Chapter 1

Blocks, sequences, bow ties, and worms

1.1 Langford sequences

Even before learning the numbers, children like to arrange objects in a line. That seems like a perfect way to begin our combinatorial journey, so let's look in on a child playing with blocks. Writing in 1958 [**125**], Dudley Langford tells us that one day, years ago, "... my son, then a little boy, was playing with some coloured blocks. There were two of each colour ...," and here is what Langford saw.

The boy arranged them with one block between the two red ones,

R		**R**

two blocks between the two blue ones,

R	**B**	**R**		**B**

and three blocks between the two yellow ones.

Y	**R**	**B**	**R**	**Y**	**B**

Replace the colors by the number of blocks between,

3	**1**	**2**	**1**	**3**	**2**

and we have an arrangement of a pair of 1's, a pair of 2's, and a pair of 3's in which the two 1's are one unit apart, the two 2's are two units apart, and the two 3's are three units apart. We now call this arrangement a *Langford sequence* or a *Langford arrangement* of order 3. More generally, a *Langford sequence of order n* is an arrangement of two copies of each of the integers $1, 2, \ldots, n$ such that for each k, the pair of k's are separated by k intervening integers. Later in 1958, Bang [**10**] showed that there is a Langford sequence of order n if and only if $n \equiv 0$ or $-1 \pmod 4$. (Recall that $a \equiv b \pmod m$ means that $a - b$ is a multiple of m.)

The proof that Langford sequences of order n exist when $n \equiv 0$ or $-1 \pmod 4$ consists of giving an algorithm for producing them. Dave Moore's website `https://dialectrix.com/langford/dave-moore.html` contains a description of how to construct Langford sequences. We now give a proof that the congruence condition is necessary.

Theorem 1.1.1. *Let n be a positive integer for which there exists a Langford sequence of order n. Then* $n \equiv 0$ *or* $-1 \pmod 4$.

Proof. Suppose there is a Langford sequence of order n. There are $2n$ positions in this sequence, namely $1, 2, \ldots, 2n$; we sum these positions in two ways.

First, we see that $1 + 2 + \cdots + 2n = 2n(2n+1)/2 = 2n^2 + n$.

Second, for $1 \le j \le n$, the left position L_j and right position R_j of j differ by $j + 1$, because L_j and R_j are separated by j integers. Hence,

$$\begin{aligned} 2n^2 + n &= \sum_{j=1}^{n} L_j + R_j = \sum_{j=1}^{n} L_j + \sum_{j=1}^{n} (L_j + j + 1) \\ &= 2\sum_{j=1}^{n} L_j + \sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\ &= 2\sum_{j=1}^{n} L_j + \frac{n(n+1)}{2} + n. \end{aligned}$$

Thus $n(n+1)/2 = 2n^2 - 2\sum_{j=1}^{n} L_j$, and so $n(n+1)/2$ is an even integer, which happens only for $n \equiv 0$ or $-1 \pmod 4$. □

The problem generalizes for triples, quadruples, etc., in a straightforward way. Thus, a *Langford* (s, n)*-sequence* is a sequence consisting of s

appearances of each i, for $1 \le i \le n$, in which consecutive occurrences of i are separated by i elements of the sequence. In this case only necessary conditions on s for a solution to exist are known. For example, if a $(3, n)$ sequence exists, then $n \equiv -1, 0, \text{or } 1 \pmod 9$—see [**166**] for further details. For example, here is a sequence of triples $i \cdots i \cdots i$ $(1 \le i \le 9)$ arranged such that consecutive appearances of the number d are separated by d numbers:

1 9 1 2 1 8 2 4 6 2 7 9 4 5 8 6 3 4 7 5 3 9 6 8 3 5 7.

The study of Langford (s, n) sequences and variations thereof continues to be an active area of research. John Dillon's paper [**60**] is a good place to start.

1.2 Partitioning sets of integers

Here is a variation on the preceding theme. To the string of blocks

Y	**R**	**B**	**R**	**Y**	**B**

,

we add two green ones with no blocks between,

Y	**R**	**B**	**R**	**Y**	**B**	**G**	**G**

.

Again, replace the colors by the numbers of blocks between,

3	**1**	**2**	**1**	**3**	**2**	**0**	**0**

,

and then add one to each number,

4	**2**	**3**	**2**	**4**	**3**	**1**	**1**

.

What results is a so-called *Skolem sequence*, that is, an arrangement of $2n$ numbers consisting of two copies of each of the integers $1, 2, \ldots, n$ such that for each k, the pair of k's in the arrangement are separated by $k - 1$ intervening integers.

Now, let's attach labels from 1 to 8 that indicate the position of each of the eight numbers in the Skolem sequence.

4	**2**	**3**	**2**	**4**	**3**	**1**	**1**
1	2	3	4	5	6	7	8

In so doing, we find that we have solved a problem Skolem was studying [**177**] in those cases when there is a solution, namely $n \equiv 0$ or $1 \pmod 4$. The problem is to partition the numbers from 1 to $2n$ into n pairs whose differences are the numbers 1 to n. Here is how Skolem did it:

For $i = 1, 2, \dots, n$, let a_i and b_i denote the positions of the first and second occurrences of the number i in the sequence. Then $b_i = a_i + i$ for $1 \le i \le n$ and no a_i can equal a b_j, so the set of pairs $P_n = \{(a_i, b_i) : 1 \le i \le n\}\}$ partition the set $\{1, \dots, n\}$. For the preceding Skolem sequence of length 8, we have $a_1 + 1 = 7 + 1 = 8 = b_1$, and similarly $2 + 2 = 4$, $3 + 3 = 6$, and $1 + 4 = 5$. Thus,

$$8 - 7 = 1, \quad 4 - 2 = 2, \quad 6 - 3 = 3, \quad \text{and} \quad 5 - 1 = 4.$$

We will encounter Skolem sequences again in Chapter 6, where we find out why Skolem cared about the equation $x + y = z$.

If we do not restrict ourselves to a finite set of integers, we can partition the positive integers into two sequences A and B with the differences between corresponding terms given in the bottom line of Figure 1.1.

A	1	**3**	4	6	**8**	9	11	12	14	16	17	19	**21**	$\cdots$
B	2	**5**	7	10	**13**	15	18	20	23	26	28	31	**34**	$\cdots$
difference	1	**2**	3	4	**5**	6	7	8	9	10	11	12	**13**	$\cdots$

Figure 1.1. Two Beatty sequences that partition the integers

A and B are examples of *Beatty sequences*, which are sequences of the form $\{\lfloor n\alpha \rfloor : n = 1, 2, \dots\}$, where α is an irrational number greater than 1. In this example, $A = \{\lfloor n\phi \rfloor\}$ and $B = \{\lfloor n\phi^2 \rfloor\}$, where $\phi = (1 + \sqrt{5})/2$ is the famous *golden section*.

In Figure 1.2, the filled circles are the first seven elements of $\{\lfloor n\phi \rfloor\}$ and the open circles are the first four elements of $\{\lfloor n\phi^2 \rfloor\}$. Notice that

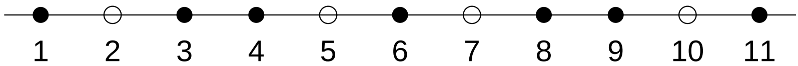

Figure 1.2. The first eleven positive integers, with $\lfloor n\phi \rfloor$ as filled circles and $\lfloor n\phi^2 \rfloor$ as open circles

since $1 < \phi < 2$, the difference between successive elements of sequence A is always either 1 or 2. Similarly, in sequence B it is either 2 or 3.

We make two observations. First, since ϕ is the positive root of the polynomial $x^2 - x - 1$, we see that $\phi^2 = \phi + 1$, and so dividing by ϕ^2 tells us that $1/\phi + 1/\phi^2 = 1$. Second, we see that the Beatty sequences A and B together contain every positive integer without repetition; two such Beatty sequences are called *complementary*. These two observations are at the heart of the following far-from-obvious theorem.

Theorem 1.2.1 (Beatty's Theorem). *Let α and β be any pair of irrational numbers greater than* 1 *satisfying* $1/\alpha + 1/\beta = 1$. *Then the sequences* $\{\lfloor n\alpha \rfloor\}$ *and* $\{\lfloor n\beta \rfloor\}$ *contain every positive integer without repetition.*

Proof. Let α and β be irrational numbers greater than 1 such that $1/\alpha + 1/\beta = 1$.

First, we show that the sequences $\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$ have no elements in common. Suppose, to the contrary, that there exist positive integers m, i, and j such that $m = \lfloor i\alpha \rfloor = \lfloor j\beta \rfloor$. Since α and β are irrational, we know that

$$m < i\alpha < m + 1 \text{ and } m < j\beta < m + 1, \text{ so}$$
$$\frac{m}{\alpha} < i < \frac{m+1}{\alpha} \text{ and } \frac{m}{\beta} < j < \frac{m+1}{\beta}.$$

Adding these inequalities, we see that

$$m = m \cdot 1 = m\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) < i + j < (m+1)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = m + 1,$$

contradicting the assumption that i and j are positive integers.

Second, we show that the sequences exclude no positive integer. Again, suppose the positive integer m is excluded. Then there exist positive integers x, i, and j such that

$$i\alpha < m < m + 1 < (i + 1)\alpha \text{ and } j\beta < m < m + 1 < (j + 1)\beta.$$

(The inequalities are strict because, as before, α and β are irrational.) Dividing the sets of inequalities by α and β, respectively, and adding them leads to the string of inequalities

$$i + j < m\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = m < m + 1 = (m + 1)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) < i + j + 2.$$

There cannot be two integers strictly between $i+j$ and $i+j+2$, contradicting the assumption that m was excluded. We conclude that indeed, the sequences $\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$ form a partition of the positive integers. □

Thus, the sequences come by the name "complementary" honestly. Also, restricting α and β to be greater than 1 avoids both $\lfloor n\alpha \rfloor = 0$ and $\lfloor n\beta \rfloor = 0$.

Samuel Beatty is one of the many curious figures that have appeared in the world of combinatorics. A Canadian by birth, he entered the University of Toronto as a student in 1903 and stayed there for the rest of his professional life, eventually becoming the university chancellor. He was the only doctoral student of John Charles Fields, in honor of whom the Fields Medal is named. By all accounts a beloved teacher, mentor, and strong supporter of his students, he is best known for a problem that appeared as Problem 3173 in the March 1926 issue of the *American Mathematical Monthly* [**12**], which we now state.

> If x is a positive irrational number and y is its reciprocal, prove that the sequences
>
> $$(1 + x), 2(1 + x), 3(1 + x), \dots \text{ and}$$
> $$(1 + y), 2(1 + y), 3(1 + y), \dots$$
>
> contain one and only one number between each pair of consecutive positive integers.

An equivalent restatement is that if x is a positive irrational number and y is its reciprocal, prove that the sequences

$$\lfloor (1 + x) \rfloor, \lfloor 2(1 + x) \rfloor, \lfloor 3(1 + x) \rfloor, \dots \text{ and}$$
$$\lfloor (1 + y) \rfloor, \lfloor 2(1 + y) \rfloor, \lfloor 3(1 + y) \rfloor, \dots$$

contain each positive integer once without duplication.

Beatty's conditions on x and y transform to ours by setting $\alpha = 1 + x$ and $\beta = 1 + y = (x + 1)/x$. For then,

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{1+x} + \frac{x}{x+1} = \frac{1+x}{1+x} = 1.$$

Problem 3173 is arguably the most studied problem ever to appear in the *Monthly*.

Are there other ways to generate complementary sequences? Indeed there are, and here is one of those ways, as described in James Tanton's delightful book named *Mathematics Galore!* [**193**]:

Begin with any nondecreasing sequence $\mathcal{P} = \{p_1, p_2, \dots\}$ of positive integers, such as the following sequence:

$$2, 2, 3, 5, 8, 11, 11, 11, 13, 17, 19, 23, \dots.$$

Now define the *frequency sequence* $\mathcal{Q} = \{q_1, q_2, \dots\}$ of nonnegative integers, where q_k is the number of entries in $\mathcal{P}$ less than k. Thus,

$$\begin{aligned} q_1 &= (\text{number of elements in } \mathcal{P} \text{ less than } 1) = 0,\\ q_2 &= (\text{number of elements in } \mathcal{P} \text{ less than } 2) = 0,\\ q_3 &= (\text{number of elements in } \mathcal{P} \text{ less than } 3) = 2,\\ q_4 &= (\text{number of elements in } \mathcal{P} \text{ less than } 4) = 3, \dots, \end{aligned}$$

and so $\mathcal{Q} = \{0, 0, 2, 3, 3, 4, 4, 4, 5, 5, 5, 8, 8, 9, \dots\}$. Next, find the frequency sequence of $\mathcal{Q}$; this turns out to be $\{2, 2, 3, 5, 8, 11, 11, 11, 13, \dots\}$. But that is the sequence we started with—that is, the frequency sequence of the frequency sequence of $\mathcal{P}$ is $\mathcal{P}$ itself!

Finally, construct the two sequences $\mathcal{P}^* = \{ p_n + n : n = 1, 2, \dots \}$ and $\mathcal{Q}^* = \{ q_n + n : n = 1, 2, \dots \}$; then

$$\begin{aligned} \mathcal{P}^* &= \{3, 4, 6, 9, 13, 17, 18, 19, \dots \},\\ \mathcal{Q}^* &= \{1, 2, 5, 7, 8, 10, 11, 12, 14, 15, 16, 20, \dots \}, \end{aligned}$$

and we see that $\mathcal{P}^*$ and $\mathcal{Q}^*$ are complementary sequences. For a proof that this method works every time, see [**193**, p. 29].

Let $\mathcal{P}$ be a nondecreasing sequence of positive integers, let $\mathcal{Q}$ be its frequency sequence, and let $\mathcal{P}^*$ and $\mathcal{Q}^*$ be constructed from $\mathcal{P}$ and $\mathcal{Q}$ as in the preceding paragraph. It is possible that $\mathcal{P}^*$ and $\mathcal{Q}^*$ are a pair of complementary Beatty sequences. But is that always the case? No. Consider, for example, the sequence of positive integers $\mathcal{P} = \mathbb{Z}^+$. We see that the frequency sequence of $\mathbb{Z}+$ is the sequence $\mathcal{Q}$ of nonnegative integers. Then

$$\mathcal{P}^* = \{2, 4, 6, 8, 10, 12, \dots\} = 2\mathbb{Z}, \text{ and}$$
$$\mathcal{Q}^* = \{1, 3, 5, 7, 9, 11, \dots\} = 2\mathbb{Z} + 1,$$

the even and odd numbers, respectively. Suppose $2\mathbb{Z} = \{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \dots\}$ for some irrational number α. Then $\alpha = 2+\delta$ for some $\delta \in (0, 1)$. If $\frac{1}{k} < \delta < \frac{1}{k-1}$, then $2k = \lfloor k\alpha \rfloor = \lfloor 2k+k\delta \rfloor = 2k+1$, contrary to assumption. Thus, the odd numbers and even numbers are complementary sequences that are not a pair of complementary Beatty sequences.

1.3 Penrose tilings

Much of the material we discuss in this section can be found in Chapter 10 of *Tilings and Patterns* (*2nd edition*) [**77**], Grunbaum and Shephard's excellent book on the subject.

The Beatty sequences A and B from the above discussion are associated, via the golden section ϕ , with the well-known *Fibonacci numbers* f_n and *Lucas numbers* L_n, defined by the recurrences

$$f_1 = 1,\ f_2 = 1, \text{ and } f_{n+1} = f_n + f_{n-1} \text{ for } n \geq 2; \text{ and}$$
$$L_1 = 1,\ L_2 = 3, \text{ and } L_{n+1} = L_n + L_{n-1} \text{ for } n \geq 2.$$

We can apply the recurrences backward as well to define f_n and L_n for $n \leq 0$. The results are displayed in the following table:

rank n	$\cdots$	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	$\cdots$
Fibonacci	$\cdots$	-3	2	-1	1	0	1	1	2	3	5	8	13	21	34	$\cdots$
Lucas	$\cdots$	7	-4	3	-1	2	1	3	4	7	11	18	29	47	76	$\cdots$

The association of ϕ with the Fibonacci numbers comes from the fact that

$$\phi = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}.$$

We prove this when we return to, and expand upon, the Fibonacci numbers in Chapter 3.

The Beatty sequences A and B associated to ϕ and ϕ^2 occur in the arrangement of *short* and *long bow ties* in the *Conway worms* that appear in one of Roger Penrose's *aperiodic tilings* of the plane with *kites* and *darts* [**74**, 6]. Don't worry: we will define the italicized terms!

A *tiling* of a surface is a covering of the surface with nonoverlapping pieces that fit together exactly—that is, the pieces do not meet except in points that are parts of common edges or in corners. The pieces are called *tiles.* Usually the tiles are copies of a finite number of compact sets known as the *prototiles*. For example, Figure 1.3 shows the tilings of the plane into congruent equilateral triangles, squares, and regular hexagons that are familiar to us all. In each of these cases there is only one prototile.

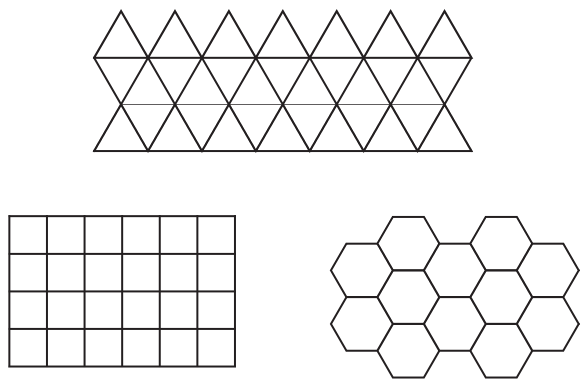

Figure 1.3. Periodic tilings of the plane by triangles, squares, and hexagons

The famous *wallpaper patterns* are tilings that use copies of a finite number of prototiles, arranged in patterns that exhibit various *symmetries*—that is, rigid motions of the entire tiling that bring it into self-coincidence. Translations, rotations, reflections, and glide-reflections are examples of such symmetries. A tiling is called *periodic* if it has at least two linearly independent translation symmetries.

Notice that it is easy to create a nonperiodic tiling by taking a periodic one (for example, the square tiling pictured above) and perturbing it slightly. For example, we could shift every row of tiles a small amount with respect to the row just above. If the distance we use is irrational, the tiling will no longer admit a nonhorizontal translational symmetry.

A finite set of prototiles is called *aperiodic* if they admit a tiling but do not admit a periodic tiling. In 1966, Robert Berger found an aperiodic set of 20,426 tiles that tile the plane and for which no periodic tiling exists. (This was his doctoral dissertation.) Smaller sets were found, and in 1971 Raphael M. Robinson found a set of six such tiles. At that point, the mathematical physicist Roger Penrose got interested in the problem, found another set of six tiles, reduced the number to four, and finally to two prototiles. In fact, he found two (closely related) pairs of prototiles with this property. We will focus on the best known one, which uses prototiles Penrose called the *kite* and the *dart*.

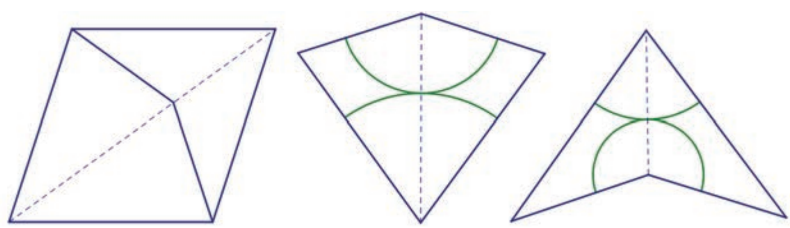

Figure 1.4. From left: the rhombus, the kite, and the dart

Kites and darts are made as follows. Begin with a rhombus $PQRS$ with edges of unit length and angles 72° and 108°. Draw the long diagonal PR, locate a point X on PR that is a golden section distance from P, draw the lines QX and SX, and erase the diagonal. Then $PQXS$ is a *kite with head X and tail P*, and $QRSX$ is a *dart with head R and tail X*. See Figure 1.4, which shows the rhombus and then the kite and the dart. Penrose tilings use the kite and dart as prototiles, but one adds a restriction: you are not allowed to put a kite and a dart together in the way we constructed them: they must not form a rhombus. This restriction can be enforced by adding small notches to the sides of the darts and kites, but we will do it in a way that seems more elegant, adding decorations to the prototiles. Once we construct the rhombus, we decorate it with four circular arcs, as we see in Figure 1.5. Suppose the sides of the rhombus have length 1. Then the circular arcs labeled G, H, K, and L have radii $r_G = 1/\varphi$, $r_H = 1/\varphi^2$, $r_K = 1/\varphi^3$, and $r_L = 1/\varphi^2$, respectively. The radii of both the dashed arcs and the solid arcs are in ratio $r_G/r_L = r_H/r_K = \varphi$. The

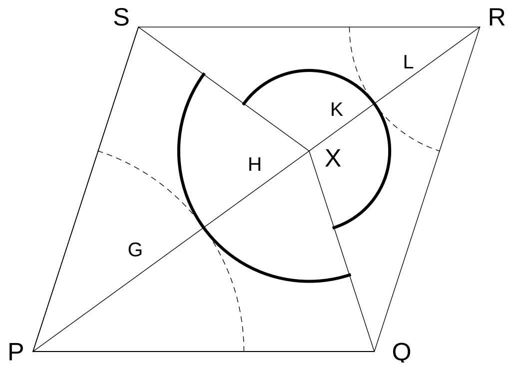

Figure 1.5. Rhombus showing the kite, the dart, and four circular arcs for constructing Penrose tilings

rule is that in a tiling the dashed arcs must connect to dashed arcs, and the solid arcs to solid arcs, making a continuous dashed (or solid) curve.

At this point, you might want to construct a set of darts and kites and start playing with them. There are also computer programs that can be used to draw these tilings. (We used *Mathematica* and a package called `PenroseTiles`.)

Figure 1.6 shows an example of such a tiling. We have removed the arcs from this picture to make the shape of the tiling clearer.

Inspecting the picture, we see several recurrent motifs: some are shaped like five-pointed stars, others like large decagons. We want to call your attention to three such patterns.

First, there is the *Ace*. It looks a little like a dart, but it is actually made up of two kites and a dart. In Figure 1.7 we have retained the arcs to show how the prototiles fit together.

Figure 1.8 shows two other patterns called *bow ties* that occur in Penrose tilings. They come in two kinds. A *short bow tie* is constructed from three kites and two darts, and a *long bow tie* is constructed from four kites and three darts.

The names of these figures all come from John Horton Conway, who became fascinated by the Penrose tilings and did a lot of work with them.

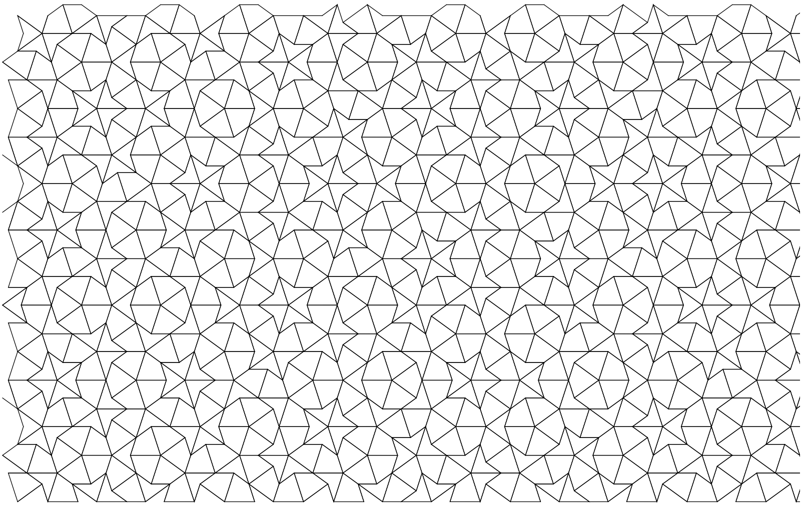

Figure 1.6. A Penrose darts-and-kites tiling

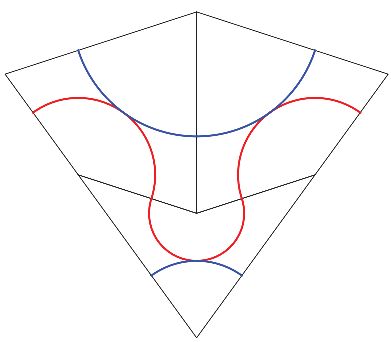

Figure 1.7. The Ace, or false dart

Conway has made notable contributions in group theory, number theory, sphere packing, symmetry, knot theory, coding theory, combinatorial game theory, analysis of algorithms, and cellular automata (such as the *Game of Life*), to name but a few. His books include the four-volume *Winning Ways* (with Elwyn Berlekamp and Richard Guy), *The Book of Numbers* with Guy, and *On Numbers and Games*. He is a force of nature.

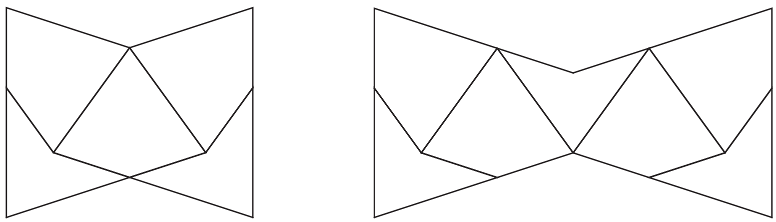

Figure 1.8. The short bow tie and the long bow tie

One of the things Conway discovered (and named) within Penrose tilings are sequences of short and long bow ties, called *Conway worms*, which occur within a Penrose tiling of the plane. In each worm, consecutive short bow ties are separated by either one or two long bow ties, as seen in Figure 1.9.

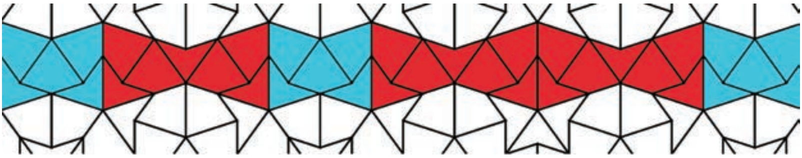

Figure 1.9. A Conway worm

There are lots of worms in several directions, with the proviso that when worms cross, they usually interrupt each other. Figure 1.10 shows a dozen worms going in various directions.

Inspecting the picture carefully, you will see that there are two kinds of worm crossings. First, it is possible for two worms to cross without disrupting each other: Notice that at the precise point of the crossing an Ace is formed, but both worms remain undisturbed; see Figure 1.11. This kind of crossing can only happen at an angle of $72^\circ = 2\pi/5$.

The other kind of crossing looks like Figure 1.12.

This time one of the worms seems to "run below" the other, so that its pattern of bow ties is interrupted. Once again, an Ace marks the crossing. This kind of crossing can occur at any angle. Of course, as Figure 1.10 shows, several worms can cross at the same point.

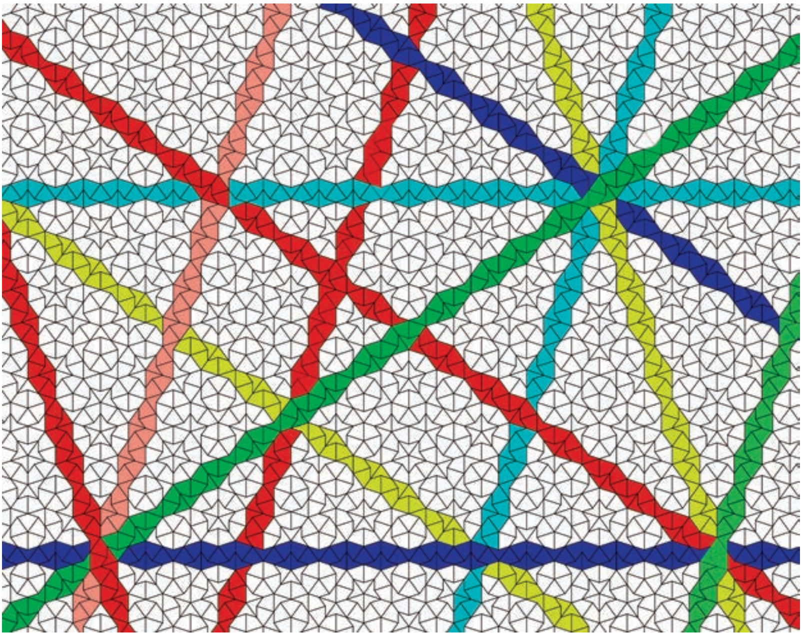

Figure 1.10. Part of a Penrose tiling with Conway worms shown

So how are worms related to the Beatty sequences A and B above? Let's first reproduce the two sequences.

A	1	**3**	4	6	**8**	9	11	12	14	16	17	19	**21**	$\cdots$
B	2	**5**	7	10	**13**	15	18	20	23	26	28	31	**34**	$\cdots$
difference	1	**2**	3	4	**5**	6	7	8	9	10	11	12	**13**	$\cdots$

As we noticed before, the differences between terms of sequence A are always 1 or 2. Think of 1 as a short bow tie and 2 as a long bow tie, and you get a worm! Figure 1.13 shows a worm and its relationship to sequence A. The numbers from sequence A appear at the places where two bow ties meet, and the bow ties correctly predict the differences. One caution is in order: both the worm and the Beatty sequence A are infinite, so there is no way to know whether we have picked the correct place to match up the two. This example works thus far, … but will it work forever?

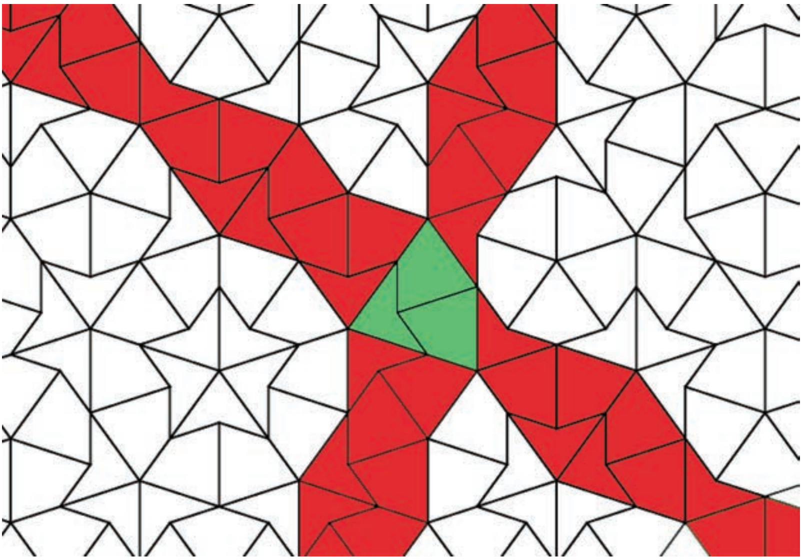

Figure 1.11. Two Conway worms crossing at an ace

Finally, why does the Beatty sequence associated to ϕ show up in the Conway worms? The reason is that the ratio of the length of a long bow tie to the length of a short bow tie is exactly $\phi : 1$.

As we noted above, Conway has done a lot of work on games. It turns out that the A Beatty sequence also has a connection (via a game invented by Willem A. Wythoff) to the fascinating world of combinatorial games. Let's see how.

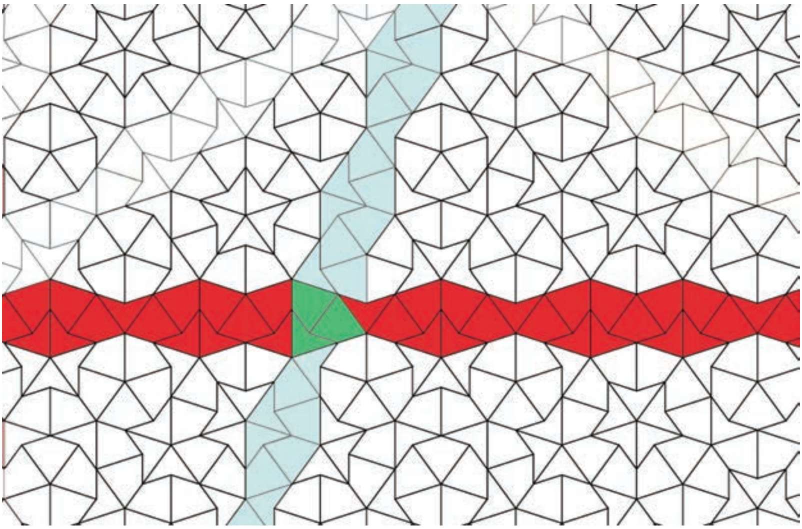

Figure 1.12. Two other worms crossing at an ace

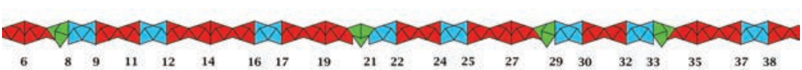

Figure 1.13. A Conway worm and a number line