$

The ratio test applies to show that the power series (1.15.13) converges for all $t \in \mathbb{R}$, yielding a solution to Airy's equation (1.15.5), with initial data (1.15.8).

A study of power series as a technique for solving ODE in a more general setting is given in §3.10.

Another useful tool is the Wronskian determinant, defined on a pair of functions y_1 and y_2 by

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1' = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}. \tag{1.15.15}$$

If y_1 and y_2 both solve (1.15.3) with $f \equiv 0$, i.e.,

$$a(t)y'' + b(t)y' + c(t)y = 0, \tag{1.15.16}$$

then substituting for y_j'' in

$$\frac{dW}{dt} = y_1 y_2'' - y_2 y_1'' \tag{1.15.17}$$

yields

$$\frac{dW}{dt} = -\frac{b(t)}{a(t)}W, \tag{1.15.18}$$

a useful first order linear equation for W. Note that if we have such y_1 and y_2, solving (1.15.16) with initial condition

$$y(t_0) = \alpha, \quad y'(t_0) = \beta, \tag{1.15.19}$$

in the form $y(t) = C_1 y_1(t) + C_2 y_2(t)$ involves finding C_1 and C_2 such that

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) &= \alpha, \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) &= \beta, \end{aligned} \tag{1.15.20}$$

which uniquely determines C_1 and C_2 precisely when $W(y_1, y_2)(t_0) \neq 0$.

In light of the existence and uniqueness statement made above (to be proved in Chapter 4), it follows that if y_1 and y_2 solve (1.15.16) and have nonvanishing Wronskian, on an interval on which a, b, and c are smooth and a is nonvanishing, then the general solution to (1.15.16) has the form $C_1 y_1 + C_2 y_2$.

Recall that the Wronskian arose in the previous section, in the treatment of the method of variation of parameters. This treatment is extended to a much more general setting in Chapter 3.

Exercises

Equations of the form

$$at^2 \frac{d^2 x}{dt^2} + bt \frac{dx}{dt} + cx = 0 \tag{1.15.21}$$

are called Euler equations.

1. Show that $x(t) = t^r = e^{r \log t}$ solves (1.15.21) for $t > 0$ provided r satisfies

$$ar(r-1) + br + c = 0. \tag{1.15.22}$$

2. Show that if (1.15.22) has two distinct solutions r_1 and r_2, then

$$C_1 t^{r_1} + C_2 t^{r_2}$$

is the general solution to (1.15.21) on $t \in (0, \infty)$.

3. Show that if r is a double root of (1.15.22), then

$$C_1 t^r + C_2 (\log t) t^r$$

is the general solution to (1.15.21) for $t \in (0, \infty)$.

4. Find the coefficients a_k in the power series expansion

$$x(t) = \sum_{k=0}^{\infty} a_k t^k$$

for the solution to the Airy equation

$$\frac{d^2 x}{dt^2} - tx = 0, \tag{1.15.23}$$

with initial data

$$x(0) = 0, \quad x'(0) = 1.$$

Show that this power series converges for all t.

5. Show that the Wronskian of two solutions to the Airy equation (1.15.23) solves the equation

$$\frac{dW}{dt} = 0.$$

1.16. Bessel's equation

Here we construct solutions to Bessel's equation

$$\frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} + \Bigl(1 - \frac{\nu^2}{t^2}\Bigr)x = 0. \tag{1.16.1}$$

This is a very important equation, whose roots in partial differential equations are discussed in Appendix 1.A. Note that if the factor $(1-\nu^2/t^2)$ in front of x had the term 1 dropped, one would have the Euler equation

$$t^2x'' + tx' - \nu^2 x = 0, \tag{1.16.2}$$

with solutions

$$x(t) = t^{\pm\nu}, \tag{1.16.3}$$

as seen in (1.15.21)–(1.15.22). In light of this, we are motivated to set

$$x(t) = t^{\nu}y(t), \tag{1.16.4}$$

and study the resulting differential equation for y,

$$\frac{d^2y}{dt^2} + \frac{2\nu+1}{t}\frac{dy}{dt} + y = 0. \tag{1.16.5}$$

This might seem only moderately less singular than (1.16.1) at $t = 0$, but in fact it has a smooth solution. To obtain it, let us note that if $y(t)$ solves (1.16.5), so does $y(-t)$, hence so does $y(t) + y(-t)$, which is even in t. Thus, we look for a solution to (1.16.5) in the form

$$y(t) = \sum_{k=0}^{\infty} a_k t^{2k}. \tag{1.16.6}$$

Substitution into (1.16.5) yields for the left side of (1.16.5) the power series

$$\sum_{k=0}^{\infty}\Bigl\{(2k+2)(2k+2\nu+2)a_{k+1} + a_k\Bigr\}t^{2k}, \tag{1.16.7}$$

assuming convergence, which we will examine shortly. From this we see that, as long as

$$\nu \notin \{-1,-2,-3,\dots\}, \tag{1.16.8}$$

we can fix $a_0 = a_0(\nu)$ and solve recursively for a_{k+1}, for each $k \ge 0$, obtaining

$$a_{k+1} = -\frac{1}{4}\,\frac{a_k}{(k+1)(k+\nu+1)}. \tag{1.16.9}$$

Given (1.16.8), this recursion works, and one can readily apply the ratio test to show that the power series (1.16.6) converges for all $t \in \mathbb{R}$.

We will find it useful to produce an explicit solution to (1.16.9). For this, it is convenient to write

$$a_k = \alpha_k \beta_k \gamma_k, \tag{1.16.10}$$

with

$$\alpha_{k+1} = -\frac{1}{4}\alpha_k, \quad \beta_{k+1} = \frac{\beta_k}{k+1}, \quad \gamma_{k+1} = \frac{\gamma_k}{k+\nu+1}. \tag{1.16.11}$$

Clearly the first two equations have the explicit solutions

$$\alpha_k = \Big(-\frac{1}{4}\Big)^k \alpha_0, \quad \beta_k = \frac{\beta_0}{k!}. \tag{1.16.12}$$

We can solve the third if we have in hand a function $\Gamma(z)$ satisfying

$$\Gamma(z+1) = z\Gamma(z). \tag{1.16.13}$$

Indeed, the Euler gamma function $\Gamma(z)$, discussed in Appendix 1.B, is a smooth function on $\mathbb{R} \setminus \{0, -1, -2, \dots\}$ that satisfies (1.16.13). With this function in hand, we can write

$$\gamma_k = \frac{\tilde{\gamma}_0}{\Gamma(k+\nu+1)}, \tag{1.16.14}$$

and putting together (1.16.10)–(1.16.14) yields

$$a_k = \Big(-\frac{1}{4}\Big)^k \frac{\tilde{a}_0}{k!\Gamma(k+\nu+1)}. \tag{1.16.15}$$

We initialize this with $\tilde{a}_0 = 2^{-\nu}$. There results the solution $y(t) = \mathcal{J}_\nu(t)$ to (1.16.5), and $x(t) = J_\nu(t) = t^\nu \mathcal{J}_\nu(t)$ to (1.16.1), given by

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k+\nu}. \tag{1.16.16}$$

Supplementing the regularity of $\Gamma(z)$ on $\mathbb{R} \setminus \{0, -1, -2, \dots\}$, we will see in Appendix 1.B that

$$\frac{1}{\Gamma(z)} \text{ is well defined and smooth in } z \in \mathbb{R} \text{ vanishing for } z \in \{0, -1, -2, \dots\}. \tag{1.16.17}$$

Consequently (1.16.16) is a valid solution to (1.16.1) for $t \in (0, \infty)$, for each $\nu \in \mathbb{R}$. In fact,

$$J_\nu \text{ and } J_{-\nu} \text{ solve (1.16.1), for } \nu \in \mathbb{R}. \tag{1.16.18}$$

The function J_ν is called a *Bessel function.*

Let us examine the behavior of $J_\nu(t)$ as $t \searrow 0$. We have

$$J_\nu(t) = \frac{1}{\Gamma(\nu+1)} \Big(\frac{t}{2}\Big)^\nu + O(t^{\nu+1}), \quad \text{as } t \searrow 0. \tag{1.16.19}$$

As long as ν satisfies (1.16.8), the coefficient $1/\Gamma(\nu+1)$ is nonzero. Furthermore,

$$J_{-\nu}(t) = \frac{1}{\Gamma(1-\nu)} \Big(\frac{t}{2}\Big)^{-\nu} + O(t^{-\nu+1}), \quad \text{as } t \searrow 0, \tag{1.16.20}$$

and as long as $\nu \notin \{1, 2, 3, \dots\}$, the coefficient $1/\Gamma(1-\nu)$ is nonzero. In particular, we see that

$$\text{If } \nu \notin \mathbb{Z},\ J_\nu \text{ and } J_{-\nu} \text{ are linearly independent solutions to (1.16.1) on } (0, \infty). \tag{1.16.21}$$

In contrast to this, we have the following:

$$\text{If } n \in \mathbb{Z},\ J_n(t) = (-1)^n J_{-n}(t). \tag{1.16.22}$$

To see this, we assume $n \in \{1, 2, 3, \dots\}$, and note that

$$\frac{1}{\Gamma(k-n+1)} = 0, \quad \text{for} \ \ 0 \le k \le n-1. \tag{1.16.23}$$

We use this, together with the restatement of (1.16.16) that

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k+\nu}, \tag{1.16.24}$$

which follows from the identity $\Gamma(k+1) = k!$, to deduce that, for $n \in \mathbb{N}$,

$$\begin{aligned} J_{-n}(t) &= \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k-n+1)} \Big(\frac{t}{2}\Big)^{2k-n} \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+n}}{\Gamma(\ell+1)\Gamma(\ell+n+1)} \Big(\frac{t}{2}\Big)^{2\ell+n} \\ &= (-1)^n J_n(t). \end{aligned} \tag{1.16.25}$$

Consequently $J_\nu(t)$ and $J_{-\nu}(t)$ are linearly independent solutions to (1.16.1) as long as $\nu \notin \mathbb{Z}$, but this fails for $\nu \in \mathbb{Z}$. We now seek a family of solutions $Y_\nu(t)$ to (1.16.1) with the property that J_ν and Y_ν are linearly independent solutions, for all $\nu \in \mathbb{R}$. The key to this construction lies in an analysis of the Wronskian,

$$W_\nu(t) = W(J_\nu, J_{-\nu})(t) = J_\nu(t) J'_{-\nu}(t) - J'_\nu(t) J_{-\nu}(t). \tag{1.16.26}$$

By (1.15.10), we have

$$\frac{dW_\nu}{dt} = -\frac{1}{t} W_\nu, \tag{1.16.27}$$

hence

$$W_\nu(t) = \frac{K(\nu)}{t}. \tag{1.16.28}$$

To evaluate $K(\nu)$, we calculate

$$\begin{aligned} W(J_\nu, J_{-\nu}) &= W(t^\nu \mathcal{J}_\nu, t^{-\nu} \mathcal{J}_{-\nu}) \\ &= W(\mathcal{J}_\nu, \mathcal{J}_{-\nu}) - \frac{2\nu}{t} \mathcal{J}_\nu(t) \mathcal{J}_{-\nu}(t). \end{aligned} \tag{1.16.29}$$

Since $\mathcal{J}_\nu(t)$ and $\mathcal{J}_{-\nu}(t)$ are smooth in t, so is $W(\mathcal{J}_\nu, \mathcal{J}_{-\nu})$, and we deduce from (1.16.28)–(1.16.29) that

$$W_\nu(t) = -\frac{2\nu}{t} \mathcal{J}_\nu(0) \mathcal{J}_{-\nu}(0). \tag{1.16.30}$$

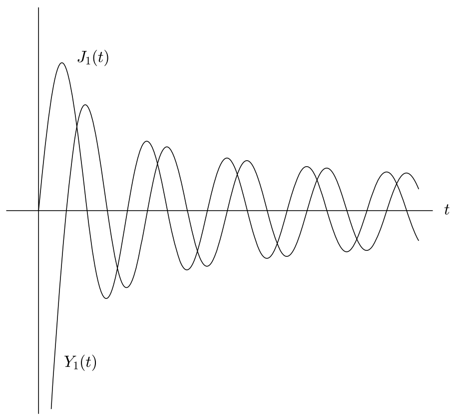

Figure 1.16.1. Graphs of $J_1(t)$ and $Y_1(t)$

Now, since $\mathcal{J}_\nu(0) = 1/2^\nu\Gamma(\nu+1)$, we have

$$\begin{aligned} \nu\mathcal{J}_\nu(0)\mathcal{J}_{-\nu}(0) &= \frac{\nu}{\Gamma(\nu+1)\Gamma(1-\nu)} \\ &= \frac{1}{\Gamma(\nu)\Gamma(1-\nu)}. \end{aligned} \tag{1.16.31}$$

An important gamma function identity, stated in Appendix 1.B, is

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu}. \tag{1.16.32}$$

Hence (1.16.30)–(1.16.31) yields

$$W(J_\nu, J_{-\nu})(t) = -\frac{2}{\pi}\,\frac{\sin \pi\nu}{t}. \tag{1.16.33}$$

This motivates the following. For $\nu \notin \mathbb{Z}$, set

$$Y_\nu(t) = \frac{J_\nu(t)\cos \pi\nu - J_{-\nu}(t)}{\sin \pi\nu}. \tag{1.16.34}$$

Note that, by (1.16.25), numerator and denominator both vanish for $\nu \in \mathbb{Z}$. Now, for $\nu \notin \mathbb{Z}$, we have

$$\begin{aligned} W(J_\nu, Y_\nu)(t) &= -\frac{1}{\sin \pi\nu} W(J_\nu, J_{-\nu})(t) \\ &= \frac{2}{\pi t}. \end{aligned} \tag{1.16.35}$$

Consequently, for $n \in \mathbb{Z}$, we set

$$Y_n(t) = \lim_{\nu \to n} Y_\nu(t) = \frac{1}{\pi}\Big[\frac{\partial J_\nu(t)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(t)}{\partial \nu}\Big]\Big|_{\nu=n}, \tag{1.16.36}$$

and we also have (1.16.35) for $\nu \in \mathbb{Z}$. The functions Y_ν are called Bessel functions of the second kind. See Figure 1.16.1 for graphs of $J_1(t)$ and $Y_1(t)$.

Another construction of a solution to accompany $J_n(t)$ is given in Chapter 3, in formulas (3.11.65)–(3.11.79).

We end this section with the following integral formula for $J_\nu(t)$, which plays an important role in further investigations, such as the behavior of $J_\nu(t)$ for large t.

Proposition 1.16.1. *If* $\nu > -1/2$,

$$J_\nu(t) = \frac{(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} (1-s^2)^{\nu-1/2} e^{ist}\, ds. \tag{1.16.37}$$

Proof. To verify (1.16.37), we replace e^{ist} by its power series, integrate term by term, and use some identities from Appendix 1.B. To begin, the integral on the right side of (1.16.37) is equal to

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_{-1}^{1} (ist)^{2k} (1-s^2)^{\nu-1/2}\, ds. \tag{1.16.38}$$

The identity (1.B.17) implies

$$\int_{-1}^{1} s^{2k}(1-s^2)^{\nu-1/2}\, ds = \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}, \tag{1.16.39}$$

so the right side of (1.16.37) equals

$$\frac{(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \sum_{k=0}^{\infty} \frac{1}{(2k)!} (it)^{2k} \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}. \tag{1.16.40}$$

As seen in (1.B.7), we have

$$\Gamma\Big(\frac{1}{2}\Big)(2k)! = 2^{2k} k!\, \Gamma\Big(k+\frac{1}{2}\Big), \tag{1.16.41}$$

so (1.16.40) is equal to

$$\Big(\frac{t}{2}\Big)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k}, \tag{1.16.42}$$

which agrees with our formula (1.16.16) for $J_\nu(t)$. □

See §3.11 for general results on ODE with regular singular points, with reference to the study of Bessel's equation. Further material on Bessel functions, including a study for large values of the argument t, and also for complex values, can be found in Chapter 7 of [**47**].

Exercises

1. Show that the Bessel functions J_ν satisfy the following recursion relations:

$$\frac{d}{dt}\Big(t^\nu J_\nu(t)\Big) = t^\nu J_{\nu-1}(t), \quad \frac{d}{dt}\Big(t^{-\nu} J_\nu(t)\Big) = -t^{-\nu} J_{\nu+1}(t),$$

or equivalently

$$\begin{aligned} J_{\nu+1}(t) &= -J'_\nu(t) + \frac{\nu}{t} J_\nu(t), \\ J_{\nu-1}(t) &= J'_\nu(t) + \frac{\nu}{t} J_\nu(t). \end{aligned}$$

2. Show that $\mathcal{J}_{-1/2}(t) = \sqrt{2/\pi}\cos t$, and deduce that

$$J_{-1/2}(t) = \sqrt{\frac{2}{\pi t}}\cos t, \quad J_{1/2}(t) = \sqrt{\frac{2}{\pi t}}\sin t.$$

Deduce from Exercise 1 that, for $n \in \mathbb{Z}^+$,

$$\begin{aligned} J_{n+1/2}(t) &= (-1)^n \Big\{\prod_{j=1}^{n}\Big(\frac{d}{dt} - \frac{j-1/2}{t}\Big)\Big\}\frac{\sin t}{\sqrt{2\pi t}}, \\ J_{-n-1/2}(t) &= \Big\{\prod_{j=1}^{n}\Big(\frac{d}{dt} - \frac{j-1/2}{t}\Big)\Big\}\frac{\cos t}{\sqrt{2\pi t}}. \end{aligned}$$

Hint. The differential equation (1.16.5) for $\mathcal{J}_{-1/2}$ is $y'' + y = 0$. Since $\mathcal{J}_{-1/2}(t)$ is even in t, $\mathcal{J}_{-1/2}(t) = C\cos t$, and the evaluation of C comes from $\mathcal{J}_{-1/2}(0) = \sqrt{2}/\Gamma(1/2) = \sqrt{2/\pi}$, thanks to (1.B.6).

3. Show that the functions Y_ν satisfy the same recursion relations as J_ν, i.e.,

$$\frac{d}{dt}\Big(t^\nu Y_\nu(t)\Big) = t^\nu Y_{\nu-1}(t), \quad \frac{d}{dt}\Big(t^{-\nu} Y_\nu(t)\Big) = -t^{-\nu} Y_{\nu+1}(t).$$

4. The Hankel functions $H_\nu^{(1)}(t)$ and $H_\nu^{(2)}(t)$ are defined to be

$$H_\nu^{(1)}(t) = J_\nu(t) + iY_\nu(t), \quad H_\nu^{(2)}(t) = J_\nu(t) - iY_\nu(t).$$

Show that they satisfy the same recursion relations as J_ν, i.e.,

$$\frac{d}{dt}\Big(t^\nu H_\nu^{(j)}(t)\Big) = t^\nu H_{\nu-1}^{(j)}(t), \quad \frac{d}{dt}\Big(t^{-\nu} H_\nu^{(j)}(t)\Big) = -t^{-\nu} H_{\nu+1}^{(j)}(t),$$

for $j = 1, 2$.

5. Show that

$$H_{-\nu}^{(1)}(t) = e^{\pi i\nu} H_\nu^{(1)}(t), \quad H_{-\nu}^{(2)}(t) = e^{-\pi i\nu} H_\nu^{(2)}(t).$$

6. Show that $Y_{1/2}(t) = -J_{-1/2}(t)$, and deduce that

$$H^{(1)}_{1/2}(t) = -i\sqrt{\frac{2}{\pi t}}e^{it}, \quad H^{(2)}_{1/2}(t) = i\sqrt{\frac{2}{\pi t}}e^{-it}.$$

1.17. Higher order linear equations

A linear differential equation of order n has the form

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_0(t)x = f(t). \tag{1.17.1}$$

If $a_j(t)$ are continuous for t in an interval I containing t_0, and $a_n(t)$ is nonvanishing on this interval, one has a unique solution to (1.17.1) given an initial condition of the form

$$x(t_0) = \alpha_0,\ x'(t_0) = \alpha_1, \dots,\ x^{(n-1)}(t_0) = \alpha_{n-1}. \tag{1.17.2}$$

(As with (1.15.1)–(1.15.2), this also follows from a general result that will be established in Chapter 4.) If $a_j(t)$ are all constant, the equation (1.17.1) has the form

$$a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_0 x = f(t). \tag{1.17.3}$$

It is homogeneous if $f \equiv 0$, in which case one has

$$a_n\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_0 x = 0. \tag{1.17.4}$$

We assume $a_n \neq 0$.

Methods developed in §§1.9–1.10 have natural extensions to (1.17.4) and (1.17.3). The function $x(t) = e^{rt}$ solves (1.17.4) provided r satisfies the characteristic equation

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0. \tag{1.17.5}$$

The fundamental theorem of algebra guarantees that (1.17.5) has n roots, i.e., there exist $r_1, \dots, r_n \in \mathbb{C}$ such that

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_0 = a_n(r - r_1)\cdots(r - r_n). \tag{1.17.6}$$

A proof of this theorem is given in §2.C. These roots $r_1, \dots, r_n$ may or may not be distinct. If they are distinct, the general solution to (1.17.4) has the form

$$x(t) = C_1 e^{r_1 t} + \cdots + C_n e^{r_n t}. \tag{1.17.7}$$

If r_j is a root of multiplicity k, one has solutions to (1.17.4) of the form

$$C_1 e^{r_j t} + C_2 t e^{r_j t} + \cdots + C_k t^{k-1} e^{r_j t}. \tag{1.17.8}$$

This observation can be used to yield a fresh perspective on what makes the calculations in §1.10 work. Consider for example the equation

$$ax'' + bx' + cx = e^{\kappa t}. \tag{1.17.9}$$

The right side solves the equation $(d/dt - \kappa)e^{\kappa t} = 0$, so any solution to (1.17.9) also solves

$$\left(\frac{d}{dt} - \kappa\right)\left(a\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right)x = 0, \tag{1.17.10}$$

a homogeneous equation whose characteristic polynomial is

$$q(r) = (r - \kappa)(ar^2 + br + c) = (r - \kappa)p(r). \tag{1.17.11}$$

If κ is not a root of $p(r)$, then certainly (1.17.9) has a solution of the form $Ae^{\kappa t}$. If κ is a root of $p(r)$, then it is a double (or, perhaps, triple) root of $q(r)$, and (1.17.8) applies, leading one to (1.10.17) or (1.10.25).

One can also extend the method of variation of parameters to higher order equations (1.17.3), though the details get grim.

The equations (1.17.1)–(1.17.4) can each be recast as $n \times n$ first order systems of differential equations, and all the results on these equations are special cases of results to be covered in Chapter 3, so we will say no more here, except to advertise that this transformation leads to a much simplified approach to the method of variation of parameters.

Exercises

1. Assume the existence and uniqueness results for the solution to (1.17.1) stated in the first paragraph of this section. Show that there exist n solutions u_j to

$$a_n(t)u_j^{(n)}(t) + a_{n-1}(t)u^{(n-1)}(t) + \cdots + a_0(t)u_j(t) = 0$$

on I such that every solution to (1.17.1) with $f \equiv 0$ can be written uniquely in the form

$$x(t) = C_1 u_1(t) + \cdots + C_n u_n(t).$$

For general continuous f, let x_p be a particular solution to (1.17.1). Show that if $x(t)$ is an arbitrary solution to (1.17.1), then there exist unique constants $C_j,\ 1 \le j \le n$, such that

$$x(t) = x_p(t) + C_1 u_1(t) + \cdots + C_n u_n(t).$$

This is called the general solution to (1.17.1).
Hint. Require $u_j^{(k-1)}(t_0) = \delta_{jk},\ 1 \le k \le n$, where $\delta_{jk} = 1$ for $j = k,\ 0$ for $j \ne k$.

2. Find the general solution to each of the following equations for $x = x(t)$.

(a)

$$\frac{d^4x}{dt^4} - x = 0.$$

(b)

$$\frac{d^3x}{dt^3} - x = 0.$$

(c)

$$x''' - 2x'' - 4x' + 8x = 0.$$

(d)

$$x''' - 2x'' + 4x' - 8x = 0.$$

(e)

$$x''' + x = e^t.$$

3. For each of the cases (a)–(e) in Exercise 1 of §1.10, produce a third or fourth order homogeneous differential equation solved by $x(t)$.

Exercises 4–6 will exploit the fact that if the characteristic polynomial (1.17.6) factors as stated there, then the left side of (1.17.4) is equal to

$$a_n\Bigl(\frac{d}{dt} - r_1\Bigr)\cdots\Bigl(\frac{d}{dt} - r_n\Bigr)x = a_n\prod_{j=1}^{n}\Bigl(\frac{d}{dt} - r_j\Bigr)x.$$

4. Show that

$$\Bigl(\frac{d}{dt} - r_j\Bigr)(e^{rt}u) = e^{rt}\Bigl(\frac{d}{dt} - r_j + r\Bigr)u,$$

and more generally

$$\prod_{j=1}^{n}\Bigl(\frac{d}{dt} - r_j\Bigr)(e^{rt}u) = e^{rt}\prod_{j=1}^{n}\Bigl(\frac{d}{dt} - r_j + r\Bigr)u.$$

5. Suppose r_j is a root of multiplicity k of (1.17.6). Show that $x(t) = e^{r_j t}u$ solves (1.17.4) if and only if

$$\prod_{\{\ell : r_\ell \neq r_j\}}\Bigl(\frac{d}{dt} - r_\ell + r_j\Bigr)\Bigl(\frac{d}{dt}\Bigr)^k u = 0.$$

Use this to show that functions of the form (1.17.8) solve (1.17.4).

6. In light of Exercise 5, use an inductive argument to show the following. Assume the roots $\{r_j\}$ of (1.17.6) are

$$r_\nu, \quad \text{with multiplicity} \quad k_\nu, \quad 1 \le \nu \le m, \quad k_1 + \cdots + k_m = n.$$

Then the general solution to (1.17.4) is a linear combination of

$$t^{\ell_\nu} e^{r_\nu t}, \quad 0 \le \ell_\nu \le k_\nu - 1, \quad 1 \le \nu \le m.$$

1.18. The Laplace transform

The Laplace transform provides a tool to treat nonhomogeneous differential equations of the form

$$c_n\frac{d^n f}{dt^n} + c_{n-1}\frac{d^{n-1} f}{dt^{n-1}} + \cdots + c_0 f(t) = g(t), \tag{1.18.1}$$

for $t \ge 0$, with initial data

$$f(0) = a_0, \dots, f^{(n-1)}(0) = a_{n-1}, \tag{1.18.2}$$

for certain classes of functions g. It is defined as follows. Assume $f : \mathbb{R}^+ \to \mathbb{C}$ is integrable on $[0, R]$ for all $R < \infty$, and satisfies

$$\int_0^\infty |f(t)|e^{-at}\, dt < \infty, \quad \forall\, a > A, \tag{1.18.3}$$

for some $A \in \mathbb{R}$. We define the Laplace transform of f by

$$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}\, dt, \quad \operatorname{Re} s > A. \tag{1.18.4}$$

By our hypotheses, this integral is absolutely convergent for each s in the half-plane $H_A = \{s \in \mathbb{C} : \operatorname{Re} s > A\}$. For our current purposes, it will suffice to take s real, in (A, ∞). Note that, for such s,

$$\frac{d}{ds}\mathcal{L}f(s) = \mathcal{L}g(s), \quad g(t) = -tf(t). \tag{1.18.5}$$

If we assume that f' is continuous on $[0, \infty)$ and

$$|f(t)| + |f'(t)| \le C_\varepsilon e^{(A+\varepsilon)t}, \quad \text{for} \ \ t \ge 0, \tag{1.18.6}$$

for each $\varepsilon > 0$, we can integrate by parts and get

$$\mathcal{L}f'(s) = s\mathcal{L}f(s) - f(0), \tag{1.18.7}$$

and similar hypotheses for higher derivatives of f gives

$$\mathcal{L}f^{(k)}(s) = s^k\mathcal{L}f(s) - s^{k-1}f(0) - \cdots - f^{(k-1)}(0). \tag{1.18.8}$$

Hence, if f satisfies an ODE of the form (1.18.1)–(1.18.2) and if $f, f', \dots f^{(n-1)}$ all satisfy (1.18.6), and g satisfies (1.18.3), we have

$$p(s)\mathcal{L}f(s) = \mathcal{L}g(s) + q(s), \tag{1.18.9}$$

with

$$\begin{aligned} p(s) &= c_n s^n + c_{n-1}s^{n-1} + \cdots + c_0, \\ q(s) &= c_n(a_0 s^{n-1} + \cdots + a_{n-1}) + \cdots + c_1 a_0. \end{aligned} \tag{1.18.10}$$

If all the roots of $p(s)$ satisfy $\operatorname{Re} s < B$, we have

$$\mathcal{L}f(s) = \frac{\mathcal{L}g(s) + q(s)}{p(s)}, \quad \operatorname{Re} s > C = \max(A, B). \tag{1.18.11}$$

Making use of (1.18.11) to solve (1.18.1)–(1.18.2) brings in two problems, which we now state.

I. The recognition problem. Given the right side of (1.18.11), i.e., given

$$\frac{\mathcal{L}g(s) + q(s)}{p(s)} = R(s), \tag{1.18.12}$$

find a function $f_1 : [0, \infty) \to \mathbb{C}$, such that

$$\mathcal{L}f_1(s) = R(s), \quad \text{for} \ \ \operatorname{Re} s > C. \tag{1.18.13}$$

II. The uniqueness problem. Given f and $f_1 : [0, \infty) \to \mathbb{C}$, both satisfying (1.18.3), one wants to know that

$$\mathcal{L}f(s) = \mathcal{L}f_1(s),\ \forall\, s > A \Longrightarrow f = f_1 \text{ on } [0, \infty). \tag{1.18.14}$$

The uniqueness problem has a satisfactory solution. As long as f and f_1 satisfy the hypotheses just stated, the result (1.18.14) is true. The proof of this can be found in §3.3 of [**47**]. In addition there are inversion formulas. Here is one, established in §3.3 of [**47**].

Proposition 1.18.1. *Assume f and f' are continuous on $[0,\infty)$, and*

$$|f(t)| + |f'(t)| \le Ce^{At}, \quad t \ge 0. \tag{1.18.15}$$

Then, for $t > 0$,

$$tf(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d}{ds}\mathcal{L}f(B+i\xi)e^{t(B+i\xi)}\,d\xi, \tag{1.18.16}$$

as long as $B > A$, with an absolutely convergent integral on the right side.

In light of the uniqueness, if f satisfies (1.18.3), we say

$$g = \mathcal{L}f \Longrightarrow f = \mathcal{L}^{-1}g, \tag{1.18.17}$$

and call $\mathcal{L}^{-1}$ the inverse Laplace transform.

Generally speaking, for functions $R(s)$ that arise in (1.18.12), calculation of the integral

$$\int_{-\infty}^{\infty} R'(B+i\xi)e^{it\xi}\,d\xi \tag{1.18.18}$$

is not so easy, though methods of *residue calculus*, discussed in §4.1 of [**47**] can be effective. For the purpose of using (1.18.11) to solve (1.18.1)–(1.18.2), by finding f that satisfies

$$\mathcal{L}f(s) = R(s), \tag{1.18.19}$$

with $R(s)$ as in (1.18.12), it is useful to have a collection of functions that are known Laplace transforms, in order to solve the recognition problem.

To start our collection, we consider the Laplace transform of e^{at},

$$\int_0^{\infty} e^{at}e^{-st}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt = \frac{1}{s-a}. \tag{1.18.20}$$

If a is real, this is valid for $\operatorname{Re} s > a$. However, using results from §1.1, we find it useful to note that (1.18.20) holds for *complex* a, as long as $\operatorname{Re} s > \operatorname{Re} a$. We can apply this to

$$f(t) = \cos at = \frac{1}{2}(e^{iat} + e^{-iat}), \tag{1.18.21}$$

for $a \in \mathbb{R}$, to get

$$\begin{aligned} \mathcal{L}f(s) &= \frac{1}{2}\Big(\frac{1}{s-ia} + \frac{1}{s+ia}\Big) \\ &= \frac{s}{s^2+a^2}. \end{aligned} \tag{1.18.22}$$

Similar techniques yield the table of Laplace transforms presented in Table 1.

If $a \in \mathbb{R}$, the range of validity of (a)–(b) is $\operatorname{Re} s > 0$, and that of (c)–(d) is $\operatorname{Re} s > |a|$.

Table 1. Table of Laplace transforms

	$f(t)$	$\mathcal{L}f(s)$
(a)	$\sin at$	$a/(s^2+a^2)$
(b)	$\cos at$	$s/(s^2+a^2)$
(c)	$\sinh at$	$a/(s^2-a^2)$
(d)	$\cosh at$	$s/(s^2-a^2)$

Laplace transforms of other functions, such as $e^{-bt}\cos at$, etc., can be identified via the identity

$$\mathcal{L}(e^{-bt}f)(s) = \mathcal{L}f(s+b). \tag{1.18.23}$$

Also, one can turn (1.18.5) around, to write

$$\mathcal{L}(tf)(s) = -\frac{d}{ds}\mathcal{L}f(s), \tag{1.18.24}$$

and, inductively,

$$\mathcal{L}(t^n f)(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}f(s). \tag{1.18.25}$$

For example,

$$\begin{aligned}\mathcal{L}(t^n e^{at})(s) &= (-1)^n \frac{d^n}{ds^n}(s-a)^{-1} \\ &= \frac{n!}{(s-a)^{n+1}},\end{aligned} \tag{1.18.26}$$

for $a \in \mathbb{C}$, $\operatorname{Re} s > \operatorname{Re} a$. In particular,

$$f(t) = t^n \Longrightarrow \mathcal{L}f(s) = n!\, s^{-(n+1)}. \tag{1.18.27}$$

Of course, by (1.18.23), the result (1.18.26) follows from its special case (1.18.27). A natural generalization of (1.18.16) arises from taking

$$f_z(t) = t^{z-1}, \quad z > 0. \tag{1.18.28}$$

We get

$$\begin{aligned}\mathcal{L}f_z(s) &= \int_0^\infty e^{-st} t^{z-1}\, dt \\ &= \Bigl(\int_0^\infty e^{-t} t^{z-1}\, dt\Bigr) s^{-z} \\ &= \Gamma(z) s^{-z},\end{aligned} \tag{1.18.29}$$

where

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad z > 1, \tag{1.18.30}$$

is the Gamma function, which plays a role in §1.16, via (1.16.13)–(1.16.16), and is treated in Appendix 1.B. Let us note that (1.18.24) implies

$$\mathcal{L}f_{z+1}(s) = -\frac{d}{ds}\mathcal{L}f_z(s), \tag{1.18.31}$$

Table 2. Further Laplace transforms

	$f(t)$	$\mathcal{L}f(s)$
(e)	t^{z-1}	$\Gamma(z)s^{-z}$
(f)	$\log t$	$-(\gamma+\log s)/s$
(g)	$(\log t)t^{z-1}$	$(\Gamma'(z)-\Gamma(z)\log s)/s^z$
(h)	$t^{z-1}e^{at}$	$\Gamma(z)(s-a)^{-z}$

which in view of (1.18.29) is equivalent to the identity

$$\Gamma(z+1)=z\Gamma(z). \tag{1.18.32}$$

Also comparison of (1.18.27) and (1.18.29), with $z=n+1$, yields

$$\Gamma(n+1)=n! \tag{1.18.33}$$

We can obtain another Laplace transform identity by applying d/dz to (18.28), noting that, since $s^{-z}=e^{-z\log s}$,

$$\frac{d}{dz}s^{-z}=-(\log s)s^{-z}, \quad s>0, \tag{1.18.34}$$

with an analogous formula for $(d/dz)t^{z-1}$:

$$\frac{d}{dz}t^{z-1}=(\log t)t^{z-1}. \tag{1.18.35}$$

Hence (1.18.29) yields

$$f(t)=(\log t)t^{z-1}\Rightarrow \mathcal{L}f(s)=\big(\Gamma'(z)-\Gamma(z)\log s\big)s^{-z}. \tag{1.18.36}$$

In particular,

$$\begin{aligned} f(t)=\log t\Rightarrow \mathcal{L}f(s)&=(\Gamma'(1)-\log s)s^{-1}\\ &=-\frac{\gamma+\log s}{s}, \end{aligned} \tag{1.18.37}$$

where $\gamma=-\Gamma'(1)$ is known as Euler's constant. Taking $s=1$ in (1.18.37), we have the formula

$$\gamma=-\int_0^\infty(\log t)e^{-t}\,dt. \tag{1.18.38}$$

Collecting these results, we complement the table of Laplace transforms compiled in Table 1 with that in Table 2. Note that (h) follows from (e), via (1.18.23). One has similar variants of (f)–(g).

Another function to consider is the *impulse function*

$$\begin{aligned} \chi_I(t)&=1, \quad \text{if } \ t\in I,\\ &\ 0, \quad \text{if } \ t\notin I. \end{aligned} \tag{1.18.39}$$

where $I=[a,b]$ is an interval, with $0\le a<b<\infty$. We have

$$\mathcal{L}\chi_I(s)=\int_a^b e^{-st}\,dt=\frac{e^{-as}-e^{-bs}}{s}. \tag{1.18.40}$$

Let us apply the Laplace transform method to the following initial value problem. Take $k,a,\alpha_0,\alpha_1\in\mathbb{R}$, and consider

$$f''(t)+k^2f(t)=\cos at, \quad f(0)=\alpha_0,\ f'(0)=\alpha_1. \tag{1.18.41}$$

From (1.18.8),

$$\mathcal{L}f''(s) = s^2\mathcal{L}f(s) - \alpha_0 s - \alpha_1, \tag{1.18.42}$$

and since $\mathcal{L}(\cos at)(s) = s/(s^2 + a^2)$, (1.18.11) becomes

$$\mathcal{L}f(s) = \frac{s}{(s^2 + k^2)(s^2 + a^2)} + \frac{\alpha_0 s + \alpha_1}{s^2 + k^2}. \tag{1.18.43}$$

The last term on the right is the Laplace transform of

$$\alpha_0 \cos kt + \frac{\alpha_1}{k}\sin kt. \tag{1.18.44}$$

It remains to write the first term on the right side of (1.18.43) as a Laplace transform. For this, we apply the method of partial fractions. To start, we try

$$\frac{s}{(s^2 + k^2)(s^2 + a^2)} = \frac{\alpha s + \beta}{s^2 + a^2} + \frac{\gamma s + \delta}{s^2 + k^2}, \tag{1.18.45}$$

with unknowns $\alpha, \beta, \gamma, \delta$. Multiplying through by $(s^2 + k^2)(s^2 + a^2)$ and equating coefficients of various powers of s leads to four linear equations in these four unknowns. Two of them yield $\alpha = -\gamma$ and $\beta = -\delta$, and then the other two become

$$(k^2 - a^2)\alpha = 1, \quad (k^2 - a^2)\beta = 0. \tag{1.18.46}$$

If $k^2 \neq a^2$, these are uniquely solvable, for $\alpha = (k^2 - a^2)^{-1}$, $\beta = 0$, and (1.18.49) becomes

$$\frac{s}{(s^2 + k^2)(s^2 + a^2)} = \frac{1}{k^2 - a^2}\Big(\frac{s}{s^2 + a^2} - \frac{s}{s^2 + k^2}\Big). \tag{1.18.47}$$

This is the Laplace transform of

$$\varphi_{a,k}(t) = \frac{1}{k^2 - a^2}(\cos at - \cos kt). \tag{1.18.48}$$

Then the solution to the differential equation (1.18.41) is

$$f(t) = \varphi_{a,k}(t) + \alpha_0 \cos kt + \frac{\alpha_1}{k}\sin kt. \tag{1.18.49}$$

This approach fails for $k^2 = a^2$, paralleling the situation we encountered in examining (1.11.4). One way to treat this exceptional case is to pass to the limit in (1.18.48), obtaining

$$\begin{aligned} \varphi_{k,k}(t) &= \lim_{a\to k} \varphi_{a,k}(t) \\ &= \lim_{a\to k} \frac{1}{k+a}\,\frac{\cos at - \cos kt}{k - a} \\ &= \frac{t}{2k}\sin kt. \end{aligned} \tag{1.18.50}$$

Another approach is to refine the method of partial fractions. In lieu of (1.18.45), we have

$$\begin{aligned} \frac{s}{(s^2 + k^2)^2} &= \frac{s}{(s + ik)^2(s - ik)^2} \\ &= \frac{i}{4k}\Big(\frac{1}{(s + ik)^2} - \frac{1}{(s - ik)^2}\Big). \end{aligned} \tag{1.18.51}$$

Using (1.18.26), with $n = 1$, we have

$$\mathcal{L}^{-1}\Big(\frac{1}{(s \pm ik)^2}\Big)(t) = te^{\mp ikt}. \tag{1.18.52}$$

Hence the right side of (1.18.51) is the Laplace transform of

$$\frac{i}{4k}(te^{-ikt} - te^{ikt}) = \frac{t}{2k}\sin kt, \tag{1.18.53}$$

and again we obtain the conclusion of (1.18.50), from a different perspective.

In light of this analysis, and recalling (1.18.12), we are motivated to compute the inverse Laplace transform of functions of the form $q(s)/p(s)$, where $p(s)$ is a polynomial of degree n, say

$$p(s) = s^n + c_{n-1}s^{n-1} + \cdots + c_0, \tag{1.18.54}$$

and $q(s)$ is a polynomial of degree $\le n-1$. The polynomial $p(s)$ has complex roots $r_1, \dots, r_m$, of multiplicity $k_1, \dots, k_m$, and we can write (1.18.54) as

$$p(s) = (s - r_1)^{k_1} \cdots (s - r_m)^{k_m}, \quad k_1 + \cdots + k_m = n. \tag{1.18.55}$$

This is a consequence of the fundamental theorem of algebra, which is proved in Appendix 2.C. The following is an incisive result on the method of partial fractions.

Proposition 1.18.2. *If $p(s)$ is a polynomial of the form* (1.18.55)*, with $\{r_1, \dots, r_m\}$ distinct, and if $q(s)$ is a polynomial of degree $\le n - 1$, then there exist unique $a_{j\ell} \in \mathbb{C}$, for $1 \le \ell \le m,\ 1 \le j \le k_\ell$, such that*

$$\frac{q(s)}{p(s)} = \sum_{\ell=1}^{m}\sum_{j=1}^{k_\ell} \frac{a_{j\ell}}{(s - r_\ell)^j}. \tag{1.18.56}$$

Proof. We use some concepts developed in Chapter 2. The set of collections $(a_{j\ell})$ of the form

$$\{a_{j\ell} \in \mathbb{C} : 1 \le j \le k_\ell, 1 \le \ell \le m\}$$

forms a vector space V_0, of dimension $k_1 + \cdots + k_m = n$. Meanwhile, the space $\mathcal{P}_{n-1}$ of polynomials $q(s)$ of degree $\le n - 1$ is also a vector space of dimension n. Now the correspondence in (1.18.56) yields a well defined linear map T from V_0 to $\mathcal{P}_{n-1}$, given by $T(a_{j\ell}) = q(s)$, the numerator in the left side of (1.18.56), and one can verify that this map is one-to-one. Hence (cf. Corollary 2.3.7 of Chapter 2), this map is also onto, and this gives Proposition 1.18.2. □

Given the representation (1.18.56), we deduce from (1.18.26) that

$$\mathcal{L}^{-1}\Big(\frac{q}{p}\Big)(t) = \sum_{\ell=1}^{m}\sum_{j=1}^{k_\ell} \frac{a_{j\ell}}{(j-1)!} t^{j-1} e^{r_\ell t}. \tag{1.18.57}$$

Taking $q(s) = 1$, we obtain a function $\varphi(t)$, of the form (1.18.57), such that

$$\mathcal{L}\varphi(s) = \frac{1}{p(s)}. \tag{1.18.58}$$

Then the solution $f(t)$ to (1.18.1)–(1.18.2) is equal to $\mathcal{L}^{-1}(q/p)(t)$ plus $f_0(t)$, satisfying

$$\mathcal{L}f_0(s) = \frac{\mathcal{L}g(s)}{p(s)} = \mathcal{L}\varphi(s)\mathcal{L}g(s). \tag{1.18.59}$$

The following result provides a useful integral formula for f_0.

Proposition 1.18.3. *Let φ and g satisfy* (1.18.3)*, and set*

$$\varphi * g(t) = \int_0^t \varphi(t-\tau)g(\tau)\,d\tau. \tag{1.18.60}$$

Then, for $s > A$,

$$\mathcal{L}(\varphi * g)(s) = \mathcal{L}\varphi(s)\mathcal{L}g(s). \tag{1.18.61}$$

Proof. Given (1.18.60), we have

$$\begin{aligned}
\mathcal{L}(\varphi * g)(s) &= \int_0^\infty e^{-st}\int_0^t \varphi(t-\tau)g(\tau)\,d\tau\,dt \\
&= \int_0^\infty \int_0^t e^{-s(t-\tau)}e^{-s\tau}\varphi(t-\tau)g(\tau)\,d\tau\,dt \\
&= \int_0^\infty \int_\tau^\infty e^{-s(t-\tau)}e^{-s\tau}\varphi(t-\tau)g(\tau)\,dt\,d\tau \\
&= \mathcal{L}\varphi(s)\int_0^\infty e^{-s\tau}g(\tau)\,d\tau \\
&= \mathcal{L}\varphi(s)\mathcal{L}g(s),
\end{aligned} \tag{1.18.62}$$

as asserted. □

Recall that the method of variation of parameters, discussed in §1.14, also leads to an integral formula involving an integral over $[0,t]$. In fact, the method of variation of parameters and the use of the Laplace transform discussed here can both be understood as special cases of a general method, involving *Duhamel's formula*, arising when the equations are recast as first-order systems. This is explained in §3.9 and §3.B.

Exercises

1. Compute the inverse Laplace transform of the following functions.

$$(a) \quad \frac{1}{s^4-1},$$

$$(b) \quad \frac{s+1}{s^3+3s^2+2s}.$$

2. Use the Laplace transform to solve the following initial value problems.

$$(a)\quad f''(t) + 3f'(t) + 2f(t) = e^{-t}\sin t, \quad f(0) = 0,\ f'(0) = 1,$$
$$(b)\quad f^{(4)}(t) - f(t) = \sin t, \quad f^{(j)}(0) = 0 \text{ for } 0 \le j \le 3.$$

3. Show that

$$f(t) = \frac{\sin t}{t} \Longrightarrow \mathcal{L}f(s) = \frac{\pi}{2} - \tan^{-1} s.$$

Hint. By (1.18.5),

$$\frac{d}{ds}\mathcal{L}f(s) = -\mathcal{L}(tf)(s) = -\frac{1}{s^2+1}.$$

Integrate, and find the constant of integration using

$$\lim_{s\to\infty} \mathcal{L}f(s) = 0.$$

4. Compute the Laplace transform of

$$\frac{1-\cos t}{t^2}.$$

1.A. The genesis of Bessel's equation: PDE in polar coordinates

Bessel functions, the subject of §1.16, arise in the natural generalization of the equation

$$\frac{d^2u}{dx^2} + k^2 u = 0, \tag{1.A.1}$$

with solutions $\sin kx$ and $\cos kx$, to partial differential equations

$$\Delta u + k^2 u = 0, \tag{1.A.2}$$

where Δ is the Laplace operator, acting on a function u on a domain $\Omega \subset \mathbb{R}^n$ by

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{1.A.3}$$

We can eliminate k^2 from (1.A.2) by scaling. Set $u(x) = v(kx)$. Then equation (1.A.2) becomes

$$(\Delta + 1)v = 0. \tag{1.A.4}$$

We specialize to the case $n = 2$ and write

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{1.A.5}$$

For a number of special domains $\Omega \subset \mathbb{R}^2$, such as circular domains, annular domains, angular sectors, and pie-shaped domains, it is convenient to switch to polar coordinates (r, θ), related to (x, y)-coordinates by

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{1.A.6}$$

In such coordinates,

$$\Delta v = \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Big)v. \tag{1.A.7}$$

A special class of solutions to (1.A.4) has the form

$$v = w(r)e^{i\nu\theta}. \tag{1.A.8}$$

By (1.A.7), for such v,

$$(\Delta + 1)v = \Big[\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w\Big]e^{i\nu\theta}, \tag{1.A.9}$$

so (1.A.4) holds if and only if

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w = 0. \tag{1.A.10}$$

This is Bessel's equation (1.16.1) (with different variables).

Note that if v solves (1.A.4) on $\Omega \subset \mathbb{R}^2$ and if Ω is a circular domain or an annular domain, centered at the origin, then ν must be an integer. However, if Ω is an angular sector or a pie-shaped domain, with vertex at the origin, ν need not be an integer.

In n dimensions, the Laplace operator (1.A.3) can be written

$$\Delta v = \Big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S\Big)v, \tag{1.A.11}$$

where Δ_S is a second order differential operator acting on functions on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, called the Laplace-Beltrami operator. Generalizing (1.A.8), one looks for solutions to (1.A.4) of the form

$$v(x) = w(r)\psi(\omega), \tag{1.A.12}$$

where $x = r\omega,\ r \in (0,\infty),\ \omega \in S^{n-1}$. Parallel to (1.A.9), for such v,

$$(\Delta + 1)v = \Big[\frac{d^2w}{dr^2} + \frac{n-1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w\Big]\psi(\omega), \tag{1.A.13}$$

provided

$$\Delta_S\psi = -\nu^2\psi. \tag{1.A.14}$$

The equation

$$\frac{d^2w}{dr^2} + \frac{n-1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w = 0 \tag{1.A.15}$$

is a variant of Bessel's equation. If we set

$$\varphi(r) = r^{n/2-1}w(r), \tag{1.A.16}$$

then (1.A.15) is converted into the Bessel equation

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + \Big(1 - \frac{\mu^2}{r^2}\Big)\varphi = 0, \quad \mu^2 = \nu^2 + \Big(\frac{n-2}{2}\Big)^2. \tag{1.A.17}$$

The study of solutions to (1.A.14) gives rise to the study of spherical harmonics, and from there to other special functions, such as Legendre functions.

The search for solutions of the form (1.A.12) is a key example of the method of separation of variables for partial differential equations. It arises in numerous other contexts. Here are a couple of other examples:

$$(\Delta - |x|^2 + k^2)u = 0, \tag{1.A.18}$$

and

$$\Big(\Delta + \frac{K}{|x|} + k^2\Big)u = 0. \tag{1.A.19}$$

The first describes the n-dimensional quantum harmonic oscillator. The second (for $n = 3$) describes the quantum mechanical model of a hydrogen atom, according to Schrödinger. Study of these equations leads to other special functions defined by differential equations, such as Hermite functions and Whittaker functions.

Much further material on these topics can be found in books on partial differential equations, such as [**45**] (particularly Chapters 3 and 8).

1.B. Euler's gamma function

We saw in (1.16.13) the need for a function $\Gamma(z)$ satisfying

$$\Gamma(z+1) = z\Gamma(z). \tag{1.B.1}$$

Here we produce a function that has this property, namely

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \text{for} \;\; z > 0. \tag{1.B.2}$$

To check (1.B.1) for $z > 0$, we apply integration by parts.

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty e^{-t} t^z \, dt \\ &= -\int_0^\infty \Big(\frac{d}{dt} e^{-t}\Big) t^z \, dt \\ &= \int_0^\infty e^{-t} \Big(\frac{d}{dt} t^z\Big)\, dt \\ &= z\Gamma(z), \end{aligned} \tag{1.B.3}$$

since $dt^z/dt = zt^{z-1}$.

The integral (1.B.2) is readily evaluated for $z = 1$, yielding

$$\Gamma(1) = 1. \tag{1.B.4}$$

Then repeated use of (1.B.3) gives

$$\Gamma(k+1) = k!, \quad \text{for} \;\; k \in \mathbb{Z}^+. \tag{1.B.5}$$

There is also a useful formula for $\Gamma(1/2)$, given by

$$\begin{aligned} \Gamma\Big(\frac{1}{2}\Big) &= \int_0^\infty e^{-t} t^{-1/2}\, dt \\ &= 2\int_0^\infty e^{-x^2}\, dx \\ &= \sqrt{\pi}, \end{aligned} \tag{1.B.6}$$

the last identity by (1.2.26). Then repeated use of (1.B.3) gives

$$\begin{aligned} \Gamma\Big(k + \frac{1}{2}\Big) &= \frac{2k-1}{2}\,\frac{2k-3}{2} \cdots \frac{1}{2}\,\Gamma\Big(\frac{1}{2}\Big) \\ &= 2^{-2k}\,\frac{(2k)!}{k!}\,\sqrt{\pi}. \end{aligned} \tag{1.B.7}$$

Having (1.B.1), we can extend $\Gamma(z)$ to be well defined and smooth on the larger set $\mathbb{R} \setminus \{0, -1, -2, \dots\}$. To see this, rewrite (1.B.1) as

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1). \tag{1.B.8}$$

Having $\Gamma(z)$ defined and smooth on $z \in (0, \infty)$, by (1.B.2), we see that the right side of (1.B.8) is defined and smooth for $z \in (-1, \infty)$, except for a pole at $z = 0$. This extends $\Gamma(z)$ to $z \in (-1, \infty) \setminus \{0\}$. Then the right side of (1.B.8) is defined and smooth for $z \in (-2, \infty)$, except for poles at $z = 0$ and $z = -1$. This argument can be continued. Let us further note that, by (1.B.2),

$$\Gamma(z) > 0 \quad \text{for} \quad z > 0, \tag{1.B.9}$$

so $1/\Gamma(z)$ is defined and smooth for $z \in (0, \infty)$. Rewriting (1.B.8) as

$$\frac{1}{\Gamma(z)} = \frac{z}{\Gamma(z+1)} \tag{1.B.10}$$

and arguing as above, we have $1/\Gamma(z)$ defined and smooth for all $z \in \mathbb{R}$, vanishing precisely for $z \in \{0, -1, -2, \dots\}$.

We derive another identity that is useful for the treatment of Bessel functions in §1.16, involving the beta function $B(x, y)$, defined for $x, y > 0$ by

$$\begin{aligned} B(x,y) &= \int_0^1 s^{x-1}(1-s)^{y-1}\, ds \\ &= \int_0^\infty (1+u)^{-x-y} u^{x-1}\, du, \end{aligned} \tag{1.B.11}$$

the latter identity via the change of variable $u = s/(1-s)$. Our asserted identity is

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.B.12}$$

To prove this, note that since

$$\Gamma(z)p^{-z} = \int_0^\infty e^{-pt} t^{z-1}\, dt, \tag{1.B.13}$$

we have

$$(1+u)^{-x-y} = \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-(1+u)t} t^{x+y-1}\, dt, \tag{1.B.14}$$

so

$$\begin{aligned} B(x,y) &= \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-t} t^{x+y-1} \int_0^\infty e^{-ut} u^{x-1}\, du\, dt \\ &= \frac{\Gamma(x)}{\Gamma(x+y)} \int_0^\infty e^{-t} t^{y-1}\, dt \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \end{aligned} \tag{1.B.15}$$

as asserted.

For closer contact with (1.16.38), note that setting $s = t^2$ in (1.B.11) gives

$$B(x,y) = 2\int_0^1 t^{2x-1}(1-t^2)^{y-1}\, dt, \tag{1.B.16}$$

so, if $k \in \mathbb{Z}^+$ and $\nu > -1/2$,

$$B\Big(k+\frac{1}{2}, \nu+\frac{1}{2}\Big) = \int_{-1}^{1} t^{2k}(1-t^2)^{\nu-1/2}\, dt. \tag{1.B.17}$$

There is much more that can be said about the gamma function, such as that it extends to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$, with $1/\Gamma(z)$ defined and smooth for all $z \in \mathbb{C}$ (which permits one to define $J_\nu(z)$ for complex ν). We refer the reader to [**28**], §4.3 of [**47**], or Chapter 3, Appendix A of [**45**], for further material. We mention the following identity, of use in (1.16.33), whose proof can be found in these references:

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu}. \tag{1.B.18}$$

Note that both sides are defined and smooth for $\nu \in \mathbb{R} \setminus \mathbb{Z}$, with singularities on $\mathbb{Z}$.

1.C. Differentiating power series

Here we establish continuity and differentiability properties for a power series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k. \tag{1.C.1}$$

We allow the coefficients a_k to be complex numbers. To start, we assume this series converges for some nonzero $t = t_0$. This implies that the terms in this series are uniformly bounded for $t = t_0$,

$$|a_k t_0^k| \le B < \infty, \quad \forall k. \tag{1.C.2}$$

The following result establishes convergence for all smaller $|t|$.

Proposition 1.C.1. *Given* (1.C.2), *the series* (1.C.1) *converges absolutely for* $|t| < T = |t_0|$.

Proof. Pick $S \in (0, T)$, and assume $|t| \le S$. Then

$$|a_k t^k| \le |a_k T^k| \Big(\frac{S}{T}\Big)^k \le B r^k, \tag{1.C.3}$$

where $r = S/T \in (0, 1)$. Hence, for each $n \in \mathbb{N}$, if $|t| \le S$,

$$\sum_{k=0}^{n} |a_k t^k| \le B \sum_{k=0}^{n} r^k. \tag{1.C.4}$$

Now we can evaluate the geometrical series on the right,

$$\begin{aligned} S_n = \sum_{k=0}^{n} r^k &\Rightarrow r S_n = \sum_{k=1}^{n+1} r^k \\ &\Rightarrow (1-r) S_n = 1 - r^{n+1} \\ &\Rightarrow S_n = \frac{1-r^{n+1}}{1-r}. \end{aligned} \tag{1.C.5}$$

Consequently,

$$\begin{aligned} 0 < r < 1 &\Rightarrow r^{n+1} \searrow 0 \ \text{ as } \ n \to \infty \\ &\Rightarrow S_n \nearrow \frac{1}{1-r} \ \text{ as } \ n \to \infty. \end{aligned} \tag{1.C.6}$$

This establishes the asserted absolute convergence. □

Similar arguments also lead to the following.

Proposition 1.C.2. *In the setting of Proposition 1.C.1, if* $0 < S < T$*, the series* (1.C.1) *converges uniformly on* $|t| \le S$.

Proof. For each $n \in \mathbb{N}$, write

$$\begin{aligned} f(t) &= \sum_{k=0}^{n} a_k t^k + \sum_{k=n+1}^{\infty} a_k t^k \\ &= S_n(t) + R_n(t). \end{aligned} \tag{1.C.7}$$

The claim is that $S_n(t) \to f(t)$, uniformly on $|t| \le S$. Indeed, for $|t| \le S$,

$$\begin{aligned} |R_n(t)| &\le \sum_{k=n+1}^{\infty} |a_k t^k| \\ &\le B \sum_{k=n+1}^{\infty} r * k \\ &= B r^{n+1} \sum_{\ell=0}^{\infty} r^\ell \\ &= B \frac{r^{n+1}}{1-r}, \end{aligned} \tag{1.C.8}$$

yielding $|R_n(t)| \to 0$ uniformly for $|t| \le S$. □

Before continuiug our study of the power series (1.C.1), we pause to note that calculations above involving the geometric series (1.C.8) enable us to establish the following result, known as the ratio test.

Proposition 1.C.3. *Let* $a_k \in \mathbb{C}$ *and assume there exist* $N < \infty$ *and* $r < 1$ *such that*

$$k \ge N \Longrightarrow \left| \frac{a_{k+1}}{a_k} \right| \le r. \tag{1.C.9}$$

Then the series $\sum_{k \ge 0} a_k$ *is absolutely convergent.*

Proof. From (1.C.9) we have, by induction,

$$|a_{N+\ell}| \le r^\ell |a_N|. \tag{1.C.10}$$

Hence

$$\begin{aligned} \sum_{\ell=0}^{\infty} |a_{N+\ell}| &\le |a_N| \sum_{\ell=0}^{\infty} r^\ell \\ &= \frac{|a_N|}{1-r}. \end{aligned} \tag{1.C.11}$$

This yields absolute convergence. □

We now state the main result of this appendix.

Proposition 1.C.4. *If the power series* (1.C.1) *converges for* $|t| < R$*, then* f *is differentiable in* $t \in (-R, R)$*, and, for such* t*,*

$$f'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}. \tag{1.C.12}$$

Proof. It suffices to show that (1.C.12) holds for $|t| \le S$, for each $S < R$. Pick $T \in (S, R)$, and note that the estimate (1.C.3) holds, when $|t| \le S$, with $r = S/T < 1$. Hence, for $|t| \le S$,

$$\begin{aligned} |k a_k t^{k-1}| &\le \frac{k}{T} |a_k T^k| \Big(\frac{S}{T}\Big)^{k-1} \\ &\le \frac{B}{T} k r^{k-1}. \end{aligned} \tag{1.C.13}$$

Now the ratio test applies to $\sum_{k\ge 1} k r^{k-1}$, given $r < 1$, so the series

$$g(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} \tag{1.C.14}$$

is absolutely convergent, and also uniformly convergent, for $|t| \le S$. It remains to show that $g(t) = f'(t)$ for $|t| \le S$, or equivalently that

$$\int_0^t g(s)\, ds = f(t) - f(0). \tag{1.C.15}$$

This is a consequence of the following result. □

Proposition 1.C.5. *Given* $b_k \in \mathbb{C}$*, assume*

$$g(t) = \sum_{k=0}^{\infty} b_k t^k \tag{1.C.16}$$

is absolutely convergent, for $|t| < R$*. Then, for* $|t| < R$*,*

$$\int_0^t g(s)\, ds = \sum_{k=0}^{\infty} \frac{b_k}{k+1} t^{k+1}. \tag{1.C.17}$$

Proof. It is elementary that the series on the right side of (1.C.17) converges for $|t| < R$. Call the sum $F(t)$. As before, pick $S < T < R$. For $n \in \mathbb{N}$, write

$$\begin{aligned} g(t) &= \sum_{k=0}^{n} b_k t^k + \sum_{k=n+1}^{\infty} b_k t^k \\ &= g_n(t) + R_n(t). \end{aligned} \tag{1.C.18}$$

As in Proposition 1.C.2, we have $g_n(t) \to g(t)$ and $R_n(t) \to 0$, uniformly for $|t| \le S$, especially

$$\max_{|t|\le S} |R_n(t)| \le \varepsilon_n \to 0. \tag{1.C.19}$$

Clearly, for $|t| < R$,

$$\int_0^t g_n(s)\, ds = \sum_{k=0}^{n} \frac{b_k}{k+1} t^{k+1} \to F(t), \tag{1.C.20}$$

as $n \to \infty$. Meanwhile,

$$\Big| \int_0^t R_n(s)\, ds \Big| \le R\varepsilon_n. \tag{1.C.21}$$

Taking $n \to \infty$ in (1.C.18)–(1.C.21) yields

$$\int_0^t g(s)\, ds = F(t), \tag{1.C.22}$$

as asserted. This proves Proposition 1.C.5, so we have Proposition 1.C.4. □

Having (1.C.12), we can iterate, computing the derivative of $f'(t)$, as

$$f''(t) = \sum_{k=2}^{\infty} k(k-1)a_k\, t^{k-2}, \tag{1.C.23}$$

and so on,

$$f^{(n)}(t) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)a_k\, t^{k-n}. \tag{1.C.24}$$

In particular,

$$f^{(n)}(0) = n!\, a_n, \quad \text{hence} \quad a_n = \frac{f^{(n)}(0)}{n!}. \tag{1.C.25}$$

We have the following.

Proposition 1.C.6. *If $f(t)$ is given by a convergent power series* (1.C.1) *for $|t| < T$, $T > 0$, then*

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k. \tag{1.C.26}$$

Frequently, one can turn this around, take a function $f : (-T, T) \to \mathbb{R}$, compute $f^{(k)}(0)$, and investigate whether (1.C.26) holds. Here is an important class of functions for which this works. Take $r \in \mathbb{R}$, and set

$$f(t) = (1-t)^{-r}. \tag{1.C.27}$$

We have

$$\begin{aligned} f'(t) &= r(1-t)^{-r-1}, \\ f''(t) &= r(r+1)(1-t)^{-r-2}, \\ &\;\vdots \\ f^{(n)}(t) &= r(r+1)\cdots(r+n-1)\,(1-t)^{-r-n}, \end{aligned} \tag{1.C.28}$$

hence

$$f^{(n)}(0) = r(r+1)\cdots(r+n-1). \tag{1.C.29}$$

Claim. For $r \in \mathbb{R}$, we have

$$(1-t)^{-r} = \sum_{k=0}^{\infty} \frac{r(r+1)\cdots(r+k-1)}{k!}\, t^k, \quad \text{for} \quad |t| < 1. \tag{1.C.30}$$

In other words, (1.C.1) holds with

$$a_k = \frac{r(r+1)\cdots(r+k-1)}{k!}. \tag{1.C.31}$$

Note that

$$a_{k+1} = \frac{r+k}{k}\, a_k, \tag{1.C.32}$$

so the ratio test implies that the right side of (1.C.30) is absolutely convergent for $|t| < 1$, i.e., we have a well defined continuous (and differentiable) function

$$g(t) = \sum_{k=0}^{\infty} \frac{r(r+1)\cdots(r+k-1)}{k!}\, t^k. \tag{1.C.33}$$

Our claim is therefore that

$$g(t) = (1-t)^{-r}. \tag{1.C.34}$$

One approach to this is to estimate the remainder $R_n(t)$ in the expansion

$$f(t) = \sum_{k=0}^{n} \frac{f^{(n)}(0)}{k!}\, t^k + R_n(t). \tag{1.C.35}$$

A discussion of this appears in §4.3 of [**50**]. Here is another approach. We can apply Proposition 1.C.4 to $g(t)$ to obtain

$$(1-t)g'(t) = rg(t), \tag{1.C.36}$$

and then calculate

$$\begin{aligned} \frac{d}{dt}(1-t)^r g(t) &= (1-t)^r g'(t) - r(1-t)^{r-1} g(t) \\ &= (1-t)^{r-1}\Big\{(1-t)g'(t) - rg(t)\Big\} \\ &= 0, \end{aligned} \tag{1.C.37}$$

and deduce (1.C.34), hence (1.C.30).

For an application of (1.C.30), with $r = 1/2$, see (1.6.60).

Remark. Note the parallel between the use of (1.C.37) to prove (1.C.30) and the use of (1.1.10) to prove (1.1.13).