Preface

Partial Differential Equations
Open Windows into Our Complex World

Partial differential equations (PDEs) are a fundamental component of modern mathematics and science. In any system wherein the independent variables (e.g., space and time) take on a continuum of values, the laws governing the system usually result in a partial differential equation (PDE) for the quantity, or quantities, of interest. Since models used to approximate many physical, biological, and economic systems are PDE-based, having some proficiency in PDEs can provide a means of addressing these complex systems. In pure mathematics, PDEs are intrinsically related to fundamental structures in analysis, geometry, and probability.

This text presents a one- to two-semester course on PDEs for **senior undergraduates** in mathematics, physics, and the other sciences.

0.1. Who Should Take a First Course in PDEs?

There are primarily three groups. Students might find themselves identifying with more than one group. Listed below are these groups that we identify, each having different reasons, and therefore focuses, for pursuing studies in PDEs.

[1] **Students of "Pure" Mathematics**

Reason for an interest in PDEs: PDEs are fundamental mathematical objects in the fields of mathematical analysis, geometry, probability, and dynamical systems.

Their focus: Acquiring a foundation in PDEs and the appropriate mathematical tools in order to pursue future studies (perhaps at the graduate level) in mathematics. The primary focus of students of pure mathematics lies in finding and proving **theorems** related to existence, uniqueness, stability, and properties of the solutions (e.g., regularity, integrability, asymptotic behavior). They also benefit from studying theoretical structures in PDEs in order to gain insight into other areas of mathematics and the connections between them.

[2] **Students of “Applied” Mathematics**

Reason for an interest in PDEs: PDEs are fundamental mathematical objects in the continuum modeling of any complex system.

Their focus: Acquiring a foundation in PDEs and the appropriate mathematical tools in order to pursue further courses in scientific computing and mathematical modeling. In particular, applied mathematics students are interested in the **development** and application of analytical and computational tools in order to gain insight into the behavior of many complex systems.

[3] **Students of the Physical Sciences (Physics and Chemistry), Computer Science, Statistics, Biology, Engineering, and Economics**

Reason for an interest in PDEs: As for group [2], PDEs are fundamental mathematical objects in the continuum modeling of any complex system. They are ubiquitous throughout the physical sciences and engineering and are increasingly appearing in such fields as mathematical biology, mathematical medicine, economics, data science and machine learning, image and signal processing, and finance.

Their focus: Increasingly, students in this group want direct information about a relevant PDE, for example, an exact, approximate, or numerical form of the solution. They are less interested in methodology (tools) and issues related to their correctness or preciseness, but rather they want **direct answers** to their problems. Today, these “direct answers” do not usually come from hand-done calculations, but rather from the computer. **But here is the catch:** Fruitful interactions with the computer (i.e., what you input and how you interpret the output) require some foundation in PDEs and the associated mathematical objects and tools.

0.2. A Text for All Three Groups: Grounding in Core Concepts and Topics

It is still quite common that undergraduate PDE courses are based upon one method: **separation of variables** (also known as Fourier’s method) for boundary value problems. For all cohorts of students, an undergraduate PDE text or course which focuses mainly on separation of variables has only limited appeal. Indeed, our experience has shown that an **entire course** devoted to the extensive details behind this technique can be summarized as follows:

- It is of limited interest to budding pure mathematicians in their preparation for further studies in the analysis and geometry of PDEs. With regard to the basic premise behind separation of variables, they would be better served with a “complete” and proper mathematical theory for eigenfunction expansions associated with certain differential operators (something which is usually only presented at the graduate level).
- It is of limited use to budding applied mathematicians, who require many analytical and computational techniques as well as a broad exposure to the character and behavior of different classes of PDEs and their solutions.

- It is of limited use to budding scientists (the future practitioners of PDEs), who are increasingly more interested in computational aspects of the theory. Moreover, much of the relevant phenomena that modern science aims to tackle is inherently nonlinear and these classical techniques have a rather limited scope.
- It gives the undergraduate student from all three camps the false impression that PDEs are an old-fashioned subject rooted in long and tedious calculations with infinite series and special functions. Nothing could be further from the truth, as PDEs form a major component of modern mathematics and science.

In our view, a clear need for a solid **grounding** in objects, concepts, tools, and structures which are ubiquitous in the theory of PDEs is common to **all three groups**. Separation of variables and Fourier series are certainly included in these core concepts/tools, but our previous point is that they should encompass chapters, **not** full books or courses. By grounding, we mean a foundation (or basis) to move forward in future courses, research, and computational excursions with PDEs. This foundation can only be achieved by developing a solid understanding of, and appreciation for, certain **core material**.

It is true that we have chosen **not** to address, in any detail, computational aspects of PDEs, an increasingly large subject of cardinal importance to the second and third groups. However, it is our feeling that all successful excursions into the computation of PDEs, from the development of novel methods to the direct application of well-known methods, require a firm grasp of the core material introduced in this text.

The need for a grounding in objects (such as the delta function), concepts, tools, and structures is particularly important in our present day when information is both ubiquitous and readily (in fact, instantaneously) accessible. Indeed, more and more, Wikipedia provides perfectly suitable definitions of objects and results relevant to any science course. **We live in an age where information is plentiful and cheap, but understanding anything well is priceless.**

This text is by no means encyclopedic nor comprehensive, but it contains core material on which to construct a one- or two-semester course. Our vision of the core material is based upon the following:

(i) **First-order equations**, the notion of a **characteristic**, and the method of characteristics. Here, one appreciates the fundamental difference between linear and nonlinear PDEs (Chapter 2).

(ii) The nature of **wave propagation** as dictated by the second-order **wave equation**, its predictions of causality in space dimensions one (Chapter 3) and three (Chapter 4).

(iii) The **Fourier transform**, its properties and uses (Chapter 6).

(iv) The nature of **diffusion** as described by the **diffusion equation**, its consequences and relationship to basic objects and concepts in probability (Chapter 7).

(v) The nature of **harmonic functions** and PDEs involving the **Laplacian** (Chapter 8).

(vi) The **fundamental solution** of the Laplacian and why it is so "fundamental". The notion of a **Green's function** for boundary value problems involving the Laplacian (Chapter 10).

(vii) **Fourier series** and the **Separation of Variables Algorithm** (Chapters 11 and 12).

Fundamental to most of these topics is a clear grasp of the **delta function** in one and several space variables, in particular:

- why this object is not a function in the usual sense, yet encapsulates concentration at a point;
- how it appears in differentiating (in a generalized sense) functions with discontinuities in both one and several independent variables;
- how it appears as the limit of different sequences of functions which concentrate;
- its crucial role in Fourier series and the Fourier transform;
- its crucial role in fundamental solutions (Green's functions) for the diffusion, Laplace, and Poisson equations.

To this end, we have not been shy about tackling **distributions** and, in particular, what it means to differentiate a function in the sense of distributions, as well as to take a limit of functions in the sense of distributions. These ideas form an important component of this text and are presented in Chapters 5 and 9.

Also fundamental to **all** these topics is a solid understanding of basic objects and techniques in **advanced calculus**. It is our hope that in learning these fundamentals, the student will, at the very least, acquire a grounding and proficiency in the geometric and physical meaning of the gradient, the divergence, and the power of integration and differentiation for functions of several variables.

Throughout this text, we have tried hard to clearly explain all steps, often with pre-motivations and post-reflections. In many ways, we have aimed towards a text for **self-learning**. We can at times be wordy in this exposition and have no scruples against repeating a key point or argument several times. Information (facts, methods, examples) is everywhere and cheap(!); again, our sole reason for writing this book is to facilitate understanding.

0.3. Basic Structure of the Text

0.3.1. Presentation and Modularization of Material. The first thing many will notice when faced with this text is its **length**. To this end, several comments are in order:

- We have strived to **modularize** and streamline material — be it concepts, results, or ideas. Each chapter is based upon a particular equation, class of equations, or a particular idea or tool. Chapters begin with an introduction, end with a summary, and are comprised of **sections** which, in turn, are sometimes divided into **subsections**. Each section (or subsection) is the basic **unit/module** of the text and addresses a particular issue/topic in a short and concise fashion. The vast majority of these modules (sections or subsections) are no more than a few pages, hence, the rather long table of contents!

- It is true that many sections do not contribute the same weight towards what we call the core. Hence, we have chosen to highlight with a **bullet** • all sections which we deem to be the "basics" (fundamentals) for a first course in PDEs. Of course, one will not be able to cover all these bulleted sections in one semester! These bulleted sections also provide the necessary prerequisite material for further chapters. The additional (nonbulleted) sections are obviously there for a reason and, in our view, they are important; but they can be browsed, or read carefully, at the discretion of the reader/instructor.
- **The long table of contents** thus provides an essential and invaluable guide to navigating this text, and the reader/instructor is encouraged to refer to it often.

0.3.2. Choice for the Orderings of Chapters. Whereas the reader/instructor need not follow our sequence of chapters (see the next subsection), let us briefly address our philosophy behind this order. The most contentious choice is to leave Fourier series and separation of variables for boundary value problems (Chapters 11 and 12) to the end. While the majority of these two chapters can be covered at any time and, indeed, are often the first topics one encounters in a PDE course, we feel that the inherent character of the **wave, diffusion, and Laplace equations (the so-called "big three")** are obscured by simply focusing on infinite series solution representations for the respective boundary value problems. It is far more enlightening to first consider the big three divorced from any boundary condition and focus, respectively, on the central character of wave propagation, diffusion, and harmonic functions.

After a short introductory chapter (Chapter 1) on general definitions for PDEs, the book begins not with one of the big three equations, but rather with a long chapter on general first-order equations and the method of characteristics (Chapter 2). There are many reasons for this: (i) It is natural to start with PDEs involving only first-order derivatives; (ii) there is a general notion (a characteristic) central to **all** first-order PDEs coupled with a general method (the method of characteristics) to solve them; (iii) characteristics are directly connected with topics and concepts from previous courses, namely the gradient and directional derivative from Calculus III and ordinary differential equations (ODEs); (iv) the subject allows us to address not just linear PDEs, but nonlinear PDEs as well. Whereas the remaining chapters will all focus on linear PDEs, nonlinear PDEs are fundamental in mathematics and the sciences, and we feel it is beneficial to introduce them early on, even if the study is confined to first-order equations.

After first-order equations, we address the second-order wave equation, the member of the big three whose structure is closest to the characteristics of the previous chapter. We study the wave equation first in one space dimension (Chapter 3) and then in three and two space dimensions (Chapter 4).

After the wave equation, we take a two-chapter **detour** before tackling the diffusion and Laplace equations. **Why?** Our methods for solving and analyzing first-order equations and the second-order wave equation are all based on a property which we can loosely sum up in one phrase: "**finite propagation speed**". To handle (i.e., solve) PDEs with finite propagation speed, the basic tools and objects from advanced calculus, e.g., differentiation and integration, suffice. These tools are all based upon **local** (in space) calculations. To address the diffusion, Laplace, and Poisson equations we must embrace a **new paradigm** wherein this principle of "finite propagation speed" is **false**. Here the solution to these problems at any point in the domain will involve a certain weighted average of **all** the data (initial values or boundary values). There are several ideas/concepts that are fundamental to address these classes of PDEs: **concentration** and the effect singularities have on differentiation and **nonlocal operations**. They are most likely **new** to the reader and will be analyzed, respectively, with the following mathematical objects, tools, and machineries: **the Dirac delta function, distributions in one space dimension** (Chapter 5), and **convolution** and **the Fourier transform** (Chapter 6). However, as outlined in the next subsection, it is only the very basics of Chapters 5 and 6 which are required for the remainder of the text; in fact, very little of Chapter 6 is actually required. That said, we feel strongly that having a solid grounding in the delta function and the Fourier transform will prove tremendously useful to all students in future studies/applications.

With these concepts in hand, we address the ubiquitous diffusion equation (Chapter 7). Next, we present Chapter 8 (with almost no prerequisites) on the Laplacian and properties of harmonic functions (solutions to Laplace's equation). This opens the way to solving boundary value problems involving the Laplacian (Chapter 10). Here, the crucial tools are the fundamental solution and Green's functions, functions whose Laplacian is concentrated at a multidimensional delta function. Hence, we first pause to address partial differentiation in the sense of distributions (Chapter 9).

0.3.3. Codependence and Different Orderings of Chapters. The reader/instructor need not follow our precise ordering of the chapters. Let us first document the required prerequisites from previous chapters. Here we do not include any reference to Chapter 1 (basic PDE definitions and terminology) which contains definitions and notions used throughout the text.

Chapter 2. First-Order PDEs and the Method of Characteristics.
Prerequisites: None.

Chapter 3. The Wave Equation in One Space Dimension.
Prerequisites: Basic notion of a characteristic and the transport equation in 1D, which can be found in the Prelude to Chapter 2 (Section 2.1).

Chapter 4. The Wave Equation in Three and Two Space Dimensions.
Prerequisites: The wave equation in 1D — Chapter 3 (bulleted sections).

Chapter 5. The Delta "Function" and Distributions in One Space Dimension.
Prerequisites: None.

Chapter 6. The Fourier Transform.
Prerequisites: Definition of the delta function δ_0 and convergence in the sense of distributions to δ_0 as in Sections 5.2 to 5.5.

Chapter 7. The Diffusion Equation.
Prerequisites: The solution formula derived using the Fourier transform (Section 6.8), **or** this can be derived directly using similarity solutions (cf. Exercise **7.5**). Definition of the delta function δ_0 and convergence in the sense of distributions to δ_0 found in Sections 5.2 to 5.5. The basic notion of convolution discussed in Sections 6.3.1 and 6.3.2.

Chapter 8. The Laplacian, Laplace's Equation, and Harmonic Functions.
Prerequisites: None.

Chapter 9. Distributions in Higher Dimensions and Partial Differentiation in the Sense of Distributions.
Prerequisites: Chapter 5 (bulleted sections).

Chapter 10. The Fundamental Solution and Green's Functions for the Laplacian.
Prerequisites: Sections 8.1 and 8.5 and Sections 9.1, 9.3, and 9.5.

Chapter 11. Fourier Series.
Prerequisites: Only a very basic notion of the delta function (cf. Section 5.2).

Chapter 12. The Separation of Variables Algorithm for Boundary Value Problems.
Prerequisites: Chapter 11 (bulleted sections).

This relatively weak codependence presents many avenues for the reader/instructor. For example:

(i) Fourier series and the basics of separation of variables can be covered at any stage. Chapter 8 on Laplace's equation and harmonic functions can also be covered at any stage.

(ii) With a very short introduction to the delta function, convergence in the sense of distributions (Sections 5.2 to 5.5), and convolution (Sections 6.3.1 and 6.3.2), one can go directly to Chapter 7 on the diffusion equation.

While it is true that, except for the short Sections 6.3.1 and 6.3.2 introducing convolution, the long Chapter 6 on the Fourier transform is not required for the vast majority of this text; this material is vital for further studies in PDE, analysis, applied math, and science in general.

0.4. Prerequisites

Given that the scope of this text is quite vast, it is remarkable that from a purely technical point of view, the prerequisites, essentially **proficiency in advanced calculus**, are rather minimal.

0.4.1. Advanced Calculus and the Appendix. One of the main difficulties students encounter in a PDE course (even at the graduate level) is "indigestion" of basic multivariable calculus. In particular, the student should be, or become, as the course progresses, comfortable and proficient with the geometric and physical meaning of the following:

- bulk (volume) and surface integrals,
- the gradient of a function of several variables and directional derivatives,

- the divergence of a vector field and the notion of flux,
- the Divergence Theorem.

In the Appendix to this book we detail the necessary concepts and tools from advanced calculus that we will need. We strongly advise the reader to read the appropriate parts as needed. Some readers may benefit by reading the first few sections of the Appendix before embarking on the subsequent chapters. To further this point, we often start a chapter or a section by reminding the reader of the relevant section(s) of the Appendix.

Basic exposure to ordinary differential equations (ODEs) is also important, in particular, what exactly an ODE is and why, in general, they are so difficult to solve. We will occasionally need some very basic techniques to solve simple ODEs.

There are some proofs in this text, and on occasion, exposure to a first undergraduate-level course in real analysis could prove helpful (for example, epsilon delta proofs and uniform convergence of functions). However, this is not necessary for the vast majority of the material, and when we do use language/approaches from real analysis, we attempt to be as self-contained and gentle as possible; see our comments below on breadth and nonrigidity!

0.4.2. Breadth and Nonrigidity.

"*Those who know nothing of foreign languages know nothing of their own.*"[a] - **Johann Wolfgang von Goethe**

[a]The beauty of this translated quote by the great German poet Goethe (1749–1832) is that the sentence resonates equally well with "languages" replaced by ... academic disciplines, doctrines, religions, ... The original reads, "*Wer fremde Sprachen nicht kennt, weiß nichts von seiner eigenen*", and is taken from *Maximen und Reflexionen* (1833).

PDEs are an intrinsically multidisciplinary subject and a *first course* should embrace this wonderful trait. This does require a certain amount of breadth and nonrigidity from the student. Some mathematics students may initially be put off by "too much physics" while some nonmath students may initially complain about too much mathematical rigor and the occasional dreaded "proof". Math majors must keep in mind that PDEs are intimately connected to physics, and physical intuition will go a long way in guiding us through the analysis. To the nonmath majors, there is sometimes a need for precision, and being precise can mean being rigorous, i.e., proving things. In particular, on occasion there is a fundamental need for mathematical precision to provide meaning to otherwise ill-defined and confusing objects (such as the delta function), where informal intuition and calculations may not be sufficient to gain the necessary proficiency. Moreover, science has taken on a huge computational component and **interactions with the computer require a degree of precision**.

For all cohorts of students, it is our opinion that what future academics, scientists, engineers, and quantitative analysts will need is breadth, flexibility, and diversity; being rigid at an early stage of one's education can prove rather toxic in the future. A first course in PDEs presents an ideal way to *foster* this diverse perspective. It is worth noting that this message is hardly novel; look no further than the **opening quotes at**

the beginning of the book wherein two founding giants of the past make an eloquent case about the following:

- Mathematics (as a discipline) **needs** nature (physics and the other sciences) to direct, guide, and illuminate it.
- Mathematics is the basis for the underlying structure of nature and all the sciences **need** mathematics to make quantitative and qualitative assertions and conclusions.

0.5. Acknowledgments

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Finally it is important to acknowledge that over the years we have learned much about the subject from two excellent modern texts:

- At the undergraduate level, **Walter A. Strauss's** *Partial Differential Equations: An Introduction*, Wiley.
- At the mathematics graduate level, **Lawrence Craig Evans's** *Partial Differential Equations*, American Mathematical Society.

While the novelties in the exposition, content, organization, and style of this text will, hopefully, speak for themselves, it would be impossible for the text not to, on occasion, share certain similarities in approaches and style with these two works. Indeed, we owe much to these two authors.