
PREFACE

There are numerous excellent textbooks presenting the basics of abstract algebra for college students. Why write another?

Most of the existing texts have a foundational character. They seem designed to lay a firm foundation for future graduate courses in abstract algebra. This is a noble goal and well served by many of the exemplars. There is however a large audience for an undergraduate abstract algebra course consisting of students who will likely never take a graduate course in abstract algebra. Notable among these are the future high school mathematics teachers. These students are better served by a course that emphasizes the roots of abstract algebra, which live in the rich soil of high school mathematics—Euclidean geometry, polynomial algebra, and trigonometry. Out of this soil spring naturally the concepts of symmetry, the complex numbers and the cyclotomic number fields, eventually blossoming into the Galois theory of equations.

The intention of this text is to emphasize the organic and historical development of the abstract theory of groups, rings, and fields from the substrate of high school mathematics. In Part I the “history” is fictitious. It is only with imaginative hindsight that we can attribute the concept of a group of motions to Euclid. In the later parts, however, the history is genuine, although the notation and terminology is updated.

Novel and exciting ideas and theorems are encountered early and often—the 2-dimensional symmetry groups in Chapter 3, Cardano’s formulas in Chapter 5, the complex numbers in Chapter 6, the Fundamental Theorem of Algebra in Chapter 7 and the 3-dimensional symmetry groups in Chapter 8. Section 3 on Number Theory features the work of Fermat. Not only his Little Theorem but also the Two Squares and Four Squares Theorems, as well as some cases of the celebrated Last Theorem and its polynomial analogue, are presented. The final Grand Synthesis section begins with a careful treatment of Gauss’ proof of the straight-edge-and-compass constructibility of the regular 17-sided polygon, and culminates with Galois’ theory of equations. Constructibility of regular polygons is a lovely topic, wonderfully down-to-earth and visual, yet laden with deep connections to subtle topics in number theory and group theory. A course that ends with Chapter 16 (even omitting some of the earlier material) will have presented a rich array of ideas to the students, all closely tied to the most elementary of questions in Euclidean geometry and the study of numbers and polynomial equations. At the end of the Introduction, I discuss some possible syllabi for semester-long and year-long courses.

Typically, Galois Theory appears as the grand finale and *raison d’être* for a first course in abstract algebra. But all too often there is not enough time to reach the finale

or to give it the attention it deserves. The abstract algebra course then becomes a series of complicated finger exercises with no beautiful sonata to play at the end. Here, too, Galois Theory is the grand finale. But along the way, the students get to play many lovely preludes, nocturnes and sonatinas; so even if the final sonata is never reached, the journey will have been filled with lovely music.

Besides giving a more organic and evolutionary development of the subject, it is the intention of this text to emphasize the connections within algebra and between algebra and other areas of mathematics, especially geometry. The different fields of mathematics are not hermetically sealed off from each other. Quite the contrary, most of the truly important achievements in mathematics have been the product of fruitful interaction of areas.

A textbook is at best a learning aid and at worst a stumbling block. Learning occurs on the dynamic interface between teacher and student. I have had the good fortune to have been inspired by many superb teachers, and wish to acknowledge a few here: Blossom Backal, who taught me high school geometry and first opened my eyes to the beauty of mathematics; Ralph G. Archibald, who taught me number theory and introduced me to mathematical research. David Goldschmidt, who taught me local group theory and gave me glimpses of a truly deep thinker at work; and my thesis advisor, Walter Feit, who taught me representation theory and forced me to figure out for myself why $V = [V, A] \oplus C_V(A)$ is Fitting's Lemma,

With specific regard to this book, I am deeply indebted to Keith Conrad, David Pollack and Inna Korchagina for their comments, corrections and encouragement. Had I saved all of their corrections more systematically, this book would have far fewer errors than it does. Also I and the book owe much to the students I have taught (and learned from) over the years of developing this material. I mention only two representatives of the many who deserve thanks: Jon Spier and Jason Petry.

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