Preface

> [Mathematics] is security. Certainty. Truth. Beauty. Insight. Structure. Architecture. I see mathematics, the part of human knowledge that I call mathematics, as one thing—one great, glorious thing. Whether it is differential topology, or functional analysis, or homological algebra, it is all one thing.... They are intimately interconnected, and they are all facets of the same thing. That interconnection, that architecture, is secure truth and is beauty. That's what mathematics is to me.
>
> –Paul Halmos, *Celebrating 50 Years of Mathematics*

Discovering surprising connections between one area of mathematics and another area is not only exciting, it is useful. Laplace transforms allow us to switch between differential equations and algebraic equations, the fundamental theorem of algebra can be proved using complex analysis, and group theory plays a fundamental role in the study of public-key cryptography. If we can think of an object from different angles, these different perspectives aid our understanding. It is in that vein that we have written this book: We look at three seemingly different stories and a hidden connection between them.

The first of our stories is complex—that is, it is about complex function theory. In complex analysis, linear fractional transformations are among the first functions that we study. We understand their geometric and function-theoretic properties, and we know that they play an important role in the study of analytic functions. In the class of linear fractional transformations, the automorphisms of the unit disk stand out for the geometric and analytic insight they provide. The functions that we focus on are a natural generalization of the disk automorphisms; they are finite products of these automorphisms and are called Blaschke products, in honor of the German geometer Wilhelm Blaschke. While we know, for example, that automorphisms of the disk map circles to circles (if we agree that lines are circles), much less is known about the geometry of Blaschke products.

The second story comes from projective geometry. Here we focus on Poncelet's closure theorem, a beautiful geometric result about conics inscribed in polygons that are themselves inscribed in conics. We devote some attention to precursors of Poncelet's theorem, including Chapple's formula and Fuss's theorem. In the final part of this book we will also consider variations on this theme, including Steiner's porism.

The final story is about the numerical range of a matrix. The numerical range is the range of a quadratic form associated with a matrix but restricted to the unit sphere of $\mathbb{C}^n$. The numerical range always contains the eigenvalues of the matrix, and it often can tell you more about a matrix than the eigenvalues can. While questions about the numerical ranges of 2×2 matrices are (relatively) easily analyzed, consideration of $n \times n$ matrices for $n > 2$ is deeper. Even 3×3 matrices lead to deep and interesting questions!

What is the hidden connection between these three stories?

Curiously, the connection is an object we have not discussed much yet: It is the ellipse. In fact, this book is really a story about the ellipse, an object studied by the Greek mathematicians Menaechmus, Euclid, Appolonius of Perga, and Pappus, among others. Later, Kepler developed a description of planetary motion, leading to his first law: The orbit of a planet is elliptical with one focus at the Sun. In spite of this long history, ellipses never cease to surprise us.

It turns out that ellipses provide a remarkable amount of information about Blaschke products. Ellipses can tell us when a Blaschke product has a nontrivial factorization with respect to composition. And when we study the dynamics of Blaschke products, ellipses can tell us what to expect. The ellipse even establishes a connection between Blaschke products and a useful tool for detecting tax fraud, known as Benford's law. But why does an ellipse know anything about these particular rational functions? That is the story we wish to tell, and it is a story that relies heavily on properties of the numerical range of a class of operators and that story, in turn, relies on projective geometry.

The first part of this book (Chapter 1 through Chapter 7) focuses on Blaschke products that are products of three disk automorphisms, ellipses that are inscribed in triangles, and 2×2 matrices. In this case, ellipses suggest directions to study. Each Blaschke product of degree 3 is naturally associated with an ellipse, as is each 2×2 matrix. The relationship between Blaschke products, 2×2 matrices, and Poncelet's theorem will be revealed in Chapter 7. This part of the book also introduces the basic ideas of two-dimensional projective geometry and provides a proof of Poncelet's theorem in the event that the circumscribing polygons are triangles. After concluding Part 1, the reader will be treated to an intermezzo (Chapter 8), as we focus on the surprising connection between Poncelet's theorem and Benford's law.

In the second part of the book (Chapter 9 through Chapter 14) we consider the connection between ellipses and Blaschke products that are products of more than three analytic disk automorphisms. We will see that though the boundaries of the numerical ranges of the matrices we study need not be elliptical, the boundaries always satisfy a Poncelet-like property. In addition, when the boundary is elliptical, it provides insight into the function-theoretic behavior of the corresponding Blaschke product. Because Poncelet's ideas are intimately connected to our presentation, we provide a recent and beautiful proof of the general theorem. This part of the book is also a bit more challenging than the first part, mathematically speaking, but we strive for a self-contained treatment, providing appropriate proofs and references.

Finally, the third part of the book provides a range of exercises and projects, from straightforward exercises for someone first entering the field to potential research problems. Each chapter has a corresponding project that is usually introduced with new material. To be prepared to work through a project, active reading is required. Projects include things we find particularly interesting; for example, we discuss Sendov's conjecture, interpolation, inversion, and the spectral radius. In addition, we provide suggestions for the creation of several algorithms that will enhance one's understanding. We hope that the reader will pick one (or more) of these projects and use it (or them) to develop a deeper appreciation of the subject, to write an honors thesis, or to begin a research project. The necessary background and a starter bibliography for each project is provided, but we encourage readers to do a thorough search of the literature.

Using this book. The first two parts of the book are meant to tell the story we have just described; a story that is meant to be read. However, this book can be used in various ways. It can serve as a reference for independent study, as the text for a capstone course in mathematics, or as a reference for a researcher. We have written this book for an active reader—paper and pencil in hand—and though there are a few exercises embedded in this reading, it is Part 3 of the text that includes a significant number of exercises as well as projects. These projects provide a range of both material and depth by including exercises well within the reach of every reader, suggested research papers, and research problems that have been open for many years.

Applets. There are several interactive applets designed to go along with the book, and we encourage using the applets while reading. If we feel that the time is ripe for experimentation, we will direct the reader to the applet using the symbol ☺.[1] Our applets illustrate various results in the text, such as the main result on the connection of Blaschke products to ellipses. The proof of Poncelet's theorem that we present depends on Pascal's and Brianchon's theorem, and we provide applets associated with each of these results. In addition, we have applets that illustrate the mapping behavior of Blaschke products (including locating fixed points in the closed unit disk), composition with Blaschke products, and these applets also produce the numerical range of particular matrices. It has been our experience that these tools are not only fun to work with, they also enhance one's understanding and are great for testing conjectures.

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[1] `http://pubapps.bucknell.edu/static/aeshaffer/v1/`

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