Preface

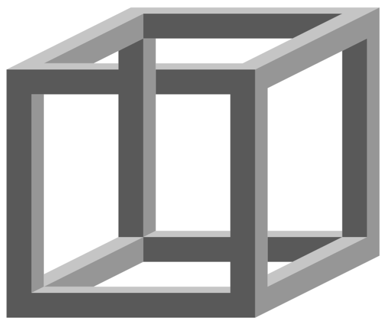

Figure 0.0.1. The impossible cube. In mathematics and in life, although it sounds contradictory, the impossible is possible. Credit: Wikimedia Commons, authors: Original—Maksim, vector—Boivie. Licensed under Creative Commons Attribution Share Alike 3.0 Unported (https://creativecommons.org/licenses/by-sa/3.0/deed.en) license.

Philosophy about learning and teaching

Lighten up about mathematics!

Have fun. What activities do people get good at without the process of improving being unpleasant? At the top of the list of such activities are perhaps *games*.[1] Why is that? Because they are fun! This leads us to the conclusion (inductive reasoning) that one of the main difficulties in exposition and teaching is to make the subject matter and presentation fun for the reader and student.

Walk the path. How do we teach infants how to speak? Well, we don't say: "This is the first rule of grammar: ...." Likewise, it doesn't make sense to start teaching mathematical proofs by saying: "This the first rule of proving theorems: ...."

[1] Disclaimer: Most of the author's childhood was spent playing sports, pinball, video games, and chess.

> There's a difference between knowing the path and walking the path.
> – Morpheus

If we have to choose, we would rather have you walk the path of proving theorems than know the path of proving theorems. But, fortunately, you do not have to choose and you can do both.

Make it natural and balanced. The best way to learn something is to make the process natural. To balance conscious thinking with subconscious thinking. Not to overthink things. To focus, but not to rush. To wander, but to come back. On one hand, ideas arise naturally, whose origins wherewith we do not know. On the other hand, ideas come from ideas. After all, imitation is the sincerest form of flattery!

Don't be formulaic. Cookbooks are comfortable to read. And by following them, we can make delicious meals! However, if we make this a cookbook on how to prove theorems, then aren't we doing the same thing as making a cookbook on how to integrate functions, except perhaps on a higher level?

Practical advice. These are our ideas, but we suggest that you take them with a grain of salt and follow your own path!

(1) You don't have to understand everything at once. Being confused and struggling are natural parts of learning.
(2) We learn by watching and we learn by doing. With respect to watching, this book may give more detailed proofs of elementary results than other books on mathematical reasoning. We compensate this by including lots of solved problems as well as exercises with hints given at the end of each chapter.
(3) Success is mostly based on passion, patience, and perseverance. In this sense, we support the Mamba Mentality or John Wick level of dedication, tempered with a relaxed attitude. If it is not for this book, then whatever you are interested in!
(4) No matter what our level is, we have the ability to greatly improve in learning a subject. Useful are:
    (a) Interest and curiosity in the subject. Ask questions while you are learning and reading!
    (b) Repetition. Imagine you are getting good at a video game. What do you do? Keep playing!
    (c) Watch and imitate (see the corresponding Yogi-isms). In the video game analogy, watch good players play and copy their techniques.

Content of this book

> Je suis désolé. ... I've heard it all before.
> – From "Sorry" by Madonna

A word to the wise: If you read this book, you will have to endure bad math puns and jokes and out-of-date pop culture references (we old-timers prefer to think of these as the classics).

Now that we got that out of the way, foremost, the material in this book is not original. Anyone potentially teaching from this book will have heard it all before. What distinguishes this book is more about what material we've selected, how we've decided to pedagogically present the material, and the informal style of writing while trying not to compromise the mathematical content of the book.

In this book we learn how to read and write mathematical proofs. One of the most beautiful subjects in mathematics is number theory. The subject of number theory ranges from the most elementary mathematics to the deepest mathematics. Number theory also provides an arena in which students already have a strong intuition for the objects of study, so this choice makes it possible to build on that existing knowledge. These properties make it an ideal subject for us to study. One reason why mathematics can be deep and advanced is that it can be abstract and complicated. We will see how abstraction can be introduced to simplify proofs and how, in order to understand concrete problems, we are forced to understand abstract notions. Complicated proofs can be broken down into smaller and far less complicated proofs. Deconstructing complicated proofs can be helpful in their understanding.

Some of the specific topics we cover are logic and implications, set theory, the arithmetic of integers, prime numbers, and algebraic structures.

Some of the themes in understanding mathematics we emphasize are conceptualization and visualization.

This book is suitable to be used as a quarter or semester college course on mathematical reasoning provided that one skips sections (and chapters!). The teacher should decide what sections to skip according to personal taste. We have put an asterisk * after certain sections which we feel may be of less priority.

Style of this book

To keep the discussion in the book lively, we do not always proceed in a linear manner. Indeed, you are likely familiar with many of the elementary rules and assumptions we use in mathematical reasoning. If we start from the beginning, you will likely be bored, at least temporarily. On the other hand, when we discuss concepts and methods that have not been introduced before, there is the risk you may not (at least initially) know what we are talking about. To remedy this, we will, when necessary or helpful, give forward references (i.e., references to material later in the book) or Wikipedia references (on the World Wide Web), which are also hyperlinked (such as the aforementioned World Wide Web and the word "hyperlinked"!) to missing items. This will facilitate looking up information in the e-version of this book.

We also include some awful jokes, bad puns, and esoteric popular culture references. Their intent is to make the book more lively, interesting, and broad. Mainly, we would like to encourage you to think about mathematics in your own way, especially writing your own proofs!

Problem solving

One of the best ways to learn mathematics (or just about anything for that matter!) is to do problems. Problems test our understanding. Problems force us to learn the material better. Problems challenge us. Problems are fun (or not, depending on your viewpoint!).

Polya [**Pol14**] has the following suggestions for "How to solve it":

(1) "You have to understand the problem."
(2) "Find the connection between the data and the unknown."
(3) "Carry out your plan."
(4) "Examine the solution obtained."

LaTeX

This book is written in LaTeX, which is a mathematics word processing software. You may consider learning LaTeX to be able to type up solutions to problems! Here is a sample of how it works. Consider the following statements:

I love the equation $e^{\pi i} + 1 = 0$. Beyond amazing is

$$\int_a^b f'(x)dx = f(b) - f(a). \tag{0.1}$$

Here is the LaTeX code we used to write this:

```
I love the equation $e^{\pi i} + 1 = 0$.  Beyond amazing is
\begin{equation}
    \int_a^b f'(x) dx = f(b) - f(a) .
\end{equation}
```

For the reader who is interested in learning LaTeX, an internet search will yield many sources. One source is https://www.overleaf.com/learn.

Origins

This book started as notes for a mathematical proofs class we taught for many years at the University of California San Diego using the book by Eccles [**Ecc97**]. The first few years, we used the book by Fletcher and Patty [**FP96**]. As such, this book is largely influenced by Eccles's book.

Further reading

There are many directions the student may pursue after, or even before(!), reading this book. Below are a handful of classics, out of the many wonderful mathematics books in the literature:

- Munkres, James R., *Topology. Second edition.* Prentice Hall, Inc., Upper Saddle River, NJ, 2000. xvi + 537 pp.
- Massey, William S., *Algebraic topology: An introduction.* Harcourt, Brace & World, Inc., New York, 1967. xix + 261 pp.
- Rudin, Walter, *Principles of mathematical analysis. Third edition.* International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976. x + 342 pp.

- Spivak, Michael, *Calculus on manifolds. A modern approach to classical theorems of advanced calculus.* W. A. Benjamin, Inc., New York-Amsterdam, 1965. xii + 144 pp.
- Ahlfors, Lars V., *Complex analysis. An introduction to the theory of analytic functions of one complex variable. Third edition.* International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978. xi + 331 pp.
- Serre, Jean-Pierre, *A course in arithmetic.* Translated from the French. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973. viii + 115 pp.

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