

# CONTENTS

<b>1. Introduction</b> .....	1
<b>Part I. The dual group of a spherical variety</b> .....	19
<b>2. Review of spherical varieties</b> .....	21
2.1. Invariants .....	22
2.2. The dual group of a spherical variety .....	25
2.3. Toroidal compactifications .....	29
2.4. Normal bundles and boundary degenerations .....	33
2.5. Degeneration to the normal bundle; affine degeneration .....	37
2.6. Whittaker-type induction .....	41
2.7. Levi varieties .....	47
2.8. Horocycle space .....	48
2.9. The example of $\mathbf{PGL}(V)$ as a $\mathbf{PGL}(V) \times \mathbf{PGL}(V)$ variety .....	50
<b>3. Proofs of the results on the dual group</b> .....	55
3.1. The root datum of a spherical variety .....	55
3.2. Distinguished morphisms .....	59
3.3. The work of Gaitsgory and Nadler .....	60
3.4. Uniqueness of a distinguished morphism .....	63
3.5. The identification of the dual group .....	65
3.6. Commuting $SL_2$ .....	67
<b>Part II. Local theory and the Ichino-Ikeda conjecture</b> .....	69
<b>4. Geometry over a local field</b> .....	71
4.1. Measures .....	71
4.2. The measure on $X_\Theta$ .....	72
4.3. Exponential map .....	74

<b>5. Asymptotics</b>	81
5.1. The main result	81
5.2. Proof of asymptotics	86
5.3. Cartan decomposition and matrix coefficients	93
5.4. Mackey theory, the Radon transform and asymptotics	98
5.5. Support of elements in $e_{\mathfrak{g}}^*(C_c^\infty(X))$	103
<b>6. Strongly tempered varieties</b>	109
6.1. Abstract Plancherel decomposition	109
6.2. Definition; the canonical hermitian form	112
6.3. The Whittaker case and the Lapid-Mao conjecture	115
6.4. The Ichino-Ikeda conjecture	123
<b>Part III. Spectral decomposition and scattering theory</b>	131
<b>7. Results</b>	133
7.1. Plancherel decomposition and direct integrals of Hilbert spaces	134
7.2. Discrete spectrum	135
7.3. Main result	136
<b>8. Two toy models: the global picture and semi-infinite matrices</b>	139
8.1. Global picture	139
8.2. Spectra of semi-infinite matrices: scattering theory on $\mathbb{N}$	141
<b>9. The discrete spectrum</b>	149
9.1. Decomposition according to the center	149
9.2. A finiteness result	151
9.3. Variation with the central character	155
9.4. Toric families of relative discrete series	159
9.5. Unfolding	166
<b>10. Preliminaries to the Bernstein morphisms: “linear algebra”</b>	177
10.1. Basic definitions	177
10.2. Finite and polynomial functions	178
10.3. Hermitian forms	182
10.4. Measurability of eigenprojections	185
<b>11. The Bernstein morphisms</b>	189
11.1. Main result	189
11.2. Harish-Chandra-Schwartz space and temperedness of exponents	191
11.3. Plancherel formula for $X_{\Theta}$ from Plancherel formula for $X$	195
11.4. The Bernstein maps. Equivalence with Theorem 11.1.2	199
11.5. Property characterizing the Bernstein maps	201
11.6. Compatibility with composition and inductive structure of $L^2(X)$	203
11.7. Isometry	204

<b>12. Preliminaries to scattering (I): direct integrals and norms</b> .....	207
12.1. General properties of the Plancherel decomposition .....	207
12.2. Norms on direct integrals of Hilbert spaces .....	210
<b>13. Preliminaries to scattering (II): consequences of the conjecture on discrete series</b> .....	219
13.1. Isogenies of tori and affine maps on their character groups .....	219
13.2. Relationship between central characters for $X_\Theta$ and $X_\Omega$ .....	223
13.3. Canonical decomposition of maps $L^2(X_\Theta) \rightarrow L^2(X_\Omega)$ .....	226
<b>14. Scattering theory</b> .....	231
14.1. Introduction .....	231
14.2. Generic injectivity of the map $\mathfrak{a}_X^*/W_X \rightarrow \mathfrak{a}^*/W$ .....	232
14.3. The scattering theorem .....	234
14.4. Proof of the first part of Theorem 14.3.1 .....	240
14.5. Estimates .....	245
14.6. Proof of the second part of Theorem 14.3.1 .....	251
<b>15. Explicit Plancherel formula</b> .....	255
15.1. Goals .....	255
15.2. Various spaces of coinvariants .....	258
15.3. Convergence issues and affine embeddings .....	263
15.4. Normalized Eisenstein integrals and smooth asymptotics .....	273
15.5. The canonical quotient and the small Mackey restriction .....	278
15.6. Unitary asymptotics (Bernstein maps) .....	281
15.7. The group case .....	286
<b>Part IV. Conjectures</b> .....	291
<b>16. The local <math>X</math>-distinguished spectrum</b> .....	293
16.1. Recollection of the Arthur conjectures .....	293
16.2. The conjecture on the local spectrum (weak form) .....	296
16.3. A global to local argument .....	298
16.4. Proof of Theorem 16.3.2 .....	302
16.5. Pure inner forms .....	305
<b>17. Speculation on a global period formula</b> .....	311
17.1. Tamagawa measure .....	312
17.2. Factorization and the Ichino-Ikeda conjecture .....	312
17.3. Local prerequisites for the conjecture .....	314
17.4. Global conjecture .....	317
17.5. How to understand the Euler product .....	318
17.6. Everywhere discrete or unramified .....	322

<b>18. Examples</b> .....	325
18.1. Principal Eisenstein periods .....	325
18.2. Parabolic periods .....	331
18.3. The Whittaker case for $GL_n$ .....	334
18.4. Compatibility of the conjecture with unfolding .....	339
<b>A. Prime rank one spherical varieties</b> .....	345
A.1. Goals .....	345
A.2. Lie algebra versions .....	346
A.3. Reductions for the existence statement .....	346
A.4. Reductions for the uniqueness statement .....	348
A.5. Further discussion of the existence result .....	350
A.6. Proof of Lemma A.4.2 .....	351
<b>Bibliography</b> .....	353