
Contents

Foreword	ix
Preface	xi
Chapter 1. Basic Notions of Probability	1
1.1. Probability Space	1
1.2. Random Variables and Their Distributions	4
1.3. Expectation	11
1.4. Inequalities	16
1.5. Numerical Projects and Exercises	17
Exercises	19
1.6. Historical and Bibliographical Notes	21
Chapter 2. Gaussian Processes	23
2.1. Random Vectors	23
2.2. Gaussian Vectors	28
2.3. Gaussian Processes	34
2.4. A Geometric Point of View	40
2.5. Numerical Projects and Exercises	45
Exercises	46
2.6. Historical and Bibliographical Notes	49
Chapter 3. Properties of Brownian Motion	51
3.1. Properties of the Distribution	51
3.2. Properties of the Paths	54
3.3. A Word on the Construction of Brownian Motion	60

3.4. A Point of Comparison: The Poisson Process	61
3.5. Numerical Projects and Exercises	62
Exercises	64
3.6. Historical and Bibliographical Notes	66
Chapter 4. Martingales	67
4.1. Elementary Conditional Expectation	67
4.2. Conditional Expectation as a Projection	70
4.3. Martingales	80
4.4. Computations with Martingales	84
4.5. Reflection Principle for Brownian Motion	89
4.6. Numerical Projects and Exercises	91
Exercises	93
4.7. Historical and Bibliographical Notes	97
Chapter 5. Itô Calculus	99
5.1. Preliminaries	99
5.2. Martingale Transform	100
5.3. The Itô Integral	104
5.4. Itô's Formula	114
5.5. Gambler's Ruin for Brownian Motion with Drift	123
5.6. Tanaka's Formula	125
5.7. Numerical Projects and Exercises	127
Exercises	128
5.8. Historical and Bibliographical Notes	132
Chapter 6. Multivariate Itô Calculus	135
6.1. Multidimensional Brownian Motion	135
6.2. Itô's Formula	137
6.3. Recurrence and Transience of Brownian Motion	144
6.4. Dynkin's Formula and the Dirichlet Problem	146
6.5. Numerical Projects and Exercises	148
Exercises	149
6.6. Historical and Bibliographical Notes	151
Chapter 7. Itô Processes and Stochastic Differential Equations	153
7.1. Definition and Examples	153
7.2. Itô's Formula	156
7.3. Multivariate Extension	162
7.4. Numerical Simulations of SDEs	163

7.5. Existence and Uniqueness of Solutions of SDEs	165
7.6. Martingale Representation and Lévy's Characterization	170
7.7. Numerical Projects and Exercises	171
Exercises	172
7.8. Historical and Bibliographical Notes	175
Chapter 8. The Markov Property	177
8.1. The Markov Property for Diffusions	177
8.2. The Strong Markov Property	180
8.3. Kolmogorov's Equations	183
8.4. The Feynman-Kac Formula	192
8.5. Numerical Projects and Exercises	194
Exercises	195
8.6. Historical and Bibliographical Notes	198
Chapter 9. Change of Probability	199
9.1. Change of Probability for a Random Variable	199
9.2. The Cameron-Martin Theorem	202
9.3. Extensions of the Cameron-Martin Theorem	209
9.4. Numerical Projects and Exercises	213
Exercises	214
9.5. Historical and Bibliographical Notes	217
Chapter 10. Applications to Mathematical Finance	219
10.1. Market Models	220
10.2. Derivatives	221
10.3. No Arbitrage and Replication	225
10.4. The Black-Scholes Model	227
10.5. The Greeks	232
10.6. Risk-Neutral Pricing	236
10.7. Exotic Options	245
10.8. Interest Rate Models	246
10.9. Stochastic Volatility Models	251
10.10. Numerical Projects and Exercises	255
Exercises	257
10.11. Historical and Bibliographical Notes	263
Bibliography	265
Index	269