Preface

This book is primarily intended for college mathematics students who enjoyed high-school geometry and who wish to learn more about the amazing properties of lines, circles, triangles, and other geometric figures. In particular, I hope that those who are preparing to become high-school mathematics teachers will find inspiration here to help them share their enthusiasm and enjoyment of geometry with their own future students.

But why should anyone study geometry? One reason, of course, is that geometry and its descendant, trigonometry, are essential tools in engineering, architecture, navigation, and other disciplines. These practical applications, however, surely do not explain why it is that for centuries, geometry has been taught to almost every student and why, at least until recently, a person who knew no geometry was not considered to be properly educated. I think there are at least two reasons more important than usefulness that explain why geometry has been, and should continue to be, a part of the school curriculum.

Since Euclid, some 2300 years ago, geometry has been taught as a deductive science, with theorems and proofs. As a consequence, generations of geometry students have learned how to draw valid conclusions from hypotheses and how to detect and avoid invalid reasoning. In other words, by studying geometry, students can learn how to think. Of course, there are other subjects that could also be used to teach deductive reasoning, but geometry is especially effective because it seems to have the perfect balance of depth and concreteness. Many of the theorems that we prove in geometry are deep in the sense that they assert something nonobvious, and sometimes even surprising. They are concrete because students can easily draw the appropriate triangles, circles, or whatever, and they can see that what is alleged to happen actually does appear to happen.

Geometry is also beautiful, and some of its theorems are so amazing as to seem almost miraculous. In fact, much the same comment could be made about most areas of mathematics, but geometry is unique in that its "miracles" are visual, so they can readily be appreciated, even by the uninitiated. Surely, one does not need a great deal of mathematical sophistication to marvel at the fact that if a line is drawn through each vertex of any triangle, and if each of these lines is perpendicular to the opposite side of the triangle, then these three lines all go through a common point. One could argue that the fundamental theorem of calculus, for example, is equally amazing and beautiful, but unfortunately, it is not possible to appreciate it without first studying calculus. But the aesthetic value of geometry extends beyond the striking statements of its theorems.

Many of the proofs have their own subtle and elegant beauty, which, unfortunately, is a little harder to perceive. Nevertheless, we expect that most readers of this book will learn to enjoy the beauty of the proofs as well as that of the theorems.

In short, we should continue to study and teach geometry because it is a highly attractive subject and because we can learn from it something about deductive reasoning and the nature of mathematical proof. It seems clear, therefore, that as in the past, students should continue to see theorems being proved in their geometry class; they should be taught how to understand proofs, and perhaps even more important, they should learn how to invent and write proofs.

I have selected for this book some of the more spectacular theorems in plane geometry, and I have presented justifications of these facts using a variety of different techniques of proof. Mostly ignored, however, are the kind of unsurprising theorem which, while required by modern standards of mathematical rigor, can seem rather pointless to students. It requires considerable sophistication to appreciate why anyone would want to prove a fact that seems completely obvious. Even professional mathematicians, most of whom do understand the significance of these results, often find their formal axiomatic proofs somewhat dull. Learning proofs should not be an unrewarding chore; instead, we expect that students will *demand* proofs because the assertions being established are otherwise so incredible.

This book was written as a text for College Geometry, which is a course that I have taught several times at the University of Wisconsin, Madison. The course's principal audience consists of sophomore and junior undergraduate math majors who are specializing in secondary education. For them, the course is required, but there is also a substantial minority who take the course electively. Some of these students simply want to learn geometry, while others take College Geometry because they find it to be a more gentle and accessible introduction to mathematical proof than a course in abstract algebra or advanced calculus. It is my belief that for some students, the study of geometry is an excellent preparation for these more difficult abstract courses.

I have been dissatisfied with the available texts that might be used for such a course. Many include a number of interesting topics within a large sampling of assorted geometric material. But they do not put the focus where I think it belongs: on the really pretty theorems and their proofs that, in my opinion, should be at the heart of a geometry course for college students. Also, they often devote much more space than I think is appropriate to formalism and axiomatics. Most students probably do not find this especially exciting, and I share that opinion.

There also exist some wonderful books that are filled with spectacular theorems and elegant proofs, but none of these seems quite suitable as a text for this geometry course either. I have found that most students who register for College Geometry claim to remember very little from their one previous exposure to geometry in high school. For the sake of this majority, some review and acclimatization are necessary. A text is needed, therefore, that starts from a point close to the beginning and introduces the notion of proof gently, and then gets to the "good stuff" as quickly as possible. Because I could not find a book that covered the right material at the right pace, and that started from the right place, I taught the course many times without a text.

Most of my students seemed to enjoy the course, and many of them became very excited about geometry. In this book, which is an expansion of my course lecture notes, I have tried to reproduce as closely as possible the experience of the classroom, and so I hope that my readers will also find that geometry is an enjoyable and exciting subject. Most of all, I hope that those of my students and readers who are or will be teaching high-school mathematics will convey some of that excitement to their own students.

I am grateful to the following reviewers for their helpful comments:

Fred Flener, Northeastern Illinois University
Alan Hoffer, University of California, Irvine
Kathryn Lenz, University of Minnesota, Duluth
David Poole, Trent University
Ron Solomon, Ohio State University
Larry Sowder, San Diego State University
Alex Turull, University of Florida, Gainsville
Jeanne Wald, Michigan State University

I. Martin Isaacs