Contents

Preface xi

1 Introduction to inverse problems 1
- 1.1 Two integral equations . . . 2
- 1.2 A forward problem/inverse problem pair . . . 7
  - 1.2.1 Forward versus inverse problems . . . 10

2 Well-posed, ill-posed, and inverse problems 15
- 2.1 Well-posed problems . . . 16
- 2.2 Ill-posed problems, part 1 . . . 18
  - 2.2.1 The uniqueness condition and the minimum-norm solution . . . 18
  - 2.2.2 The existence condition and least-squares solutions 21
- 2.3 Ill-posed problems, part 2 . . . 27
  - 2.3.1 Stability and the generalized inverse . . . 27
  - 2.3.2 Inverse problems . . . 32
  - 2.3.3 History of the concept of inverse problems . . . 33
  - 2.3.4 An example: medical imaging and computed tomography . . . 34
- 2.4 Discretized inverse problems . . . 37
  - 2.4.1 Ill-conditioned matrix-vector equations . . . 37
  - 2.4.2 Discretization of inverse problems . . . 42

3 Tikhonov regularization 49
- 3.1 Existence of the Tikhonov solution . . . 52
- 3.2 Convergence of Tikhonov regularization . . . 60

3.3 Convergence for noisy data . . . 70
3.4 Rates of convergence . . . 73
3.4.1 Examples . . . 79
3.5 Parameter choice rules . . . 84
3.5.1 The discrepancy principle . . . 87
3.5.2 The L-curve method . . . 95
3.6 Converse results . . . 98
3.6.1 Preliminary results . . . 100
3.6.2 The convergence of Tikhonov regularization can be arbitrarily slow . . . 102
3.6.3 Tikhonov regularization can fail to converge if the regularization parameter is not chosen properly . . . 103
3.6.4 Converse results about the rate of convergence . . . 114
3.6.5 The rate of convergence in the discrepancy principle . . . 118
3.6.6 Summary . . . 121

4 Compact operators and the singular value expansion 123
4.1 Finite-dimensional problems and the singular value decomposition . . . 123
4.1.1 An example with a symmetric matrix . . . 124
4.1.2 The SVD of a matrix . . . 132
4.1.3 Using the SVD to solve a discrete inverse problem 134
4.2 Compact operators . . . 137
4.3 The spectral theorem for a compact self-adjoint operator 143
4.3.1 Orthogonal sets in Hilbert space . . . 145
4.3.2 Examples of complete orthonormal sets . . . 151
4.3.3 The spectral theorem . . . 156
4.4 The singular value expansion of a compact operator . . . 163
4.4.1 Integral operators of the first kind . . . 170
4.4.2 Integral operators of the second kind . . . 180
4.5 The generalized inverse in terms of the SVE . . . 182
4.5.1 Interpretation of the series representation of $T^\dagger$ . . . 188
4.6 Tikhonov regularization in terms of the SVE . . . 197

4.7 Convergence of Tikhonov regularization via the SVE . . 202
4.7.1 Rates of convergence 206
4.7.2 Another converse result 208
4.8 General regularization methods 210
4.8.1 Functions of T^*T 210
4.8.2 An abstract regularization method 213
4.8.3 Rates of convergence 215
4.8.4 Asymptotic regularization 219

5 Tikhonov regularization with seminorms 225
5.1 Densely defined operators 226
5.1.1 The adjoint of a closed densely defined operator . 229
5.1.2 The graph norm and L^*L 231
5.1.3 An example: the derivative operator 236
5.2 Best approximate solutions using seminorms 239
5.3 Tikhonov regularization with seminorms 254
5.4 Convergence for exact data 257
5.5 Rates of convergence 264

Epilogue 271

A Basic Hilbert space theory 279
A.1 Hilbert spaces and linear operators 279
A.2 Subspaces of Hilbert space 284
A.3 Linear functionals 288
A.4 The adjoint of a linear operator 290
A.5 The open mapping theorem and the uniform boundedness principle 294
A.6 Weak sequential convergence in Hilbert space 301

B Sobolev spaces 307
B.1 Sobolev spaces on an interval 307
B.2 Sobolev spaces in higher dimensions 313

Bibliography 317
Index 319
About the Author 321