Chapter 1

Dimensional analysis

Mathematical models are equations that express relationships between given quantities of interest. The equations may be of any type, and the quantities may be of any type, either variable or constant. In this chapter, we outline various results about the units and dimensions of quantities, which can lead to insights, and point the way towards simpler, more concise forms of any mathematical model.

1.1. Units and dimensions

Throughout our developments we consider equations involving real-valued quantities expressed in given units of some given dimension. By a **unit** for a quantity we mean a scale for its measurement, such as a foot, hour, or gram. By the **dimension** of a quantity we mean its intrinsic type, such as length, time, or mass. Whereas a unit for a quantity can be chosen arbitrarily, the dimension of a quantity is a characteristic property that is fixed.

Not all quantities have a dimension of their own. Indeed, by virtue of their definition, the dimensions of some quantities can be expressed as combinations of others. Thus only a basic set or **basis** of dimensions is required to describe a collection of quantities. For example, a standard dimensional basis for quantities arising in simple physical systems is

$$\{\text{length } (L), \text{ time } (T), \text{ mass } (M), \text{ temperature } (\Theta)\}. \tag{1.1}$$

Different bases could be considered depending on the context. In systems for which forces are important but not masses, the basis could include the dimension of force instead of mass. A similar change could be made if energies were important but not masses. In systems that include electrical quantities, the basis would be enlarged to include the dimension of electric current. As a different example, to describe quantities arising in a simple ecological system, a dimensional basis might consist of

$$\{\text{carnivore } (C), \text{ herbivore } (H), \text{ plant } (P), \text{ insect } (I), \text{ time } (T)\}. \tag{1.2}$$

To any dimensional basis we associate a corresponding **choice of units**. These units may have some standard size, or any other arbitrary, nontrivial size, and they may have some standard name, or any other arbitrary name for convenience. For example, for the dimensional basis in (1.1), one choice of units is {meter, second, kilogram, kelvin}. For the dimensional basis in (1.2), one choice of units could be {herd, flock, field, swarm, month}, where, for example, 1 herd may be defined as 20 carnivores, 1 flock may be defined as 12 herbivores, and so on. Thus we will consider real-valued quantities, with values specified in a given choice of units, in a given dimensional basis. The following notation will be used throughout.

Definition 1.1.1. *Let $q \in \mathbb{R}$ be a quantity specified in units $\{U_1, \dots, U_m\}$ in a dimensional basis $\{D_1, \dots, D_m\}$ for some $m \geq 1$. By $[q]$ we mean the dimension of q expressed as a product of powers of the basis elements, namely*

$$[q] = D_1^{a_1} D_2^{a_2} \cdots D_m^{a_m}. \tag{1.3}$$

The numbers $a_1, \dots, a_m$ are called the **dimensional exponents** *of q in the given basis. The array of exponents is denoted by $\Delta_q = (a_1, \dots, a_m) \in \mathbb{R}^m$.*

Example 1.1.1. Let $p = 3\,\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}$, $g = 9.8\,\frac{\text{m}}{\text{s}^2}$, and $q = 100\,\frac{\text{kelvin}}{\text{s}}$. These quantities are expressed in units {meter, second, kilogram, kelvin} in the dimensional basis $\{L, T, M, \Theta\}$. The dimensions and corresponding dimensional exponents for p, g and q in this basis are

$$\begin{aligned} [p] &= ML^2/T^2 = L^2T^{-2}M\Theta^0, & \Delta_p &= (2, -2, 1, 0), \\ [g] &= L/T^2 = LT^{-2}M^0\Theta^0, & \Delta_g &= (1, -2, 0, 0), \\ [q] &= \Theta/T = L^0T^{-1}M^0\Theta, & \Delta_q &= (0, -1, 0, 1). \end{aligned} \tag{1.4}$$

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Recall that, in the notation $g = 9.8\,\frac{\text{m}}{\text{s}^2}$, the number 9.8 is the numerical value of the quantity, and the tag $\frac{\text{m}}{\text{s}^2}$ is an explicit reminder of the units for the quantity. When we say that one quantity is a function of another, we mean that a relation exists between their numerical values, with respect to a given choice of units, in a given dimensional basis. Thus when we write $p = f(q)$, we mean that the numerical value of p is completely determined by the numerical value of q. The function f is simply a map from one real value to another, and may be defined by a formula or graph in the usual way.

1.2. Axioms of dimensions

We adopt the basic axioms that addition and subtraction are dimensionally meaningful only for quantities of the same dimension, whereas multiplication and division are meaningful for quantities of arbitrary dimension. To state these axioms in a more precise way, let $p, q, r, s \in \mathbb{R}$ be quantities with given units in a given dimensional basis.

The basic axiom on addition and subtraction reflects the idea that only quantities of the same dimension can be added and subtracted in a dimensionally meaningful way. Thus the statement $r = p \pm q$ has a dimensional meaning only when p and q, and hence r, have the same dimension. For instance, "1 meter + 2 meter" is a meaningful statement, whereas "1 meter + 2 second" is not.

The basic axiom on multiplication and division reflects the idea that quantities of any dimension can be multiplied and divided; indeed, this is how more complicated dimensions are derived from elementary ones. Thus the statements $r = pq$ and $s = p/q$ ($q \neq 0$) have a dimensional meaning for all p and q, and the dimensions of the results r and s are well defined in each case. Moreover, this axiom can be extended to arbitrary powers, integration, and differentiation.

Axiom 1.2.1. *Let $p, q \in \mathbb{R}$ be quantities specified in units $\{U_1, \dots, U_m\}$, in a dimensional basis $\{D_1, \dots, D_m\}$, with dimensions $[p], [q]$. Then*

(1) $[p \pm q]$ is defined if and only if $[p] = [q]$,

(2) $[pq] = [p][q]$ for all p, q,

(3) $[p/q] = [p]/[q]$ for all p, q with $q \neq 0$,

(4) $[q^\alpha] = [q]^\alpha$ for all $q > 0$ and real α,

(5) $[\int p\, dq] = [p][q]$ for any integrable function $p = f(q)$,

(6) $[dp/dq] = [p]/[q]$ for any differentiable function $p = f(q)$.

In property (4) the condition $q > 0$ ensures that q^α is defined for any power α. While it would suffice to only consider rational powers, we assume that the property holds for all real powers. The content of properties (2)–(4) can be translated to the dimensional exponents Δ_p and Δ_q in a straightforward way, namely

$$\Delta_{pq} = \Delta_p + \Delta_q, \quad \Delta_{p/q} = \Delta_p - \Delta_q, \quad \Delta_{q^\alpha} = \alpha\Delta_q. \tag{1.5}$$

1.3. Dimensionless quantities

The concept of a quantity with no dimension as defined next will play an important role throughout our developments. We note that such quantities can arise when considering combinations of other quantities, and can also arise naturally in other ways.

Definition 1.3.1. *A quantity $q \in \mathbb{R}$ is called* **dimensionless** *if its dimensional expression is $[q] = 1$, or equivalently its array of dimensional exponents is $\Delta_q = 0$, in any units in any dimensional basis.*

Example 1.3.1. (1) Let $q = ab/c$, where $a = 4\,\frac{\text{ft}}{\text{hour}}$, $b = 3\,\frac{1}{\text{hour}}$, $c = 2\,\frac{\text{ft}}{\text{hour}^2}$. Considering dimensions we have $[a] = LT^{-1}$, $[b] = T^{-1}$, and $[c] = LT^{-2}$, and we find that $[q] = [a][b]/[c] = 1$. Thus q is a dimensionless quantity; its value is $q = 6$.

(2) Let a be an arbitrary quantity with dimension $[a]$, and let $b = a + a$ and $c = a \cdot a \cdot a$. Then it is natural to rewrite these quantities as $b = 2a$ and $c = a^3$. In these latter expressions, we note that the coefficient 2 and exponent 3 are dimensionless; they are purely mathematical entities called **pure numbers**. The dimensions of b and c are $[b] = [2a] = [a]$ and $[c] = [a^3] = [a]^3$.

(3) Let θ be an arbitrary angle, which when inscribed in a circle of radius r subtends an arc of length ℓ. Then, in the radian unit of measurement, we have $\theta = \frac{\ell}{r}$ and we find $[\theta] = 1$. Hence angles and the radian unit of measurement are dimensionless.

Similarly, since it only differs in size, the degree unit of measurement is dimensionless.

(4) Any ratio of two quantities of the same dimension is dimensionless. The value of such a ratio can be expressed as a pure number, or in terms of any arbitrary dimensionless unit such as a percentage or parts-per-hundred. ∎

1.4. Change of units

Here we outline the effect of a change of units on an arbitrary quantity. For our purposes it will be sufficient to only consider changes in the dimensional units associated with a given dimensional basis, with any dimensionless units held fixed. We assume that any two units of the same dimensional type are related by a multiplicative conversion factor as introduced below.

To state the result, we consider an arbitrary quantity $q \in \mathbb{R}$, expressed in units $\{U_1, \dots, U_m\}$, in a dimensional basis $\{D_1, \dots, D_m\}$, with dimensional exponents $\Delta_q = (a_1, \dots, a_m) \in \mathbb{R}^m$.

Result 1.4.1. *If units $\{U_1, \dots, U_m\}$ are changed to $\{\widetilde{U}_1, \dots, \widetilde{U}_m\}$, then the quantity q is changed to $\tilde{q}$, where*

$$\tilde{q} = q\lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_m^{a_m}. \tag{1.6}$$

Here $\lambda_i > 0$ $(i = 1, \dots, m)$ are **unit-conversion factors***; each factor λ_i quantifies the number of units of $\widetilde{U}_i$ per unit of U_i.*

The above result follows from straightforward algebra and the axioms on dimensions regarding multiplication and division. Note that if q is dimensionless, then $\Delta_q = (0, \dots, 0)$, and we obtain $\tilde{q} = q$. Thus dimensionless quantities are not affected by a change of dimensional units.

Example 1.4.1. Let $g = 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$. This quantity is expressed in units $\{\mathrm{m},\ \mathrm{s}\}$ in the dimensional basis $\{L, T\}$. Since $[g] = LT^{-2}$, its dimensional exponents are $\Delta_g = (a_1, a_2) = (1, -2)$. If the units are changed to $\{\mathrm{km}, \mathrm{min}\}$, then the unit-conversion factors are

$$\lambda_1 = \frac{1}{1000}\,\frac{\mathrm{km}}{\mathrm{m}}, \quad \lambda_2 = \frac{1}{60}\,\frac{\mathrm{min}}{\mathrm{s}}. \tag{1.7}$$

In the new units we have

$$\tilde{g} = g\lambda_1^{a_1}\lambda_2^{a_2} = \Big(9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\Big)\Big(\frac{1}{1000}\,\frac{\mathrm{km}}{\mathrm{m}}\Big)\Big(\frac{1}{60}\,\frac{\mathrm{min}}{\mathrm{s}}\Big)^{-2} = 35.28\,\frac{\mathrm{km}}{\mathrm{min}^2}. \tag{1.8}$$

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1.5. Unit-free equations

In the modeling of various types of systems, we will usually consider a set of real-valued quantities $q_1, \dots, q_n$, specified in units $\{U_1, \dots, U_m\}$, in a dimensional basis $\{D_1, \dots, D_m\}$, for some $n \geq 2$ and $m \geq 1$. We will often seek to construct and study equations of the form

$$q_1 = f(q_2, \dots, q_n), \tag{1.9}$$

where $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is some function. The function notation above indicates that the numerical value of q_1 is completely determined by the numerical values of $q_2, \dots, q_n$ in the given units. In our pursuits, we will only consider equations that are unit-free as defined next.

Definition 1.5.1. *An equation* $q_1 = f(q_2, \dots, q_n)$ *is called* **unit-free** *if it transforms into*

$$\tilde{q}_1 = f(\tilde{q}_2, \dots, \tilde{q}_n) \tag{1.10}$$

under an arbitrary change of units on arbitrary values of $q_1, \dots, q_n$.

The key point of a unit-free equation is that the function f is unaffected by the choice of units. All the equations that we consider will be unit-free in this sense. Note that, without this property, the function f may change whenever the units are changed, and the equation would have limited value as a model. Indeed, it would be tedious to document each different version of the equation for each different choice of units. Thus a unit-free equation can be viewed as a well designed equation. Model equations derived from fundamental physical laws are naturally unit-free; they inherit this property from the laws on which they are based. In contrast, empirical equations derived from curve fitting procedures are not naturally unit-free, but can always be re-designed to have this property by introducing appropriate dimensional constants. In the most basic sense, a unit-free equation can be viewed as a dimensionally meaningful equation, consistent with the axioms of dimensions, and this property can always be achieved.

Example 1.5.1. Let x, t and g be specified in units $\{\mathrm{m},\ \mathrm{s}\}$, in the dimensional basis $\{L, T\}$, with dimensions $[x] = L$, $[t] = T$ and $[g] = LT^{-2}$, and exponents $\Delta_x = (1, 0)$, $\Delta_t = (0, 1)$ and $\Delta_g = (1, -2)$. Suppose that the value of x is determined by the values of t and g through the equation

$$x = f(t, g) = \frac{1}{2} g t^2. \tag{1.11}$$

(Unless mentioned otherwise, unnamed quantities such as the factor $\frac{1}{2}$ and exponent 2 can be interpreted as pure numbers.) To determine if the above equation is unit-free, we consider a change of units from $\{\mathrm{m},\ \mathrm{s}\}$ to arbitrary units $\{\widetilde{U}_1, \widetilde{U}_2\}$, defined by arbitrary conversion factors λ_1, λ_2. In the new units, the values of x, t and g become

$$\tilde{x} = x\lambda_1, \quad \tilde{t} = t\lambda_2, \quad \tilde{g} = g\lambda_1\lambda_2^{-2}. \tag{1.12}$$

Substitution of these expressions into $x = \frac{1}{2} g t^2$ gives

$$\left(\tilde{x}\lambda_1^{-1}\right) = \frac{1}{2}\left(\tilde{g}\lambda_1^{-1}\lambda_2^{2}\right)\left(\tilde{t}\lambda_2^{-1}\right)^2. \tag{1.13}$$

In the above, all factors with λ_1 and λ_2 cancel, and we get

$$\tilde{x} = \frac{1}{2}\tilde{g}\tilde{t}^2. \tag{1.14}$$

Thus (1.11) is unit-free since it has exactly the same form in any choice of units. The original equation $x = f(t, g)$ is transformed into $\tilde{x} = f(\tilde{t}, \tilde{g})$, with the same function f. ∎

Example 1.5.2. Let x, t and g be as before, and let c be an additional quantity, say a constant, with $[c] = T$ and $\Delta_c = (0, 1)$. For purposes of comparison, consider the two different equations

$$x = \frac{1}{2}gt^2e^{-t}, \quad x = \frac{1}{2}gt^2e^{-t/c}. \tag{1.15}$$

(Here $e^q = \exp(q)$ is the natural exponential function; the base e can be interpreted as a pure number.) Considering an arbitrary change of units as above, we get

$$\tilde{x} = \frac{1}{2}\tilde{g}\tilde{t}^2e^{-\tilde{t}\lambda_2^{-1}}, \quad \tilde{x} = \frac{1}{2}\tilde{g}\tilde{t}^2e^{-\tilde{t}/\tilde{c}}. \tag{1.16}$$

The first equation is not unit-free since it changes form: a unit-conversion factor remains in the equation and does not cancel out. In contrast, the second equation is unit-free since all the unit-conversion factors cancel. Note how the first equation becomes unit-free by introduction of the constant c. The equations in (1.15), written in units {m, s}, would be numerically the same when $c = 1$s. However, the second equation is advantageous since it would have exactly the same form in any units. ∎

Example 1.5.3. Let v and t be quantities specified in units {lb, hr}, in the dimensional basis $\{M, T\}$, with dimensions $[v] = M/T$ and $[t] = T$. Suppose that the value of v is determined by the value of t through an empirical equation

$$v = 3.7t^2 - \sin(5.4t). \tag{1.17}$$

Here we rewrite this equation in a unit-free form. To begin, we introduce constants a, b, c with values $3.7, 1, 5.4$ in units {lb, hr} and consider

$$v = at^2 - b\sin(ct). \tag{1.18}$$

We next determine the dimensions of these constants to make the equation unit-free. Accordingly, let $\Delta_a = (\alpha_1, \alpha_2)$, $\Delta_b = (\beta_1, \beta_2)$ and $\Delta_c = (\gamma_1, \gamma_2)$ be the unknown dimensional exponents. Under an arbitrary change of units with conversion factors λ_1, λ_2, using the fact that $\Delta_v = (1, -1)$ and $\Delta_t = (0, 1)$, we get, after dividing out the conversion factors from the left side of the equation,

$$\tilde{v} = \lambda_1^{1-\alpha_1}\lambda_2^{-3-\alpha_2}\, \tilde{a}\tilde{t}^2 - \lambda_1^{1-\beta_1}\lambda_2^{-1-\beta_2}\, \tilde{b}\sin(\lambda_1^{-\gamma_1}\lambda_2^{-1-\gamma_2}\, \tilde{c}\tilde{t}). \tag{1.19}$$

Note that the unit-free condition will be satisfied when the exponents of all the conversion factors in the above expression are zero, which requires $\Delta_a = (1, -3)$, $\Delta_b = (1, -1)$ and $\Delta_c = (0, -1)$. Thus the dimensions of the constants are completely determined, and in units {lb, hr} we have

$$a = 3.7\,\frac{\text{lb}}{\text{hr}^3}, \quad b = 1\,\frac{\text{lb}}{\text{hr}}, \quad c = 5.4\,\frac{1}{\text{hr}}. \tag{1.20}$$

In any other units, the equation would be $\tilde{v} = \tilde{a}\tilde{t}^2 - \tilde{b}\sin(\tilde{c}\tilde{t})$, where $\tilde{a}, \tilde{b}, \tilde{c}$ are the values of the constants in the new units. In our function notation, the equation in (1.18) would be written as $v = f(t, a, b, c)$. ∎

1.6. Buckingham π-theorem

Here we outline a classic result known as the Buckingham π-theorem. It states that, for any unit-free equation $q_1 = f(q_2, \dots, q_n)$, the function f cannot depend on $q_2, \dots, q_n$ in a completely arbitrary way; it can only depend on certain dimensionless combinations. For simplicity we state the result only for positive values of $q_1, \dots, q_n$. Similar results hold for nonpositive values, but at the expense of more complicated statements.

Definition 1.6.1. *By a* **power product** *of $q_1, \dots, q_n > 0$ we mean a quantity $\pi > 0$ of the form*

$$\pi = q_1^{b_1} \cdots q_n^{b_n}, \tag{1.21}$$

for some powers $b_1, \dots, b_n \in \mathbb{R}$. We say that π includes q_i if $b_i \neq 0$.

The condition that each q_i be positive ensures that π is well defined for arbitrary powers. In any dimensional basis $\{D_1, \dots, D_m\}$, we note that each quantity q_i will have dimensional exponents $\Delta_{q_i} \in \mathbb{R}^m$, and the power product π will have dimensional exponents $\Delta_\pi \in \mathbb{R}^m$. From the definition in (1.21), together with the properties that $\Delta_{pq} = \Delta_p + \Delta_q$ and $\Delta_{q^\alpha} = \alpha\Delta_q$ given in (1.5), we deduce that

$$\Delta_\pi = b_1\Delta_{q_1} + \cdots + b_n\Delta_{q_n} = Av, \tag{1.22}$$

where $A = (\Delta_{q_1}, \Delta_{q_2}, \dots, \Delta_{q_n}) \in \mathbb{R}^{m\times n}$ and $v = (b_1, \dots, b_n) \in \mathbb{R}^n$. Here all one-dimensional arrays are considered as columns, and we assume $n \geq 2$ and $m \geq 1$ with $n \geq m$.

Given $q_1, \dots, q_n$ we will be interested in forming power products π that are dimensionless. In this respect, we note that

$$\pi \text{ dimensionless} \quad \Leftrightarrow \quad \Delta_\pi = 0 \quad \Leftrightarrow \quad Av = 0. \tag{1.23}$$

Furthermore, we will only be interested in nontrivial power products $\pi \not\equiv 1$, which correspond to $v \neq 0$. The following result, which essentially is a definition, characterizes the dimensionless power products that we seek.

Result 1.6.1. *If $Av = 0$ has a total of k independent solutions $v_1, \dots, v_k$, then a total of k independent dimensionless power products $\pi_1, \dots, \pi_k$ can be formed. Any such $\pi_1, \dots, \pi_k$ is called a* **full set***. This set is further called* **normalized** *if π_1 includes q_1 (with power $b_1 = 1$), and $\pi_2, \dots, \pi_k$ do not include q_1.*

Recall that, through the usual process of row reduction, any nontrivial solution of $Av = 0$ will be expressed in terms of certain free variables. If there are k free variables, then there are k independent solutions v, and hence k independent dimensionless power products π. While any independent choices of the free variables can be made to form a full set of solutions, a deliberate choice of these variables is required to form a normalized set. Specifically, the normalization condition requires that the first solution have $b_1 = 1$, and any other solutions have $b_1 = 0$.

Example 1.6.1. Let $x, t, g, h, m > 0$ be quantities with dimensions $[x] = L$, $[t] = T$, $[g] = LT^{-2}$, $[h] = LT^{-3}$ and $[m] = M$. A dimensional basis is $\{L, T, M\}$, and the dimensional exponent matrix in this basis is

$$A = (\Delta_x, \Delta_t, \Delta_g, \Delta_h, \Delta_m) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.24}$$

An arbitrary power product has the form $\pi = x^{b_1} t^{b_2} g^{b_3} h^{b_4} m^{b_5}$. The equation $Av = 0$, where $v = (b_1, \dots, b_5)$, has two free variables, and the general solution is

$$b_1 = -b_3 - b_4, \quad b_2 = 2b_3 + 3b_4, \quad b_5 = 0, \quad b_3, b_4 \text{ free.} \tag{1.25}$$

Since there are two free variables, there are two independent solutions. For one solution we choose $b_3 = -1, b_4 = 0$, which gives $v_1 = (1, -2, -1, 0, 0)$, and hence $\pi_1 = x/(gt^2)$. For a second solution we choose $b_3 = -1, b_4 = 1$, which gives $v_2 = (0, 1, -1, 1, 0)$, and hence $\pi_2 = th/g$. By choice, we arranged for π_1 to include x with an exponent of unity, and for π_2 to exclude x. Hence π_1, π_2 is a full set of dimensionless power products for x, t, g, h, m, and this set is normalized with respect to x. Note that m will not be included in any dimensionless power product. ∎

The next result shows that any unit-free equation can only depend on dimensionless power products. No assumption on the form or continuity properties of the function f are required. For simplicity, the results are stated only for positive quantities; similar results can be derived to account for negative and zero quantities.

Result 1.6.2. [π-theorem] *Let $\pi_1, \dots, \pi_k > 0$ be a full set of dimensionless power products for $q_1, \dots, q_n > 0$ where $k \geq 1$ and $n \geq 2$. If the set $\pi_1, \dots, \pi_k$ is normalized, then any unit-free equation $q_1 = f(q_2, \dots, q_n)$ for some function f, is equivalent to an equation*

$$\pi_1 = \phi(\pi_2, \dots, \pi_k) \tag{1.26}$$

for some function ϕ. In the case that $k = 1$, the function ϕ reduces to some constant C.

The normalization condition ensures that the reduced equation (1.26) is explicit, just as the original equation, in terms of q_1. We remark that a more general form of the theorem states that any unit-free equation in the general implicit form $F(q_1, \dots, q_n) = 0$ is equivalent to $\Phi(\pi_1, \dots, \pi_k) = 0$, without any normalization condition on the power products. Also, if the only dimensionless power product for quantities $q_1, \dots, q_n$ is trivial, then the only unit-free relation among these quantities is trivial; in this case, the set $q_1, \dots, q_n$ would need to be enlarged in order for a nontrivial unit-free relation to exist. The proof of the theorem is based on a change of variable argument that exploits the unit-free condition and the definition of the power products, which will be outlined after some examples.

Example 1.6.2. Let $x, t, g, h, m > 0$ be quantities as in the previous example. A normalized set of power products for these quantities is $\pi_1 = x/(gt^2)$ and $\pi_2 = th/g$. Thus any unit-free equation of the form $x = f(t, g, h, m)$ must be equivalent to an equation of the form

$$\pi_1 = \phi(\pi_2) \quad \text{or} \quad \frac{x}{gt^2} = \phi\Big(\frac{th}{g}\Big), \tag{1.27}$$

which can be rearranged to yield

$$x = gt^2 \cdot \phi\Big(\frac{th}{g}\Big). \tag{1.28}$$

Thus x cannot depend on m, and must depend on t, g and h in a specific way. If $\phi(0)$ is defined, then the special case when h has a fixed value of zero can be considered, and the relation becomes $x = \beta g t^2$, where $\beta = \phi(0)$ is a dimensionless constant. ∎

Example 1.6.3. Here we explicitly find the reduced form of the unit-free equation

$$u = f(x, t, \alpha, \beta) = \frac{\alpha x}{t} e^{-x^2/(\beta t^2)}, \tag{1.29}$$

where $[u] = \Theta$, $[x] = L$, $[t] = T$, $[\alpha] = \Theta T/L$, and $[\beta] = L^2/T^2$. A dimensional basis is $\{\Theta, L, T\}$, and the dimensional exponent matrix is

$$A = (\Delta_u, \Delta_x, \Delta_t, \Delta_\alpha, \Delta_\beta) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix}. \tag{1.30}$$

An arbitrary power product is $\pi = u^{b_1} x^{b_2} t^{b_3} \alpha^{b_4} \beta^{b_5}$. The equation $Av = 0$, where $v = (b_1, \ldots, b_5)$, has two free variables. By choosing these variables in a similar way as before, we obtain the full, normalized set $\pi_1 = u/(\alpha\sqrt{\beta})$ and $\pi_2 = x/(t\sqrt{\beta})$. By the π-theorem, the original equation $u = f(x, t, \alpha, \beta)$ must be equivalent to $\pi_1 = \phi(\pi_2)$ for some function ϕ. Here this result can be verified directly due to the explicit form of the original equation. Specifically, dividing the equation by $\alpha\sqrt{\beta}$, and then substituting, we obtain

$$u = \frac{\alpha x}{t} e^{-x^2/(\beta t^2)} \quad \Leftrightarrow \quad \frac{u}{\alpha\sqrt{\beta}} = \frac{x}{t\sqrt{\beta}} e^{-x^2/(\beta t^2)} \quad \Leftrightarrow \quad \pi_1 = \pi_2 e^{-\pi_2^2}. \tag{1.31}$$

∎

Example 1.6.4. A simple theory of sound waves in a gas proposes that the speed of sound $v > 0$ should depend on only the mass density $\rho > 0$, pressure $p > 0$, and viscosity $\mu > 0$ so that

$$v = f(\rho, p, \mu), \tag{1.32}$$

for some function f. Here we use the π-theorem to find an equivalent and possibly simpler form of (1.32) assuming that it is unit-free. This can be viewed as an important first step in exploring any new or proposed relation of interest.

The quantities v, ρ, p, μ have dimensions $[v] = L/T$, $[\rho] = M/L^3$, $[p] = M/(LT^2)$ and $[\mu] = M/(LT)$. A dimensional basis is $\{L, T, M\}$, and the dimensional exponent matrix in this basis is

$$A = (\Delta_v, \Delta_\rho, \Delta_p, \Delta_\mu) = \begin{pmatrix} 1 & -3 & -1 & -1 \\ -1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{1.33}$$

An arbitrary power product has the form $\pi = v^{b_1} \rho^{b_2} p^{b_3} \mu^{b_4}$. The equation $Av = 0$, where $v = (b_1, \ldots, b_4)$, has one free variable, and the general solution is

$$b_1 = -2b_3, \quad b_2 = -b_3, \quad b_4 = 0, \quad b_3 \text{ free}. \tag{1.34}$$

Since there is only one free variable, there is only one independent solution. For this solution we choose $b_3 = -1/2$, which gives $v_1 = (1, 1/2, -1/2, 0)$, and hence $\pi_1 = v\sqrt{\rho/p}$. Hence π_1 is a full set of independent dimensionless power products for v, ρ, p, μ, and this set is normalized with respect to v.

By the π-theorem, the equation in (1.32) must be equivalent to

$$\pi_1 = C \quad \text{or} \quad v = C\sqrt{\frac{p}{\rho}}, \tag{1.35}$$

where $C > 0$ is some dimensionless constant. Thus any experimental investigation of sound waves under the given hypothesis should be aimed at (1.35), and the determination of the unknown constant C.

Note that, even though an unknown constant is involved, there is valuable, direct information implied by (1.35). For instance, it implies that the speed of sound must be independent of the viscosity, and would increase with pressure at fixed density, and decrease with density at fixed pressure. Moreover, the speed of sound would remain unchanged when pressure and density are both increased or decreased in a simultaneous way. ∎

Sketch of proof: Result 1.6.2. Let $q_1, \dots, q_n > 0$ and $\pi_1, \dots, \pi_k > 0$ be given, and let $A \in \mathbb{R}^{m\times n}$ be the dimensional exponent matrix whose columns are $\Delta_{q_\sigma} = (a_{1,\sigma}, \dots, a_{m,\sigma}) \in \mathbb{R}^m$, where $\sigma = 1, \dots, n$. Note that, to each dimensionless power product π_ρ, there is an independent solution $v_\rho = (b_{1,\rho}, \dots, b_{n,\rho}) \in \mathbb{R}^n$ of $Av = 0$, where $\rho = 1, \dots, k$. Thus the row-reduced form of $Av = 0$ has k columns without pivots, which correspond to the free variables, and $n - k$ columns with pivots.

Due to the normalization condition, we have $\pi_1 = q_1 q_2^{b_{2,1}} \cdots q_n^{b_{n,1}}$, and any remaining power products $\pi_2, \dots, \pi_k$ involve only $q_2, \dots, q_n$. In view of this, we consider $A' \in \mathbb{R}^{m\times(n-1)}$, defined to be the submatrix of A obtained by omitting the first column, and $v' \in \mathbb{R}^{n-1}$, defined to be the subvector of v obtained by omitting the first entry. The assumption that $Av = 0$ has a full set of k independent solutions that satisfy the normalization condition implies that $A'v' = 0$ has precisely $k - 1$ independent solutions, and hence precisely as many free variables. Consequently, the row-reduced form of $A'v' = 0$ has $n - k$ columns with pivots, so that A' has rank $n - k$.

Let $q_1 = f(q_2, \dots, q_n)$ be given and consider an arbitrary change of units that changes $q_1, \dots, q_n$ into $\tilde{q}_1, \dots, \tilde{q}_n$, and note that $\tilde{q}_1 = f(\tilde{q}_2, \dots, \tilde{q}_n)$ by the unit-free assumption. In view of the above expression for π_1, we introduce the function

$$F(q_2, \dots, q_n) = f(q_2, \dots, q_n) q_2^{b_{2,1}} \cdots q_n^{b_{n,1}}$$

so that $\pi_1 = F(q_2, \dots, q_n)$. Similarly, beginning from the analogous expression for $\widetilde{\pi}_1$, we find $\widetilde{\pi}_1 = F(\tilde{q}_2, \dots, \tilde{q}_n)$. Because it is dimensionless, we have $\pi_1 = \widetilde{\pi}_1$, which implies $F(q_2, \dots, q_n) = F(\tilde{q}_2, \dots, \tilde{q}_n)$. Thus the function F is invariant under an arbitrary change of units.

To establish the result of the theorem, we consider different cases depending on the number k of power products. In the case when $k = 1$, we consider the change of unit relations $\tilde{q}_\sigma = q_\sigma \lambda_1^{a_{1,\sigma}} \cdots \lambda_m^{a_{m,\sigma}}$ for $\sigma = 2, \dots, n$, where $\lambda_1, \dots, \lambda_m$ are the conversion factors. From this we obtain the log-linear system $\ln(\tilde{q}_\sigma/q_\sigma) = a_{1,\sigma} \ln \lambda_1 + \cdots + a_{m,\sigma} \ln \lambda_m$.

In matrix form, we have $A'^T u = g'$, where $u = (\ln\lambda_1, \ldots, \ln\lambda_m) \in \mathbb{R}^m$ and $g' = (\ln(\tilde{q}_2/q_2), \ldots, \ln(\tilde{q}_n/q_n)) \in \mathbb{R}^{n-1}$. When $k = 1$, the rank of A' and A'^T is $n-1$, and the columns of A'^T span $\mathbb{R}^{n-1}$. Thus, for arbitrary old values $q_2, \ldots, q_n$, we can always find a change of units to obtain any specified new values $\tilde{q}_2, \ldots, \tilde{q}_n$, say a value of one for each. The required change of units can be found by setting $\tilde{q}_2 = 1, \ldots, \tilde{q}_n = 1$ in this log-linear system and solving for the conversion factors $\lambda_1, \ldots, \lambda_m$. Due to the invariance property of F, for arbitrary $q_2, \ldots, q_n$ we get $\pi_1 = F(1, \ldots, 1) = C$, where C is some fixed constant, which establishes the result for this case.

In the case when $k = n$, the system $A'v' = 0$ has $n-1$ independent solutions $v'_\rho = (b_{2,\rho}, \ldots, b_{n,\rho}) \in \mathbb{R}^{n-1}$ for $\rho = 2, \ldots, n$. Let $B' \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix whose columns are these solutions, and note that it is square and has full rank, and hence is invertible. For this case, we consider the power products $\pi_\rho = q_2^{b_{2,\rho}} \cdots q_n^{b_{n,\rho}}$ for $\rho = 2, \ldots, n$, and obtain the log-linear system $\ln\pi_\rho = b_{2,\rho}\ln q_2 + \cdots + b_{n,\rho}\ln q_n$. In matrix form, we have $B'^T w' = h'$, where $w' = (\ln q_2, \ldots, \ln q_n) \in \mathbb{R}^{n-1}$ and $h' = (\ln\pi_2, \ldots, \ln\pi_n) \in \mathbb{R}^{n-1}$. Since B'^T is invertible, we find that $q_2, \ldots, q_n$ can be uniquely expressed in terms of $\pi_2, \ldots, \pi_n$. This implies that $\pi_1 = F(q_2, \ldots, q_n) = \phi(\pi_2, \ldots, \pi_n)$, for some function ϕ, which establishes the result for this case.

In the case when $1 < k < n$, the system $A'v' = 0$ has $k-1$ independent solutions $v'_\rho = (b_{2,\rho}, \ldots, b_{n,\rho}) \in \mathbb{R}^{n-1}$ for $\rho = 2, \ldots, k$, and has rank $n-k$ as noted earlier. Without loss of generality, up to a reordering of $q_2, \ldots, q_n$, we may suppose that the pivots in the system $A'v' = 0$ all occur in the leading $n-k$ columns, whereas the free variables all occur in the latter $k-1$ columns. We now consider the $n-k$ change of unit relations $\tilde{q}_\sigma = q_\sigma \lambda_1^{a_{1,\sigma}} \cdots \lambda_m^{a_{m,\sigma}}$ for $\sigma = 2, \ldots, n-k+1$, and again consider the system $\ln(\tilde{q}_\sigma/q_\sigma) = a_{1,\sigma}\ln\lambda_1 + \cdots + a_{m,\sigma}\ln\lambda_m$. Since the dimensional exponent vectors $(a_{1,\sigma}, \ldots, a_{m,\sigma})$ are the leading $n-k$ columns of A', they are independent. Thus for arbitrary $q_2, \ldots, q_{n-k+1}$ we can find $\lambda_1, \ldots, \lambda_m$ to achieve $\tilde{q}_2 = 1, \ldots, \tilde{q}_{n-k+1} = 1$. We next consider the $k-1$ power products $\pi_\rho = q_2^{b_{2,\rho}} \cdots q_n^{b_{n,\rho}}$ for $\rho = 2, \ldots, k$. Since $\pi_\rho = \tilde{\pi}_\rho$ and $\tilde{q}_2 = 1, \ldots, \tilde{q}_{n-k+1} = 1$, we get the reduced expressions $\pi_\rho = \tilde{q}_{n-k+2}^{b_{n-k+2,\rho}} \cdots \tilde{q}_n^{b_{n,\rho}}$, which leads to the system $\ln\pi_\rho = b_{n-k+2,\rho}\ln\tilde{q}_{n-k+2} + \cdots + b_{n,\rho}\ln\tilde{q}_n$. For each ρ, we note that $(b_{n-k+2,\rho}, \ldots, b_{n,\rho})$ are the $k-1$ free variables from the system $A'v' = 0$, which were independently chosen to generate the solution set. Thus this log-linear system is square and has full rank, and we find that $\tilde{q}_{n-k+2}, \ldots, \tilde{q}_n$ can be uniquely expressed in terms of $\pi_2, \ldots, \pi_k$. This implies $\pi_1 = F(\tilde{q}_2, \ldots, \tilde{q}_n) = F(1, \ldots, 1, \tilde{q}_{n-k+2}, \ldots, \tilde{q}_n) = \phi(\pi_2, \ldots, \pi_k)$, for some function ϕ, which establishes the result. ∎

1.7. Case study

Setup. To illustrate the preceding results on dimensional methods, and the process of modelling a simple mechanical system, we study the motion of a pendulum released from rest. Figure 1.1 illustrates the system, which consists of a string of length ℓ, with one end attached to a fixed support point, and the other end attached to a ball of mass m. We assume the string is always in tension and hence straight, and we let θ denote the angle between the string and a vertical line through the support point, and arbitrarily take the positive direction to be counter-clockwise. We assume that gravitational acceleration g is directed in the downward, vertical direction. When the ball is raised

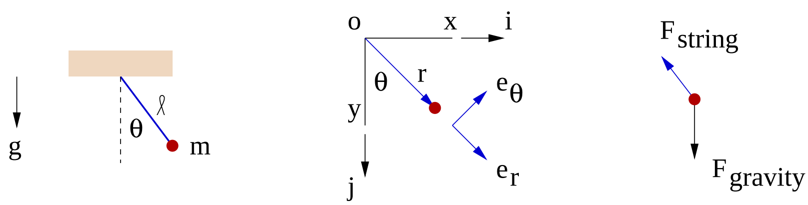


Figure 1.1.

and released from the rest conditions $\theta = \theta_0$ and $\frac{d\theta}{dt} = 0$ at time $t = 0$, the ball will swing back-and-forth in a periodic motion. We seek to understand various aspects of this motion; for example, how the period depends on the parameters m, g, ℓ, and θ_0.

Outline of model. We assume that the motion occurs in a plane and introduce an origin and x, y coordinates as shown. The standard unit vectors in the positive x and y directions are denoted by $\vec{i}$ and $\vec{j}$, and the position vector for the ball is denoted by $\vec{r}$. It will be convenient to introduce unit vectors $\vec{e}_r$ and $\vec{e}_\theta$ that are parallel and perpendicular to $\vec{r}$. For any angle θ, the components of these vectors are $\vec{r} = \ell \sin\theta\, \vec{i} + \ell\cos\theta\, \vec{j}$, $\vec{e}_r = \sin\theta\, \vec{i} + \cos\theta\, \vec{j}$, and $\vec{e}_\theta = \cos\theta\, \vec{i} - \sin\theta\, \vec{j}$. By differentiating the position with respect to time, we obtain the velocity and acceleration vectors

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \ell\cos\theta\frac{d\theta}{dt}\,\vec{i} - \ell\sin\theta\frac{d\theta}{dt}\,\vec{j},\\ \frac{d^2\vec{r}}{dt^2} &= \left[\ell\cos\theta\frac{d^2\theta}{dt^2} - \ell\sin\theta\left(\frac{d\theta}{dt}\right)^2\right]\vec{i} - \left[\ell\sin\theta\frac{d^2\theta}{dt^2} + \ell\cos\theta\left(\frac{d\theta}{dt}\right)^2\right]\vec{j}.\end{aligned} \tag{1.36}$$

We assume that only two forces act on the ball: one due to gravity, and another due to the pull of the string. Thus we neglect any other forces, such as that due to air resistance. The force of gravity has the form $\vec{F}_{\text{gravity}} = mg\,\vec{j}$, and the force in the string has the form $\vec{F}_{\text{string}} = -\lambda\,\vec{e}_r$, where λ is an unknown tension, which is nonconstant in general. Note that, although the magnitude of this force is unknown, its direction is known: it is always parallel to $\vec{e}_r$. Newton's law of motion for the ball requires that the product of its mass and acceleration be equal to the sum of the applied forces, or equivalently,

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{\text{gravity}} + \vec{F}_{\text{string}}. \tag{1.37}$$

To put the above equation in a concise form, and eliminate the unknown magnitude of $\vec{F}_{\text{string}}$, we consider the vector dot-product of the above with the unit vector $\vec{e}_\theta$, namely

$$m\frac{d^2\vec{r}}{dt^2}\cdot\vec{e}_\theta = \vec{F}_{\text{gravity}}\cdot\vec{e}_\theta + \vec{F}_{\text{string}}\cdot\vec{e}_\theta. \tag{1.38}$$

By direct calculation, using the component expressions for all vectors involved, and the facts that $\vec{i}\cdot\vec{i} = 1$, $\vec{j}\cdot\vec{j} = 1$ and $\vec{i}\cdot\vec{j} = 0$, and noting that $\vec{e}_r\cdot\vec{e}_\theta = 0$ because they are perpendicular, we obtain

$$\frac{d^2\vec{r}}{dt^2}\cdot\vec{e}_\theta = \ell\frac{d^2\theta}{dt^2}, \quad \vec{F}_{\text{gravity}}\cdot\vec{e}_\theta = -mg\sin\theta, \quad \vec{F}_{\text{string}}\cdot\vec{e}_\theta = 0. \tag{1.39}$$

By substituting (1.39) into (1.38), and dividing out the mass and rearranging, we arrive at a differential equation for the pendulum motion. When the release conditions at time $t = 0$ are included, we obtain

$$\ell \frac{d^2\theta}{dt^2} + g \sin\theta = 0, \quad \frac{d\theta}{dt}\Big|_{t=0} = 0, \quad \theta|_{t=0} = \theta_0, \quad t \geq 0. \tag{1.40}$$

The equations in (1.40) form a second-order, nonlinear, initial-value problem for the pendulum angle θ as a function of time t. This function also naturally depends on the parameters g, ℓ, and θ_0 that appear in the equations, and we note that the mass m was eliminated along the way. The theory of ordinary differential equations guarantees that there exists a unique solution $\theta = f(t, g, \ell, \theta_0)$, for some function f. Moreover, provided that the initial velocity is zero and the initial angle satisfies $\theta_0 \in (0, \pi)$, this solution will be periodic in time with a period $P = F(g, \ell, \theta_0)$, for some function F. Although they can be written in terms of certain special (elliptic) functions, there are no elementary expressions for f or F. Here we use dimensional methods to find a reduced form of the period relation and examine some implications.

Reduced equation for period. The quantities P, g, ℓ, θ_0 have dimensions $[P] = T$, $[g] = L/T^2$, $[\ell] = L$ and $[\theta_0] = 1$. A dimensional basis is $\{T, L\}$, and the dimensional exponent matrix in this basis is

$$A = (\Delta_P, \Delta_g, \Delta_\ell, \Delta_{\theta_0}) = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{1.41}$$

An arbitrary power product has the form $\pi = P^{b_1} g^{b_2} \ell^{b_3} \theta_0^{b_4}$. The equation $Av = 0$, where $v = (b_1, \ldots, b_4)$, has two free variables, and the general solution is

$$b_1 = -2b_3, \quad b_2 = -b_3, \quad b_3 \text{ and } b_4 \text{ free}. \tag{1.42}$$

Since there are two free variables, there are two independent solutions. For the first solution, we choose $b_3 = -1/2$ and $b_4 = 0$, which gives $\pi_1 = P\sqrt{g/\ell}$. For the second solution, we choose $b_3 = 0$ and $b_4 = 1$, which gives $\pi_2 = \theta_0$. This is a full set of independent dimensionless power products, and is normalized with respect to P.

By the π-theorem, the period equation $P = F(g, \ell, \theta_0)$ must be equivalent to

$$\pi_1 = \phi(\pi_2) \quad \text{or} \quad P = \sqrt{\frac{\ell}{g}}\ \phi(\theta_0), \tag{1.43}$$

for some function ϕ. Thus the relation between the quantities P, g, ℓ, θ_0 is not characterized by an unknown function of three quantities $F(g, \ell, \theta_0)$, but is instead characterized by an unknown function of one quantity $\phi(\theta_0)$. Equivalently, the dependence of $F(g, \ell, \theta_0)$ on the quantities g and ℓ is completely dictated by dimensional considerations.

Some implications. The reduced form of the period relation given in (1.43) has some interesting implications, as summarized next.

(1) A single curve of π_1 versus π_2 completely determines the function ϕ, and hence the relation between the quantities P, g, ℓ, and θ_0. Thus the goal of any experiment or further analysis should be aimed at determining this curve.

(2) Consider any two pendula released from the same initial angle θ_0. Let $\{g_1, \ell_1, \theta_0\}$ be the parameters of the first pendulum, and $\{g_2, \ell_2, \theta_0\}$ be the parameters of the second. In view of (1.43), the periods of the two pendula are given by $P_1 = \sqrt{\ell_1/g_1}\ \phi(\theta_0)$ and $P_2 = \sqrt{\ell_2/g_2}\ \phi(\theta_0)$. By dividing these two expressions, we obtain a fundamental **period law** for pendula, namely

$$\frac{P_1}{P_2} = \sqrt{\frac{\ell_1 g_2}{\ell_2 g_1}}. \tag{1.44}$$

(3) The dependence of the period P on each of the quantities g, ℓ, and θ_0 can be characterized using (1.43). Specifically, for fixed g and θ_0, the period P is an increasing function of ℓ; for fixed ℓ and θ_0, the period P is a decreasing function of g; and for fixed ℓ and g, the period P increases or decreases with θ_0 depending on the function ϕ.

A detailed analysis shows that the function $\phi(\theta_0)$, $\theta_0 \in (0, \pi)$ is positive, monotone, increasing, and has the limits

$$\lim_{\theta_0 \to 0^+} \phi(\theta_0) = 2\pi, \qquad \lim_{\theta_0 \to \pi^-} \phi(\theta_0) = \infty. \tag{1.45}$$

Thus the period satisfies $P \approx 2\pi\sqrt{\ell/g}$ for a pendulum released from rest with initial angle $\theta_0 \approx 0$, which corresponds to a nearly vertical, downward position. Interestingly, the period P is arbitrarily large for initial angles $\theta_0 \approx \pi$, which corresponds to a nearly vertical, upward position. The special cases of $\theta_0 = 0$ and $\theta_0 = \pi$ correspond to rest or equilibrium positions of the system in which no motion occurs. Such states and further properties of dynamical systems will be considered in later chapters.

Reference notes

Classic references for the material presented here are the books by Birkhoff (2015) and Bridgman (1963). A recent treatment with a wealth of details and examples is given in Szirtes (2007), and a concise guide that illustrates various diverse applications is given in Lemons (2017).

Exercises

1. Let x, t, a, b be quantities in units $\{\text{m}, \text{s}\}$, and let $\tilde{x}, \tilde{t}, \tilde{a}, \tilde{b}$ be the corresponding quantities in units $\{\text{cm}, \text{hr}\}$, with dimensions $[x] = L$, $[t] = T$, $[a] = L/T$ and $[b] = T$. Change the equation from x, t, a, b to $\tilde{x}, \tilde{t}, \tilde{a}, \tilde{b}$. Is the equation unit-free?

(a) $x = \dfrac{at^3 \arctan(t)}{(b + 4t)^2}$. (b) $x = \dfrac{at^2 \arctan(t/b)}{b + 4t}$.

2. Let x, y and p, q, r, s be quantities in units $\{\text{ft}, \text{lb}\}$ in the basis $\{L, M\}$. Also, let $v = \frac{dy}{dx}$. Assuming $[x] = L$ and $[y] = M$, find the dimensions $[p]$, $[q]$, $[r]$, $[s]$ as needed to make the given equation unit-free.

(a) $y = px^2 + q\sin(rx)$.

(b) $y = p\ln(sx) + \dfrac{q}{r+x^2}$.

(c) $y = (px + qx^2)e^{x/r}$.

(d) $y = \dfrac{p + qe^{rx^2}}{s+x}$.

(e) $v = px - qx^3 - r$.

(f) $v = py - qx^2 - rxy^2$.

3. Let P, Q, R, S be quantities in given units in a basis $\{D_1, D_2\}$. Show that $P = Q + R + S$ is unit-free if and only if $[P] = [Q] = [R] = [S]$.

4. Let $x, y, z > 0$ and $p, q, r > 0$ be quantities with the following dimensions in the basis (1.1):

$$[x] = L, \quad [y] = ML/T, \quad [z] = \Theta M/(LT),$$
$$[p] = L^2, \quad [q] = M/T, \quad [r] = M/\Theta.$$

Find a reduced form of the given equation assuming it is unit-free. If not possible, explain why.

(a) $x = f(y, q, r)$.

(b) $y = f(x, z, p)$.

(c) $z = f(x, y, r)$.

(d) $y = f(x, p, q)$.

5. Let $u, v, w > 0$ and $\alpha, \beta, \gamma, \delta > 0$ be quantities with the following dimensions in the basis (1.2):

$$[u] = C/T, \quad [v] = H/T, \quad [w] = P/T,$$
$$[\alpha] = C/H, \quad [\beta] = P/H, \quad [\gamma] = H/(CT), \quad [\delta] = 1/T.$$

Find a reduced form of the given equation assuming it is unit-free. If not possible, explain why.

(a) $u = f(v, w, \alpha, \beta)$.

(b) $v = f(w, \beta, \gamma, \delta)$.

(c) $w = f(\alpha, \beta, \gamma)$.

(d) $w = f(u, v, \alpha, \beta, \delta)$.

6. An experiment to measure the temperature u in a furnace at time t is performed. A curve fitting procedure applied to the u, t data yields the empirical relation $u = 3.7t^{1.5} + 4.2t + 293.2$ in units {kelvin, minutes}. Here we explore different ways to make this relation unit-free.

(a) Consider $u = at^{1.5} + bt + c$, where a, b, c are dimensional constants with values $3.7, 4.2, 293.2$ in units {kelvin, minutes}. What must be the dimensions of a, b, c so that the equation is unit-free? What would be the values of a, b, c in units {kelvin, hours}?

(b) Alternatively, consider $u = \beta[3.7(\frac{t}{\alpha})^{1.5} + 4.2\frac{t}{\alpha} + 293.2]$, where α and β are dimensional constants, with values $\alpha = 1$ minute and $\beta = 1$ kelvin, and 3.7, 4.2 and 293.2 are dimensionless (pure) numbers. Show that this form of the equation is also unit-free.

7. Data is collected on the height y of certain trees at time t during their lifetimes. A curve fitting procedure gives the empirical relation $y = 52.4 - 52.4e^{-0.1t} - 3.3te^{-0.2t}$ in units {foot, year}. Rewriting as $y = a - be^{-ct} - pte^{-qt}$, find the dimensions of a, b, c, p, q so that the equation is unit-free. Convert the equation to units {yard, decade}.

8. According to a simple theory of growth, the ultimate radius $r > 0$ of a single-celled organism is determined by the nutrient absorption rate $a > 0$ through its surface, and nutrient consumption rate $c > 0$ throughout its volume, via a unit-free equation $r = f(a, c)$. Find a reduced form of the relation using $[a] = M/(L^2T)$ and $[c] = M/(L^3T)$.

9. A metal forming process involves a pressure $P > 0$, length $x > 0$, time $t > 0$, mass $m > 0$ and density $\rho > 0$, and is described by a unit-free equation $P = f(x, t, m, \rho)$. Find a reduced form of this relation and express the result explicitly in terms of P. Recall that $[P] = M/(LT^2)$ and $[\rho] = M/L^3$.

10. A sphere of radius $r > 0$ is immersed in a fluid of density $\rho > 0$ and viscosity $\mu > 0$. When subject to a force $q > 0$, the sphere attains a terminal velocity $v > 0$. We suppose there is a unit-free equation $v = f(r, q, \rho, \mu)$, where $[q] = MLT^{-2}$, $[\rho] = ML^{-3}$, $[\mu] = ML^{-1}T^{-1}$.

(a) Find a reduced form of $v = f(r, q, \rho, \mu)$.

(b) For fixed q, ρ, μ, show that v is proportional to a power of r.

11. A model for the digestion process in animals states that the absorption rate $u > 0$ of a given nutrient is determined by the concentration $c > 0$, residence time $\tau > 0$, and breakdown rate $r > 0$ of the nutrient in the gut, along with the volume $v > 0$ of the gut. We suppose there is a unit-free equation $u = f(c, \tau, r, v)$, where $[u] = M/T$, $[c] = M/L^3$ and $[r] = M/(L^3T)$.

(a) Find a reduced form of $u = f(c, \tau, r, v)$.

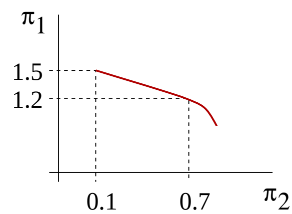
(b) For fixed c, τ, r, show that u is proportional to v.

12. In an explosion, a circular blast wave of intense pressure expands from the point of explosion into the surrounding air. A simple theory asserts that the radius $r > 0$ of the wave is determined by the elapsed time $t > 0$, the energy $E > 0$ released in the explosion, and the density $\rho > 0$ of the surrounding air, via a unit-free equation $r = f(t, E, \rho)$. Find a reduced form of this relation and show that the radius must increase with time in a nonlinear way; specifically, r increases as $t^{2/5}$. Here $[E] = ML^2/T^2$ and $[\rho] = M/L^3$.

13. In a domino toppling show, a long line of dominoes topple over, one by one, in a chain reaction. It is hypothesized that the speed $v > 0$ of the toppling wave depends on the spacing $d > 0$ and height $h > 0$ of each domino, and gravitational acceleration $g > 0$, via a unit-free equation $v = f(d, h, g)$. (The speed is assumed to be insensitive to the thickness and width of each domino.)

(a) Show that a reduced form of the speed equation is $v = \sqrt{gh}\,\phi(d/h)$ for some function ϕ.

(b) The figure below shows a plot of $\pi_1 = \phi(\pi_2)$, where $\pi_1 = v/\sqrt{gh}$ and $\pi_2 = d/h$, made with data from different domino experiments. Note that the graph is approximately linear on the interval $0.1 \leq \pi_2 \leq 0.7$. Find a linear expression for $\pi_1 = \phi(\pi_2)$ valid on this interval. Use this expression to write v in terms of d, h, g.



14. Drone airplanes of surface area s use fuel of energy content e to fly at velocity v through air of viscosity μ. A theory proposes that the fuel consumption rate k is determined by a unit-free equation $k = f(s, e, v, \mu)$, where $[k] = L^3T^{-1}$, $[e] = ML^{-1}T^{-2}$ and $[\mu] = ML^{-1}T^{-1}$.

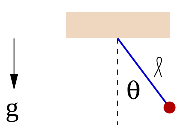
(a) Find a reduced form of $k = f(s, e, v, \mu)$.

(b) Using data from two experiments, and linear interpolation in a plot of the reduced form, find k when $(s, e, v, \mu) = (6, 3, 20, 1)$.

Data in some appropriate units

	s	e	v	μ	k
experiment 1:	5	2	10	1	1
experiment 2:	7	3	15	2	5

Mini-project. As developed in Section 1.7, a model for an ideal pendulum released from rest is



$$\ell \frac{d^2\theta}{dt^2} + g \sin\theta = 0, \quad \frac{d\theta}{dt}\Big|_{t=0} = 0, \quad \theta|_{t=0} = \theta_0, \quad t \geq 0.$$

Here θ is the pendulum angle, ℓ is the length, g is gravitational acceleration, and t is time. The above system has a unique solution $\theta = f(t, g, \ell, \theta_0)$, for some function f, and provided that the initial velocity is zero and the initial angle satisfies $\theta_0 \in (0, \pi)$, this solution will be periodic in time with a period $P = F(g, \ell, \theta_0)$, for some function F. Here we study the period relation and construct an approximate formula for it using some data. All quantities are in units of meters and seconds.

(a) As outlined in the text, show that the reduced form of the period relation is $P = \sqrt{\ell/g}\;\phi(\theta_0)$, for some function ϕ. Given that ϕ and F are both unknown, what is the conceptual advantage of the form $P = \sqrt{\ell/g}\ \phi(\theta_0)$ compared to the form $P = F(g, \ell, \theta_0)$?

(b) The table below shows experimental measurements of P for five pendula with different values of g, ℓ, and θ_0. Compute the value of ϕ for each case; make a table or plot of ϕ versus θ_0. Over the given interval of θ_0, what are the qualitative features of ϕ? Does the function appear to be increasing or decreasing? Concave up or concave down?

	g, m/s^2	ℓ, m	θ_0, rad	P, s
case 1:	9.80	0.20	$\pi/12$	0.9015
case 2:	9.80	0.10	$\pi/6$	0.6457
case 3:	9.80	0.50	$\pi/4$	1.4760
case 4:	9.80	0.30	$\pi/3$	1.1798
case 5:	9.80	0.25	$\pi/2$	1.1845

(c) Using your ϕ versus θ_0 table, and linear interpolation between entries, predict the period P for given parameter values $\{g, \ell, \theta_0\} = \{9.8, 0.5, \frac{5\pi}{12}\}$, $\{9.8, 0.7, \frac{5\pi}{12}\}$, $\{4.9, 0.6, \frac{\pi}{3}\}$. More generally, what would be an approximate formula for P, valid for any $\ell > 0$, $g > 0$ and $\theta_0 \in [\frac{\pi}{3}, \frac{\pi}{2}]$?

(d) Use Matlab or other similar software to numerically solve the pendulum differential equation, along with the initial conditions, to produce a plot of θ versus t. Run a simulation with each set of parameters $\{g, \ell, \theta_0\}$ from part (c) and directly estimate the period from the plot for each case. Do the estimates agree with the predictions from (c)?