Preface

The overall theme of this book is:

Algebraic Structures in Mathematics

Roughly speaking, an algebraic structure consists of a set of objects and a set of rules that let you manipulate the objects. Here are some examples that will be familiar to you:

Example 0.1. The objects are the numbers $1, 2, 3, \ldots$. You already know two ways to manipulate these objects, namely addition $a + b$ and multiplication $a \cdot b$.

Example 0.2. The objects are triangles in the plane, and we can be manipulate them by translations and by rotations and by reflections.

Example 0.3. The objects are functions $f : \mathbb{R} \to \mathbb{R}$, and we can manipulate them by addition $f(x) + g(x)$, by multiplication $f(x) \cdot g(x)$, and also by composition $f\big(g(x)\big)$.

Our primary goal is to take examples of this sort and generalize them, or in mathematical terminology, *axiomatize them*. To do this, we strip away everything that is not essential and reduce down to an abstract description consisting of a set with operations (such as addition and multiplication) that are required to satisfy certain rules, also known as *axioms*.[1] We will focus on the following five types of objects and their associated rules:

- **Chapters 2, 6, 12** Groups
- **Chapters 3, 7** Rings
- **Chapters 4, 10** Vector Spaces
- **Chapters 5, 8, 9** Fields
- **Chapters 11, 13** Modules

Although groups, rings, vector spaces, fields, and modules are not the same, the five topics share common themes. For each we use axioms to describe objects having an algebraic structure, and then we study maps between these objects that preserve the structure.

[1] Axioms are also sometimes called "laws." For example, you're probably familiar with the "commutative law" for addition, which says that $a + b = b + a$. But this isn't really a law, debated and approved by a legislative body! Instead, addition is a rule that explains how to combine two numbers and get a third number, and the "commutative law" is a property that we impose on the "addition rule."

Roughly speaking, each of the introductory Chapters 2–5 is organized as follows, although the order may vary slightly from chapter to chapter:

- Give an example of a certain type of algebraic structure.
- Give a formal definition, using axioms, of the algebraic structure.
- Prove a basic property directly from the definitions.
- Discuss what a map must do to "preserve the algebraic structure."
- Give additional examples.
- Investigate and prove a deeper property.

A Note on the Integrated Approach of This Book: Most algebra textbooks start with a thorough discussion of group theory, which in a typical course might occupy half a semester. They then move on to a lengthy exposition of ring theory, which occupies much of the remaining time, with some time left at the end for field theory. The effect, in my experience, is that students learn a lot about various types of trees and flowers and mosses and vines, but they do not gain perspective on how these flora combine to form a forest. The aim of this book is to reveal the forest by intertwining its constituent pieces. We thus start with (roughly) one to two week units introducing each of groups, rings, vector spaces, and fields, where each unit culminates in a "punchline," i.e., an interesting non-trivial result. Seeing these algebraic structures in quick succession, bang-bang-bang-bang, enables students to recognize their similarities and especially to see how they all fit into an axiomatic framework and to learn the importance of studying "maps that preserve structure."

This whirlwind introduction is followed by follow-up chapters in which each of the four main topics is investigated further, and in which deeper results are proven. Each of these chapters is designed to take one to two weeks, depending on how much is covered. Then, with this background under our belts, any remaining time may be used to delve more deeply into one or more of the four main topics and/or to study other types of algebraic structures such as modules or Galois theory.

In another departure from typical algebra texts, we introduce quotients first for rings and only later for groups. We follow this approach because most students find that ideals are easier than normal subgroups and because many of them will have seen the ring $\mathbb{Z}/m\mathbb{Z}$, either as a general construction or at least with $m = 12$ in the form of clock arithmetic.

And in a further attempt to prevent group theory from dominating the exposition, while at the same time emphasizing the ubiquity of group theory in all of algebra, we have at times taken a "buy now, pay later" approach. In particular, we use properties of permutations in Chapter 10 in order to give an intrinsic development of the determinant, and we use the simplicity of the alternating group $\mathcal{A}_n$ in Chapter 9 in order to prove insolvability by radicals of polynomial equations of degree at least 5, but we defer the proofs of these results to Chapter 12, which is the third chapter developing aspects of group theory. However, those who prefer to prove the requisite group theory before it is used can simply cover part or all of Chapter 12 at an earlier stage of the course.

A Note on Background and Prerequisites: The primary formal prerequisite for abstract algebra in general, and this book in particular, is a course in linear algebra. Of course, the mathematical maturity gained in a multivariable calculus class is helpful, too. For students who have taken a serious proof-based linear algebra course, the first linear algebra chapter

(Chapter 4) requires only a brief review, although my experience is that even students with an excellent background will benefit from revisiting the proof that the dimension of a vector space is well-defined. For students whose linear algebra background is primarily of the "let's mess around with matrices to solve linear equations" variety, it is important to spend more time on Chapter 4. Similarly, before covering the material on modules in Chapter 11, it is important either to briefly review or to study in detail the second linear algebra chapter (Chapter 10), depending on the students' backgrounds.

Since students' backgrounds are so varied, we have included an introductory chapter that serves two purposes. The first goal is to provide guidance on how to approach abstract mathematics, with a description of some of its constituent pieces such as definitions, axioms, and proofs. This is vital material, so as your author and teacher, I strongly urge you to:

Read Sections 1.1 and 1.2 — YES, this means you!!

This material will aid your transition from problem-solving student to proof-creating mathematician.

The second goal of Chapter 1 is to briefly present/review a number of topics such as logic, set theory, functions, number theory, and combinatorics that are used at various points in this book. There are two ways to approach this material. One is to spend the first couple of weeks covering Sections 1.3–1.9. There is nothing wrong with this, but it may feel as if you are spending a lot of time exploring side roads, while the main highway to algebra beckons. A second approach is to jump right in and start with Chapters 2–5, the first four algebra chapters.[2] This gets straight to the pizzazz, after which you can go back and cover some or all of Sections 1.3–1.9 as needed.

A Note on Textbooks versus Reference Books: There is a natural tension between a book that is designed to be an introductory text for a subject and a book that is both a textbook and a reference book. Both types of books have a role to play. An introductory text does not try to be encyclopedic, and it is often content proving special cases of important results. This makes comprehension easier for students encountering the material for the first time. An example of a textbook in this category is Herstein's *Topics in Algebra*[3], which provided inspiration for the present text. Dual text/reference books include more material, both in terms of breadth and depth, and frequently end up on the shelves of working mathematicians. Examples include Lang's *Algebra*[4] (914 pages) and Dummit and Foote's *Abstract Algebra*[5] (932 pages). If books could have mantras, the mantra of the book you are presently reading would be:

I am a Textbook, not an Encyclopedia.

A Note on Notation, Definitions, and Glossaries: For better or for worse, the subject of algebra includes a large number of definitions and a lot of notation, probably more than you're used to seeing in a typical class. We have included a list of notation, and there is a

[2] If the students are not familiar with the integers modulo m, it may be necessary to discuss that material from Section 1.8.

[3] I.N. Herstein, *Topics in algebra*, 2nd edition, Xerox College Publishing, Lexington, Mass.-Toronto, Ont., 1975.

[4] S. Lang, *Algebra*, 3rd edition, GTM 211, Springer-Verlag, New York, 2002.

[5] D.S. Dummit and R.M. Foote, *Abstract Algebra*, 3rd edition, John Wiley & Sons, Inc., Hoboken, NJ, 2004.

comprehensive index that you can use to locate where terms are defined. But I urge you to create your own glossary of terminology and your own list of notation, adding a new entry every time you encounter new material. Briefly summarizing definitions and describing notation is an excellent way to learn and remember them. And if you create a searchable electronic document, it will be a useful reference and study guide.

Mathematical Punchlines: Any good mathematics course includes numerous punchlines, those beautiful and unexpected results or relationships that are comparatively easy to state and understand but surprisingly subtle to prove. A first course in abstract algebra contains many such results, which may be viewed as goals and presented as highlights of the course. We include a brief list of some of the punchlines in this book:

- Lagrange's Theorem in GROUPS, PART 1 (Theorem 2.48)
- Fields of prime power order in FIELDS, PART 1 (Theorem 5.31), FIELDS, PART 2 (Theorem 8.28), and GALOIS THEORY (Section 9.12)
- Sylow's Theorem in GROUPS, PART 2 (Theorems 6.29 and 6.35)
- The Fundamental Theorem of Galois Theory in GALOIS THEORY (Theorem 9.52)
- Hilbert's Basis Theorem in MODULES, PART 1 (Theorem 11.43)
- The Structure Theorem for Finitely Generated Modules over Euclidean Domains in MODULES, PART 1 (Theorem 11.50)

Sample Schedules: Most instructors will want to start their course with the introduction to groups, rings, vector spaces, and fields in Chapters 2–5, followed by a more detailed study of these topics chosen from Chapters 6, 7, 8, 12. (Some selection from these latter chapters will fill a semester course; most or all of it should fit into two trimesters.) There are then many roads to travel in a subsequent semester or trimester. As an aid to the instructor, we have included three day-by-day sample schedules starting on page 523. The first is a typical 3-day-a-week first semester course that mirrors the one used by the author. The second and third describe two possiblities for a second semester course, one covering an in-depth development of Galois theory and the theory of modules, the other a course that surveys a large number of topics.

The Algebra Road Goes Ever On and On: The contents of this book provide an introduction to one of the principal branches of modern mathematics. This means that even in 500+ pages, we will explore only a limited part of the vast world of algebra. In order to whet your appetite for further algebraic adventures, the final chapter contains brief introductions to a baker's dozen additional topics in algebra and related fields. We hope that these snippets will encourage you, the mathematical traveler, to pursue the algebra road with eager feet, until it joins some larger way, where many maths and topics meet.[6]

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[6] In our slightly amended version of Bilbo and Frodo's wonderful walking song.

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