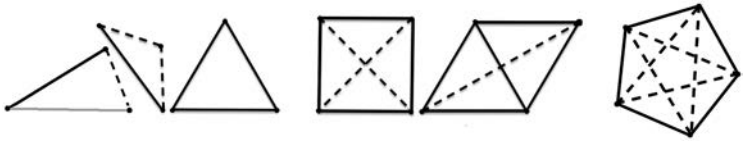
Chapter 2
The distance problem

2.1 Introduction to the distance problem

Let's play a game. Your goal is to draw three points on a blank sheet of paper, and then connect each pair of dots with a line, trying to minimize the number of *distinct* distances. How many distinct distances did you draw? If you picked three random points, then most likely you will have 3 distinct distances. If you drew an isosceles triangle, with two sides being the same length, then your drawing determined only two distances. Better still, if you drew an equilateral triangle so that all sides of the triangle are equal, then you win as you have three points determining only one distance, which is clearly the best you can do. Now, what happens if you have to select four points? What is the minimum number of distinct distances you achieve here? It turns out that a square will determine only two distances (the edge of the square and the diagonal), and this is best possible. Now what if you draw 5 points? 10 points? 100 points?



The complexity of the problem increases dramatically even for a small number of points (try this for 10 points for example). Therefore, rather than trying to find a function which would give the minimum of distances for exactly n points, it is more productive to ask about the long-term behavior of such a quantity: Roughly how many distances do we

achieve if we were to draw n points on a sheet of a paper, when n is very large? This question was originally posed by Paul Erdős in 1946 ([**49**]).

Suppose $f(n)$ denotes the minimum number of distances determined by a set of n points in $\mathbb{R}^2$, so that $f(3) = 1$ as an equilateral triangle determines a single distinct distance, $f(4) = f(5) = 2$ as shown above with the square and pentagon, and so on. What is the best asymptotic lower bound for $f(n)$, if we let n go to infinity? More precisely, given a set $E \subseteq \mathbb{R}^2$ of cardinality $|E| = n$, define

$$\Delta(E) = \{|x - y| \,:\, x, y \in E\} \subseteq \mathbb{R}, \tag{2.1}$$

where $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$ denotes the standard (Euclidean) distance. Then, the Erdős distance problem asks one for a lower bound on the quantity

$$f(n) = \min\{|\Delta(E)| \,:\, E \subseteq \mathbb{R}^2 \text{ and } |E| = n\}.$$

In [**49**], Erdős showed that one always has the bound

$$\sqrt{n} \ll f(n) \ll \frac{n}{\sqrt{\log n}}. \tag{2.2}$$

The upper bound in (2.2) came by considering the set

$$E = \left\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \,:\, 0 \leq x, y \leq \sqrt{n}\right\}$$

when n is a square. It is well known ([**90**]) that if $A(x)$ denotes the set of nonnegative integers less than or equal to x which are sums of two squares, then $|A(x)| \sim \gamma x/\sqrt{\log x}$, where $\gamma = 0.7642\ldots$ is the Landau-Ramanjuan constant. It follows that $|\Delta(E)| \leq 2|A(n)|$, so that

$$|\Delta(E)| \leq \frac{2\gamma n}{\sqrt{\log 2n}} \asymp \frac{n}{\sqrt{\log n}}. \tag{2.3}$$

The lower bound was achieved by some clever combinatorial reasoning. Essentially, Erdős' argument involved only elementary notions like the fact that two distinct circles intersect in at most 2 points, coupled with the pigeonhole principle. Details of this argument can be found in Section 6.2, and a thorough and intuitive description of these results and the progress thereafter are the subject of the excellent book *The Erdős distance problem* by Garibaldi, Iosevich, and Senger ([**61**]).

Erdős' bounds from (2.2) led him to conjecture ([**15**]) that $f(n) \gg n^{1-o(1)}$ for all sets $E \subseteq \mathbb{R}^2$ with cardinality n. The lower bound was

steadily improved ([**18**, **19**, **98**, **112**, **143**, **146**, **151**]) until more than sixty years after Erdős first wrote about this problem, a breakthrough was achieved by Guth and Katz ([**70**]). They showed that

$$f(n) \gg \frac{n}{\log n}, \tag{2.4}$$

a very near optimal bound which established the Erdős distance conjecture! Later (in Section 6.2) we will provide a brief outline of their proof which follows the so-called Elekes-Sharir framework ([**43**]), and we provide some related insights by applying analogues of their ideas to the finite field problem which we will describe below.

The Erdős distinct distance problem has been considered in higher dimensions as well. Let $f_d(n)$ denote the minimum number of Euclidean distances determined by a set of n points in $\mathbb{R}^d$. The integer lattice example giving the upper bound for $d = 2$ shows that when $d \geq 3$, we can hope for $f_d(n) \ll n^{2/d}$, and no better. More precisely, let

$$E = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d \,:\, 1 \leq x_i \leq n^{1/d} \text{ for all } i = 1, \ldots, d\}$$

for some integer n which is a perfect dth power. Legendre showed that if $B(x)$ denotes the set of positive integers which are the sum of three squares, then $B(x)$ satisfies $B(x) \sim \frac{5}{6}x$ (see Exercise 2.2). Additionally, every integer is the sum of at most four squares by Lagrange's four-square theorem. Hence,

$$f_d(n) \leq c_d n^{2/d} \ll n^{2/d}$$

for $d \geq 3$. It is further conjectured that $f_d(n) \gg n^{2/d}$ with no logarithmic loss whenever $d \geq 3$. Erdős' bound $f(n) \gg n^{1/2}$ can be adapted to higher dimensions to show that $f_d(n) \gg n^{1/d}$. The best result in general higher dimensions belongs to Solymosi and Vu ([**144**]) who showed that $f_d(n) \gg n^{\frac{2}{d} - \frac{2}{d^2+2d}}$.

2.1.1 Erdős unit-distance problem

Erdős also considered a second distance problem. Given a set of n points, how many times is it possible for a single distance to occur? Formally, we define

$$g(n) = \max_{\substack{|E|=n \\ E \subseteq \mathbb{R}^2}} |\{\|x - y\| = 1 \,:\, x, y \in E\}|\,.$$

Note that given a point set E, we can simply scale the set so that the most frequent distance achieved is 1, and hence there is no harm in replacing "most popular distance" with the distance 1. Now, Erdős showed that we must have the bound

$$g(n) \gg n^{1+c/\log\log n},$$

for some $c > 0$ by again considering the set

$$\left\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq x, y \leq \sqrt{n}\right\}$$

and using known results about the distribution of primes of the form $p = 4m + 1$ ([**48**]), since such primes can always be written as the sum of squares (see Exercise 2.4). He conjectured ([**15**]) that

$$g(n) \ll n^{1+o(1)},$$

but the best bound established thus far is

$$g(n) \ll n^{4/3}, \tag{2.5}$$

due to the work of Spencer, Szemerédi, and Trotter ([**140**]), which relied on the work of Szemerédi and Trotter on incidences:

Theorem 2.1.1 ([**148**]). *Let $P \subseteq \mathbb{R}^2$ be a set of points, and let L be a set of lines in $\mathbb{R}^2$. An incidence is a pair (p, ℓ) such that $p \in \ell$. The number of incidences determined by the sets P and L is then*

$$I(P,L) = |\{(p,\ell) \in P \times L : p \in \ell\}|.$$

Finally we have

$$I(P,L) \ll (|P|^{2/3}|L|^{2/3} + |P| + |L|).$$

See Section 7.1 for much more information on incidence theory and its applications to problems in additive combinatorial problems in particular.

Really, the result of Spencer, Szemerédi, and Trotter relied on an adaptation of the work of Szemerédi and Trotter where lines were replaced by curves with no self-intersections. This was then applied to circles in the plane. This remarkable theorem of Szemerédi and Trotter has many applications to geometric combinatorics including a result on the sum-product problem ([**41**]) and the unit-distance problem ([**140**]). We explore incidence theory in more detail in Chapter 7.

The unit-distance problem is only interesting in dimensions 3 and lower due to some arithmetic obstructions ([**89**]). In $\mathbb{R}^3$, we have that $f(n) \ll n^{2/3}$, so that $g_3(n) \gg n^{4/3}$ by the pigeonhole principle, where

$$g_3(n) = \max_{\substack{|E|=n \\ E\subseteq\mathbb{R}^3}} |\{||x-y|| = 1 : x, y \in E\}|.$$

Until recently, the best known bound was $g_3(n) \ll n^{3/2}$, but now the world record belongs to Joshua Zahl ([**160**]), as he has demonstrated that $g_3(n) \ll n^{\frac{3}{2}-\frac{1}{394}+o(1)}$.

The unit distance problem remains one of the most important unsolved problems in combinatorics. See, for example, [**1**, **15**, **93**, **108**, **117**] and the references therein for current results on the problem and its variants.

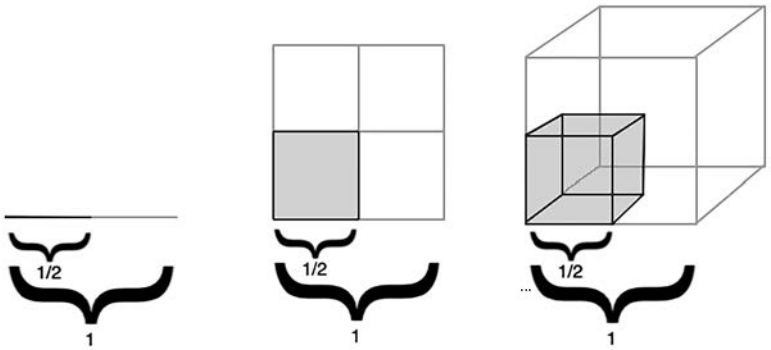
2.2 Falconer distance problem

In 1986, Falconer ([**52**]) considered a continuous version of the distance problem. While the Falconer distance problem is very similar to that of the Erdős distance problem, the Falconer distance problem requires a more technical setup, so these ideas should not be taken too seriously, especially if the notions of measure and fractal dimension are new to you.

There are two main ideas necessary to understand the gist of the Falconer distance problem. The first idea is that of (Lebesgue) measure. The measure of a set $E \subseteq \mathbb{R}^d$ is the quantity that mathematicians use which can loosely be thought of as length in $\mathbb{R}$, area in $\mathbb{R}^2$, volume in $\mathbb{R}^3$, and so on. Also, a set $E \subseteq \mathbb{R}^d$ has measure 0 if for any $\epsilon > 0$, the set can be covered by a countable (or finite) collection of open balls whose union has volume ϵ. The only fact concerning measure which we will require here is that any countable set $E \subseteq \mathbb{R}^d$ has measure 0 (see Exercise 2.5).

The second idea to convey is that of Hausdorff dimension. We loosely describe dimension in the following intuitive way. If you were to take a line segment of length 1 and scale down the line segment by 1/2, you would unsurprisingly (and perhaps uninterestingly) get a line segment of length $1/2 = (1/2)^1$. If you take a square of area 1, and scale down

each side-length to half its size, the resulting square would be of area $1/4 = (1/2)^2$. If you take a cube of area 1 and scale down each side to half its original size, you would make a cube with volume $1/8 = (1/2)^3$. You can think of those powers in the terms $(1/2)^1$ for the line segment, $(1/2)^2$ for the square, and $(1/2)^3$ for the cube as the reason why the line segment is 1-dimensional, the square is 2-dimensional, and the cube is 3-dimensional.



Now let's consider the strange and wonderful fractal that is the Cantor set. The Cantor set $\mathcal{C}$ is defined as follows: Start with the unit interval $[0,1] \subseteq \mathbb{R}$, and remove the middle third $(1/3, 2/3)$. You are left with only the union of two intervals $[0,1/3] \cup [2/3,1]$. Now from each of those intervals, remove the middle third, to be left with a union of 4 intervals, each of length $1/9$. Taking a limit of this process gives a fractal called the Cantor set[1]:

Notice that if you scale down the Cantor set by a factor of 3, you have split the Cantor set into two identical copies of itself, each $1/3$ the size of the original. This means that the dimension s of the Cantor set should satisfy $(1/3)^s = 1/2$. In this sense the Cantor set $\mathcal{C}$ has fractal dimension $\ln(2)/\ln(3) \approx 0.63093\ldots$.

[1]Image taken from [**156**].

While Hausdorff dimension is the notion of fractal dimension used in the Falconer distance problem, there are many other notions of fractal dimension, particularly Minkowski dimension, box-counting dimension, and packing dimension. We will write $\dim_H(E)$ to denote the Hausdorff dimension of the set E. See Section 7.3 for an introduction to the Minkowski dimension, and the interested reader should see [**110**] for an excellent treatise on fractal geometric including rigorous definitions of measure, fractal dimension, and much more.

Let $\Delta(E)$ be the distance set defined as before. Falconer showed that if $E \subseteq \mathbb{R}^d$ is compact and the set E has Hausdorff dimension $\dim_H(E) > \frac{d+1}{2}$, then the set $\Delta(E)$ would have positive Lebesgue measure. Furthermore, he showed that for any $s < \frac{d}{2}$, there exists a set $E \subseteq \mathbb{R}^d$ such that $\dim_H(E) = s$, and yet the distance set $\Delta(E)$ has Lebesgue measure zero. This led him to conjecture that if a set $E \subseteq \mathbb{R}^d$ is compact with $\dim_H(E) > \frac{d}{2}$, then the distance set $\Delta(E)$ would have positive measure. The details of Falconer's construction can be found in [**53**] or [**110**], for example.

The Falconer distance problem can be thought of as a generalization of the Steinhaus Theorem ([**53**]), which states that if $A \subseteq \mathbb{R}$ is of positive (Lebesgue) measure, then the set $A - A = \{a - a' : a, a' \in A\}$ contains some interval of the form $[0, \varepsilon)$ for some $\varepsilon > 0$. The Falconer distance problem has received considerable attention, most notably from Wolff, Bourgain, and Erdoğan. For a long while, the best known result in the plane was due to Wolff ([**157**]) as he demonstrated that $\Delta(E)$ has positive Lebesgue measure when $\dim_H(E) > 4/3$ for all compact sets $E \subseteq \mathbb{R}^2$, hence why we refer to the results in Chapter 4 as "Wolff's Exponent."

Interestingly, Wolff's bound was improved by Guth, Iosevich, Ou, and Wang ([**69**]) who demonstrated that if $E \subseteq \mathbb{R}^2$ is compact and has $\dim_H(E) > 5/4$, then $\Delta(E)$ has positive Lebesgue measure[2]. If $E \subseteq \mathbb{R}^2$ is compact and also s-Ahlfors-David regular for some $s \geq 1$, then Orponen ([**118**]) has shown that $\Delta(E)$ has packing dimension $\dim_p(\Delta(E)) = 1$. In higher dimensions the best known results currently belong to Du, Guth, Ou, Wang, Wilson, Zhang ([**37**]) for $d = 3$ and Du and Zhang ([**38**] for $d \geq 4$. They have shown that if $E \subseteq \mathbb{R}^3$ satisfies $\dim_H(E) > 9/5$ and if $E \subseteq \mathbb{R}^d$

[2]They actually proved a stronger so-called pinned-distance result, the likes of which we discuss in Section 5.4.

(for $d \geq 4$) satisfies $\dim_H(E) > \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}$, then $\Delta(E)$ has positive Lebesgue measure in each case. See [**47**, **110**, **157**] for a more detailed treatise of the subject.

2.3 Finite field distance problem

Finite fields have long been used as an uncomplicated setting in which one could play with Euclidean problems in an environment with fewer technical difficulties. To this end the distance problem was considered in this finite setting in order to gain some insight into the Euclidean distances problems discussed above. A finite field version of the distance problem will be described below.

We will always require the parameter q to be odd unless explicitly stated otherwise. Let $\mathbb{F}_q^d$ denote the d-dimensional vector space over the finite field with q elements. Recall that q must be the power of an odd prime[3]. For $x = (x_1, \ldots, x_d) \in \mathbb{F}_q^d$, we set

$$||x|| = x \cdot x = x^t x = x_1^2 + \cdots + x_d^2 \in \mathbb{F}_q,$$

viewing x as a column vector. The quantity $||x||$ is clearly not a norm in any analytic sense, though it should be observed that the function $f(x) = ||x||$ is preserved by rigid motions. More precisely, let $O_d(\mathbb{F}_q)$ denote the set of $d \times d$ orthogonal matrices with entries in $\mathbb{F}_q$. Then, it is easy to check that $||x|| = ||Ox||$ for any $O \in O_d(\mathbb{F}_q)$. As before, for $E \subseteq \mathbb{F}_q^d$ we define the distance set of E as

$$\Delta(E) = \{||x - y|| : x, y \in E\} \subseteq \mathbb{F}_q.$$

This notion of distance is also preserved by rigid motions in $\mathbb{F}_q^d$.

Example 2.3.1. Consider the set $\mathbb{F}_{13}^2$ consisting of the five points $E = \{(4,2); (4,3); (5,4); (7,4); (3,6)\}$. You can check that $\Delta(E) = \{0, 1, 2, 4, 5, 7, 8, 10\} \subseteq \mathbb{F}_{13}$ (see Figure 2.1).

The first result on distances in finite fields came from Bourgain, Katz, and Tao ([**14**]) who showed the following.

[3] If finite fields are new mathematical objects to you, then it may be helpful to first consider the case when q is prime, as $\mathbb{F}_q$ is indeed a finite field when q is prime.

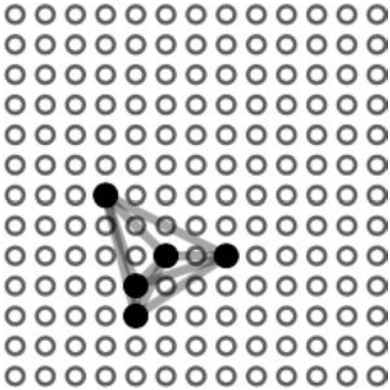


Figure 2.1. The set E from Example 2.3.1 represented by the set of integer points (x, y) where $0 \leq x, y \leq 12$

Theorem 2.3.2. *Suppose that $E \subseteq \mathbb{F}_p^2$, where $p \equiv 3 \pmod 4$ is a prime. Suppose that E has cardinality $|E| = p^\alpha$ for some $0 < \alpha < 2$. Then, there exists a quantity $\epsilon = \epsilon(\alpha)$ such that*

$$|\Delta(E)| \gg |E|^{\frac{1}{2}+\epsilon}.$$

Note that if $\alpha = 2$, then $|\Delta(E)| = |E|^{1/2}$, and no better. Notice also that if $p \equiv 1 \pmod 4$, then there exists an element $i \in \mathbb{F}_p$ such that $i^2 = -1$. Hence, we can construct a set $E = \{(x, ix) : x \in \mathbb{F}_p\}$ such that $\Delta(E) = \{0\}$, and yet $|E| = p$.

This finite field distance problem was rephrased and improved upon by Iosevich and Rudnev ([**87**]). Rather than showing an Erdős-style bound of $|\Delta(E)| > |E|^\beta$, it is more fruitful to ask how large a set $E \subseteq \mathbb{F}_q^d$ needs to be in order to ensure that $\Delta(E) = \mathbb{F}_q$, or more modestly that $|\Delta(E)| \gg q$. Therefore the finite field distance problem has flavors of both the Erdős distance problem and the Falconer distance problem, and is henceforth referred to as the Erdős-Falconer distance problem. Before we state the current conjecture, we first detail what progress has been made on this problem.

Theorem 2.3.3 ([**87**])**.** *Let $E \subseteq \mathbb{F}_q^d$ with $|E| > 2q^{\frac{d+1}{2}}$. Then, $\Delta(E) = \mathbb{F}_q$.*

We will prove this theorem in Chapter 3. In the same chapter, we will give proofs of the weaker result $|\Delta(E)| \gg q$ when $|E| \gg q^{\frac{d+1}{2}}$, which turns out to have many applications to generalizations of the distance problem. Regarding counterexamples, notice that if $q = p^2$, and since $\mathbb{F}_q$ contains a subfield isomorphic to $\mathbb{F}_p$, then we have found a set $E \subseteq \mathbb{F}_q^d$ with $|E| = p^d = q^{d/2}$, and yet $|\Delta(E)| = p = \sqrt{q}$, and no better. Even worse, adapting a counterexample from above, note that in any even dimension $d = 2k$, if -1 is a square in $\mathbb{F}_q$, then we could construct

$$E = \{(x_1, ix_1, x_2, ix_2, \dots, x_k, ix_k) : x_i \in \mathbb{F}_q\}$$

which gives a set of size $|E| = q^{d/2}$ with $\Delta(E) = \{0\}$. Using these examples and the Falconer distance conjecture as a guide, it would be reasonable to assume that if $|E| \geq Cq^{d/2}$ for a sufficiently large constant C, then one achieves $\mathbb{F}_q = \Delta(E)$. However, it was shown in [**76**] that the exponent $\frac{d+1}{2}$ is best possible at least in odd dimensions. That is, given $\mathbb{F}_q^d$, where $d \geq 3$ is odd, we can construct $E \subseteq \mathbb{F}_q^d$ such that $|E| \asymp q^{\frac{d+1}{2}-\epsilon}$ and yet $|\Delta(E)| \ll q^{1-\epsilon}$. In the Euclidean setting point-sets behave more or less the same regardless of the parity of the dimension of the ambient space. In finite fields, however, the parity of the dimension will play a much larger role. We will construct the above counterexample in Section 3.5.

Finally, some partial improvement to Theorem 2.3.3 is known but only in the case $d = 2$. Chapman, Erdoğan, Hart, Iosevich, and Koh ([**17**]), showed that if $E \subseteq \mathbb{F}_q^2$ where $q \equiv 3 \pmod 4$ and $|E| \geq q^{4/3}$, then $|\Delta(E)| \geq cq$ for some $0 < c \leq 1$. A further paper ([**7**]) was able to extend this result to the case $q \equiv 1 \pmod 4$. Combining these results, we have:

Theorem 2.3.4. *Let $E \subseteq \mathbb{F}_q^2$ such that $|E| \geq q^{4/3}$ for a sufficiently large constant C. Then, there exists an absolute constant $c \in (0, 1]$ such that $|\Delta(E)| \geq cq$.*

While this shows that the distance set is large when E is of sufficiently large cardinality (in fact we show that at the very least the distance set contains a positive proportion of the field $\mathbb{F}_q$), we still do not know which elements of $\mathbb{F}_q^*$ are in the distance set. Thus, if we care whether or not $1 \in \Delta(E)$, for example, then we must require that $|E| > 2q^{\frac{d+1}{2}}$.

In an interesting twist Murphy and Petridis ([**114**]) demonstrated that the 4/3 threshold is best possible in $\mathbb{F}_q^2$, at least for arbitrary finite fields when you want to guarantee that all elements are in the distance set $\Delta(E)$. Specifically they construct a family of sets $E \subseteq \mathbb{F}_q^2$ with $|E| = q^{4/3}$ such that $|\Delta(E)| < q$.

Other results are known to hold for certain sets E (such as Salem sets and subsets of spheres), and we collect such results in later chapters (see Subsection 3.3.1 and Theorem 5.3.6). Putting all of the results together, we now state two versions of the Erdős-Falconer distance conjecture.

Conjecture 2.3.5 (Strong Erdős-Falconer Distance Conjecture)**.** *Let $E \subseteq \mathbb{F}_q^d$, $d \geq 4$ even, be such that $|E| \geq Cq^{d/2}$ for a sufficiently large constant C. Then, $\Delta(E) = \mathbb{F}_q$.*

Conjecture 2.3.6 (Weak Erdős-Falconer Distance Conjecture)**.** *Let $E \subseteq \mathbb{F}_q^d$, $d \geq 2$ even, be such that $|E| \geq Cq^{d/2}$ for a sufficiently large constant C. Then, $|\Delta(E)| \geq cq$ for some $0 < c \leq 1$.*

In the following chapters, we will recount the progress made on the strong and weak versions of the Erdős-Falconer conjectures and its many variants thus far.

Before we prove any finite field distance result, I want to emphasize the importance and utility of studying finite field analogues of classical (Euclidean) problems. While we have not seen all of these problems just yet, I will mention a strange intertwining of results between both the finite field and Euclidean worlds. The Kakeya conjecture (see Section 7.3) asserts that any set containing a line in every direction must be appropriately large. The finite field analogue, long thought to be a difficult analytical problem, turned out to have an easy (one paragraph) algebraic solution using the so-called algebraic method (see Theorem 7.3.9). This algebraic method turned out not to be useful for the original Kakeya problem, though it did inspire G. Elekes and M. Sharir to recast the Erdős distance problem in terms of algebraic operations (see Section 6.2). This recasting of the problem allowed for the aforementioned solution of the Erdős distance problem in dimension $d = 2$ by Guth-Katz, but it did not yield any gains in higher dimensions due to some annoying algebraic obstructions there. The progress made on the Erdős

distance problem further helped to improve the Falconer distance problem in dimension $d = 2$, but also did not yield any new results in higher dimensions. The Erdős distance problem remains open for $d \geq 3$, the Erdős-unit distance problem remains open in dimensions 2 and 3, the Falconer distance problem remains open in all dimensions $d \geq 2$, and the finite field distance problem is open for all *even* dimensions $d \geq 2$. At any step in this strange winding road of progress, it was not obvious that a solution to some problem was going to assist with the solution to another, but undoubtedly, finite fields have played a key role in better understanding some combinatorial problems, even in the Euclidean setting.

2.4 Exercises: Chapter 2

Exercise 2.1. *Find a set $E \subseteq \mathbb{F}_3^2$ such that $|E| = 3$ and $1 \notin \Delta(E)$. On the other hand, prove that any set $E \subseteq \mathbb{F}_3^2$ with cardinality $|E| \geq 4$ satisfies $\Delta(E) = \mathbb{F}_3$. Moreover, find a set $E \subseteq \mathbb{F}_5^2$ such that $|E| = 10$, and yet $1 \notin \Delta(E)$. Then, prove that if $E \subseteq \mathbb{F}_5^2$ has cardinality $|E| \geq 11$, then $\Delta(E) = \mathbb{F}_5$.*

Exercise 2.2. *Let $B = \{n \in \mathbb{Z}^+ : n = x^2 + y^2 + z^2 \text{ for some } x, y, z \in \mathbb{Z}\}$, and consider its counting function $B(x) = \{n \in B : n \leq x\}$. Provide a counting argument for:*

$$\lim_{x \to \infty} \frac{B(x)}{x} = \frac{5}{6}.$$

HINT: Use the following heuristic: Legendre's 3 squares theorem states that an integer is not a sum of three squares if and only if it is of the form $4^a(8b + 7)$ for nonnegative integers a and b. Finally, notice

$$\sum_{k=0}^{\infty} \frac{1}{8 \cdot 4^k} = \frac{1}{6}.$$

Exercise 2.3. *Show that if you take $E \subseteq \mathbb{F}_2^d$ to be the set of vertices which have an even number of zero components, then $|E| = 2^{d-1}$ and yet $\Delta(E) = \{0\}$. However, using the pigeonhole-principle, show that if $|E| > 2^{d-1}$, then $\Delta(E) = \mathbb{F}_2$. This is why we emphasize that in the above theorems (and the text throughout), q is assumed to be odd.*

Exercise 2.4. *Look up and write down any proof of Fermat's Theorem on sums of two squares which states that an odd prime p is a sum of two squares if and only if $p \equiv 1 \pmod 4$.*

Exercise 2.5. *Prove that any countable set in $\mathbb{R}^d$ has measure 0, using the definitions laid out in Section 2.2.*

Exercise 2.6. *Prove that every element in $\mathbb{F}_q$ is the sum of two squares using the pigeonhole principle. Conclude that the square lattice construction in (2.3) does not yield a logarithmic loss in $\mathbb{F}_q^2$.*