# Preface

*There is a tendency in each profession to veil the mysteries in a language inaccessible to outsiders. Let us oppose this. Probability theory is used by many who are not mathematicians, and it is essential to develop it in the plainest and most understandable way possible.* [1]
– Edward Nelson (1932–2014)

The development of stochastic calculus, and in particular of Itô calculus based on Brownian motion, is one of the great achievements of twentieth-century probability. This book was developed primarily to open the field to undergraduate students. The idea for such a book came when I was assigned to teach the undergraduate course *Stochastic Calculus for Finance* in 2016 at Baruch College, City University of New York. Some of the material was also developed while teaching the course *Probability and Stochastic Processes in Finance* for the Master's in Financial Engineering program at Baruch College in 2016 and in 2019. The end product is a mathematical textbook for readers who are interested in taking a first step towards advanced probability. Specifically, it is targeted at:

(A) **Advanced undergraduate students in mathematics.** As such, the book assumes a good background in multivariable calculus (Calculus 3), in linear algebra (a first course in undergraduate linear algebra in $\mathbb{R}^d$ with some exposure to vector spaces is sufficient), and probability (e.g., a first introductory course in probability covering joint distributions). Prior knowledge of basic stochastic processes or advanced calculus (i.e., mathematical analysis) can be helpful but is not assumed.

(B) **Master's students in financial engineering.** The book may serve as a first introduction to stochastic calculus with applications to option pricing in mind. Applications are given in the last chapter. However, as detailed below, they can be introduced in parallel with the other chapters. It can also be used as a self-study guide for the basics of stochastic calculus in finance.

[1] From the preface to the Russian edition of *Radically Elementary Probability* [**Nel95**, **Nel87**, **KK15**].

Nelson's quote above sums it all: The challenge in presenting stochastic calculus at the undergraduate level is to avoid measure theory concepts (whenever possible) without reducing the subject to a litany of recipes. With this in mind, the book approaches the subject from two directions. First and foremost, there is a strong emphasis on

**NUMERICS!**

Today's undergraduate math students might not be versed in analysis, but they are usually acquainted with basic coding. This is a powerful tool to introduce the students to stochastic calculus. Every chapter contains numerical projects that bring the theoretical concepts of the chapter to life. It is strongly suggested that the students write the codes in Python in a Jupyter notebook (`https://jupyter.org`). The coding is elementary and the necessary Python knowledge can be picked up on the way. All figures in the book were generated this way. The minimal time invested to learn the necessary basic coding will open the gates to the marvels of stochastic calculus. It might not be obvious to prove Itô's formula, the arcsine law for Brownian motion, Girsanov's theorem, and the invariant distribution of the CIR model, but they are easily verified with a couple of lines of codes, as the reader will find out.

Second, from a theoretical point of view, the reader is invited to consider random variables as elements of a linear space. This allows us to rigorously introduce many fundamental results of stochastic calculus and to prove most of them. This approach is also useful to build an intuition for the subject as it relies on a geometric point of view very close to the one in $\mathbb{R}^d$. Of course, some measure theory concepts (such as convergence theorems) need to be introduced to seriously study this subject. These are presented as needed.

The general objective of the book is to present the theory of Itô calculus based on Brownian motion and Gaussian processes. In particular, the goal is to construct stochastic processes using the Itô integral and to study their properties. The last chapter presents the applications of the theory to option pricing in finance. Each chapter contains many exercises, mixing both theory and concrete examples. Exercises marked with a $\star$ are more advanced. The remarks in the text also relate the material to more sophisticated concepts.

Here are the two suggested approaches to reading this book:

**(A) How to use this book for an undergraduate course.** This book offers *two* possible pathways for a course at the undergraduate level depending on whether the student's interest is in *mathematical finance* or in *advanced probability*. The common path of these two options consists of Chapters 1 to 5, with a road fork after. The blueprint is based on a 14-week, 4-credit class with no exercise sessions.

- Chapter 1 (1.1–1.4): It reviews the elementary notions of probability theory needed along the way. This usually covers one week of material and offers sufficient opportunity to get acquainted with basic commands in Python.
- Chapter 2 (2.1–2.4): The chapter introduces the notion of Gaussian processes, based on the notion of Gaussian vectors. This is where we learn the Cholesky

decomposition to express jointly Gaussian random variables in terms of IID standard Gaussians. This concept is very useful to numerically sample a plethora of Gaussian processes, such as Brownian motion, the Ornstein-Uhlenbeck process, and fractional Brownian motion. It also sets the table for the notion of *orthogonal projection* and its relation to conditional expectation. This geometric point of view is introduced in Section 2.4. Around two weeks of classes are dedicated to this chapter.

- Chapter 3 (3.1–3.2, 3.4): This chapter studies the properties of Brownian motion in more detail. In particular it introduces the notion of quadratic variation that is central to Itô calculus. The Poisson process is also presented as a point of comparison with Brownian motion. This chapter takes up around one to two weeks.
- Chapter 4 (4.1–4.5): The class of stochastic processes known as martingales is introduced here. It is built on the conditional expectation, which is defined as a projection in the space of random variables with finite variance. Elementary martingales, such as geometric Brownian motion, are given as examples. One of the powers of Itô calculus is to give a systematic way to construct martingales using Brownian motion. Martingales are useful, as some probabilistic computations are simplified in this framework. For example, solving the gambler's ruin problem using martingales is illustrated in Section 4.4. This chapter is longer and takes about two weeks to cover.
- Chapter 5 (5.1–5.5): The Itô integral is constructed as a limit of a martingale transform of Brownian motion. The martingale transform is analogous to Riemann sums in standard calculus. Itô's formula, which can be seen as the *fundamental theorem of Itô calculus*, is also proved and numerically verified. This is where we start to explore the beautiful interplay between partial differential equations (PDE) and stochastic processes. Around two weeks are needed to study Chapter 5.

At this point, the student should have a pretty good grasp of the basic rules of Itô calculus. There are then two possible subsequent paths:

- *Mathematical finance road* (Chapters 7 and 10). The ultimate objective here is to study the basics of option pricing through the lens of stochastic calculus. The reader may jump to Chapter 7 where Itô processes, and in particular diffusions, are studied. It is then fairly straightforward to study them using the rules of Itô calculus. An important part of this chapter is the introduction of stochastic differential equations (SDE). The emphasis here should be on numerically sampling diffusions using SDEs. Suggested sections are 7.1, 7.2, and 7.4. This takes only around one week, as the reader should be fluent with the rules of Itô calculus by that point.

  The course ends with Chapter 10 on the applications in finance. The suggested sections are 10.1 to 10.6 (with minimal inputs from Section 9.2). At this level, the focus should be on the Black-Scholes model. The pricing of options can be given using the Black-Scholes PDE and the risk-neutral pricing. This is an excuse to present the Cameron-Martin theorem of Chapter 9 which ensures the existence of a risk-neutral probability for the model. A good numerical project there is the biased sampling of paths based on the Cameron-Martin theorem. If time permits, Section 10.7 on exotic options is a good ending.

- *Probability and PDE road* (Chapters 6 and 7). The ultimate objectives here are the study of the Dirichlet problem and SDEs. It is particularly suited for students that love multivariate calculus. Suggested sections are 6.1 to 6.4 and 7.1 to 7.5. The generalization of Itô's formula for multidimensional Brownian motion is done in Chapter 6. It is used to prove the transience of Brownian motion in higher dimensions. The course can then move on to Itô processes and SDEs and include multivariate examples of such processes. If time permits, the Markov property of diffusions and its relation to PDEs can be presented in Chapter 8.

**(B) How to use this book for a master's course in mathematical finance.** Besides some more advanced subjects, the main difference in a master's course is that the finance concepts could be presented concurrently with the probability material. With this in mind, here are some suggestions for how to incorporate the material of Chapter 10 within the text.

- Sections 10.1, 10.2, and 10.3 on market models, derivative products, and the no-arbitrage pricing can be presented right at the beginning of the class. It may serve as a motivation for the need of stochastic calculus. This is also a possibility at the undergraduate level.
- Chapters 2 and 3 should be covered right after, with an emphasis on numerics to instill some intuition on Brownian motion and other stochastic processes. This is also a good time to understand the notion of projection for random variables, which is necessary in Chapter 4.
- Sections 10.4, 10.5, and 10.6 on the Black-Scholes model, the Greeks, and risk-neutral pricing for Black-Scholes model (with a brief mention of the Cameron-Martin theorem) are a good fit concurrently with Chapters 4 and 5 and Sections 7.1, 7.2, and 7.4. By then, the student has all the tools to derive the Black-Scholes PDE. Exotic options from Section 10.7 can be done there too.
- Sections 10.6, 10.8, and 10.9 on general risk-neutral pricing, interest rate models, and stochastic volatility models are good companions to Chapters 8 and 9.

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